19. LC and RLC Oscillators

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Abstract

Lecture slides 19 for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island.

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Mechanical Oscillator

- law of motion: \( F = ma, \quad a = \frac{d^2 x}{dt^2} \)
- law of force: \( F = -kx \)
- equation of motion: \( \frac{d^2 x}{dt^2} = -\frac{k}{m} x \)
- displacement: \( x(t) = x_{max} \cos(\omega t) \)
- velocity: \( v(t) = -\omega x_{max} \sin(\omega t) \)
- angular frequency: \( \omega = \sqrt{\frac{k}{m}} \)
- kinetic energy: \( K = \frac{1}{2} mv^2 \)
- potential energy: \( U = \frac{1}{2} kx^2 \)
- total energy: \( E = K + U = \text{const.} \)
Electromagnetic Oscillator (LC Circuit)

- Loop rule: \( \frac{Q}{C} + L \frac{dI}{dt} = 0 \), \( I = \frac{dQ}{dt} \)

- Equation of motion: \( \frac{d^2 Q}{dt^2} = -\frac{1}{LC} Q \)

- Charge on capacitor: \( Q(t) = Q_{max} \cos(\omega t) \)

- Current through inductor: \( I(t) = -\omega Q_{max} \sin(\omega t) \)

- Angular frequency: \( \omega = \frac{1}{\sqrt{LC}} \)

- Magnetic energy: \( U_B = \frac{1}{2} LI^2 \) (stored on inductor)

- Electric energy: \( U_E = \frac{Q^2}{2C} \) (stored on capacitor)

- Total energy: \( E = U_B + U_E = \text{const.} \)
Mechanical vs Electromagnetic Oscillations

**Mechanical Oscillations**

- **position:** \( x(t) = A \cos(\omega t) \) [red]
- **velocity:** \( v(t) = -A \sin(\omega t) \) [green]
- **period:** \( \tau = \frac{2\pi}{\omega}, \quad \omega = \sqrt{\frac{k}{m}} \)

**Electromagnetic Oscillations**

- **charge:** \( Q(t) = A \cos(\omega t) \) [red]
- **current:** \( I(t) = -A \sin(\omega t) \) [green]
- **period:** \( \tau = \frac{2\pi}{\omega}, \quad \omega = \frac{1}{\sqrt{LC}} \)

**Potential Energy**

- **potential energy:** \( U(t) = \frac{1}{2} k x^2(t) \) [r]
- **kinetic energy:** \( K(t) = \frac{1}{2} m v^2(t) \) [g]
- **total energy:** \( E = U(t) + K(t) = \text{const} \)

**Electric Energy**

- **electric energy:** \( U_E(t) = \frac{1}{2C} Q^2(t) \) [r]
- **magnetic energy:** \( U_B(t) = \frac{1}{2} LI^2(t) \) [g]
- **total energy:** \( E = U_E(t) + U_B(t) = \text{const} \)
Mechanical Oscillator with Damping

- law of motion: \( F = ma, \quad a = \frac{d^2 x}{dt^2} \)
- law of force: \( F = -kx - bv, \quad v = \frac{dx}{dt} \)
- equation of motion: \( \frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \)

Solution for initial conditions \( x(0) = A, \quad v(0) = 0 \):

(a) underdamped motion: \( b^2 < 4km \)

\[
x(t) = Ae^{-bt/2m} \left[ \cos(\omega't) + \frac{b}{2m\omega'} \sin(\omega't) \right] \quad \text{with} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}
\]

(b) overdamped motion: \( b^2 > 4km \)

\[
x(t) = Ae^{-bt/2m} \left[ \cosh(\Omega't) + \frac{b}{2m\Omega'} \sinh(\Omega't) \right] \quad \text{with} \quad \Omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}
\]
Damped Electromagnetic Oscillator (RLC Circuit)

- **loop rule:** \( R I + L \frac{dI}{dt} + \frac{Q}{C} = 0 \), \( I = \frac{dQ}{dt} \)
- **equation of motion:** \( \frac{d^2 Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC} Q = 0 \)

Solution for initial conditions \( Q(0) = Q_{max}, I(0) = 0 \):

(a) **underdamped motion:** \( R^2 < \frac{4LC}{C} \)

\[
Q(t) = Q_{max} e^{-Rt/2L} \left[ \cos(\omega't) + \frac{R}{2L\omega'} \sin(\omega't) \right] \quad \text{with} \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}
\]

(b) **overdamped motion:** \( R^2 > \frac{4LC}{C} \)

\[
Q(t) = Q_{max} e^{-Rt/2L} \left[ \cosh(\Omega't) + \frac{R}{2L\Omega'} \sinh(\Omega't) \right] \quad \text{with} \quad \Omega' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}
\]
Name the $LC$ circuit with the highest and the lowest angular frequency of oscillation.
At time $t = 0$ a charge $Q = 2C$ is on each capacitor and all currents are zero.

(a) What is the energy stored in the circuit?
(b) At what time $t_1$ are the capacitors discharged for the first time?
(c) What is the current through each inductor at time $t_1$?
In these \( LC \) circuits all capacitors have equal capacitance \( C \) and all inductors have equal inductance \( L \). Sort the circuits into groups that are equivalent.
Oscillator with Two Modes

Electromagnetic:

mode #1: \[ L \frac{dI}{dt} + \frac{Q}{C} + \frac{Q}{C} + L \frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt} \]
\[ \Rightarrow \frac{dI}{dt} = -\frac{Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}} \]

mode #2: \[ L \frac{dI}{dt} + \frac{Q}{C} + \frac{2Q}{C} = 0, \quad I = \frac{dQ}{dt} \]
\[ \Rightarrow \frac{dI}{dt} = -\frac{3Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \sqrt{\frac{3}{LC}} \]

Mechanical:

mode #1: \[ \omega = \sqrt{\frac{k}{m}} \]

mode #2: \[ \omega = \sqrt{\frac{3k}{m}} \]
In the circuit shown the capacitor is without charge. When the switch is closed to position \( a \),

(a) find the initial rate \( \frac{dI}{dt} \) at which the current increases from zero,
(b) find the charge \( Q \) on the capacitor after a long time.

Then, when the switch is thrown from \( a \) to \( b \),

(c) find the time \( t_1 \) it takes the capacitor to fully discharge,
(d) find the maximum current \( I_{max} \) in the process of discharging.
In the circuit shown the capacitor is without charge and the switch is in position \( a \).

(i) When the switch is moved to position \( b \) we have an \( RL \) circuit with the current building up gradually:

\[
I(t) = \left( \frac{E}{R} \right) \left[ 1 - e^{-t/\tau} \right].
\]

Find the time constant \( \tau \) and the current \( I_{\text{max}} \) after a long time.

(ii) Then we reset the clock and move the switch from \( b \) to \( c \) with no interruption of the current through the inductor. We now have a \( LC \) circuit:

\[
I(t) = I_{\text{max}} \cos(\omega t).
\]

Find the angular frequency of oscillation \( \omega \) and the maximum charge \( Q_{\text{max}} \) that goes onto the capacitor periodically.
In the circuit shown the capacitor is without charge and the switch is in position \( a \).

(i) When the switch is moved to position \( b \) we have an \( RC \) circuit with the capacitor being charged up gradually: \( Q(t) = \mathcal{E}C[1 - e^{-t/\tau}] \).

Find the time constant \( \tau \) and the charge \( Q_{\text{max}} \) after a long time.

(ii) Then we reset the clock and move the switch from \( b \) to \( c \).

We now have a \( LC \) circuit: \( Q(t) = Q_{\text{max}} \cos(\omega t) \).

Find the angular frequency of oscillation \( \omega \) and the maximum current \( I_{\text{max}} \) that flows through the inductor periodically.