19. LC and RLC Oscillators

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Abstract
Lecture slides 19 for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island.

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Mechanical Oscillator

- law of motion: \( F = ma, \quad a = \frac{d^2x}{dt^2} \)
- law of force: \( F = -kx \)
- equation of motion: \( \frac{d^2x}{dt^2} = -\frac{k}{m}x \)
- displacement: \( x(t) = x_{max} \cos(\omega t) \)
- velocity: \( v(t) = -\omega x_{max} \sin(\omega t) \)
- angular frequency: \( \omega = \sqrt{\frac{k}{m}} \)
- kinetic energy: \( K = \frac{1}{2}mv^2 \)
- potential energy: \( U = \frac{1}{2}kx^2 \)
- total energy: \( E = K + U = \text{const} \)
Electromagnetic Oscillator (LC Circuit)

- loop rule: \[ \frac{Q}{C} + L \frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt} \]

- equation of motion: \[ \frac{d^2Q}{dt^2} = -\frac{1}{LC}Q \]

- charge on capacitor: \[ Q(t) = Q_{max} \cos(\omega t) \]

- current through inductor: \[ I(t) = -\omega Q_{max} \sin(\omega t) \]

- angular frequency: \[ \omega = \frac{1}{\sqrt{LC}} \]

- magnetic energy: \[ U_B = \frac{1}{2} LI^2 \] (stored on inductor)

- electric energy: \[ U_E = \frac{Q^2}{2C} \] (stored on capacitor)

- total energy: \[ E = U_B + U_E = \text{const.} \]
Mechanical vs Electromagnetic Oscillations

Mechanical oscillations

- **position**: \( x(t) = A \cos(\omega t) \) [red]
- **velocity**: \( v(t) = -A \sin(\omega t) \) [green]
- **period**: \( \tau = \frac{2\pi}{\omega} \), \( \omega = \sqrt{\frac{k}{m}} \)

![Graph of mechanical oscillations]

Electromagnetic oscillations

- **charge**: \( Q(t) = A \cos(\omega t) \) [red]
- **current**: \( I(t) = -A \sin(\omega t) \) [green]
- **period**: \( \tau = \frac{2\pi}{\omega} \), \( \omega = \frac{1}{\sqrt{LC}} \)

![Graph of electromagnetic oscillations]

- **potential energy**: \( U(t) = \frac{1}{2} k x^2(t) \) [r]
- **kinetic energy**: \( K(t) = \frac{1}{2} m v^2(t) \) [g]
- **total energy**: \( E = U(t) + K(t) = \text{const} \)

- **electric energy**: \( U_E(t) = \frac{1}{2C} Q^2(t) \) [r]
- **magnetic energy**: \( U_B(t) = \frac{1}{2} LI^2(t) \) [g]
- **total energy**: \( E = U_E(t) + U_B(t) = \text{const} \)
Mechanical Oscillator with Damping

- law of motion: \( F = ma, \quad a = \frac{d^2 x}{dt^2} \)
- law of force: \( F = -kx - bv, \quad v = \frac{dx}{dt} \)
- equation of motion: \( \frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \)

Solution for initial conditions \( x(0) = A, \ v(0) = 0 \):

(a) underdamped motion: \( b^2 < 4km \)

\[ x(t) = Ae^{-bt/2m} \left[ \cos(\omega' t) + \frac{b}{2m\omega'} \sin(\omega' t) \right] \quad \text{with} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \]

(b) overdamped motion: \( b^2 > 4km \)

\[ x(t) = Ae^{-bt/2m} \left[ \cosh(\Omega' t) + \frac{b}{2m\Omega'} \sinh(\Omega' t) \right] \quad \text{with} \quad \Omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} \]
Damped Electromagnetic Oscillator (RLC Circuit)

- Loop rule: \(RI + L \frac{dI}{dt} + \frac{Q}{C} = 0, \ I = \frac{dQ}{dt}\)

- Equation of motion: \(\frac{d^2 Q}{dt^2} + \frac{R}{L} \frac{dQ}{dt} + \frac{1}{LC}Q = 0\)

Solution for initial conditions \(Q(0) = Q_{max}, I(0) = 0\):

(a) Underdamped motion: \(R^2 < \frac{4L}{C}\)

\[
Q(t) = Q_{max} e^{-Rt/2L} \left[ \cos(\omega't) + \frac{R}{2L\omega'} \sin(\omega't) \right] \quad \text{with} \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}
\]

(b) Overdamped motion: \(R^2 > \frac{4L}{C}\)

\[
Q(t) = Q_{max} e^{-Rt/2L} \left[ \cosh(\Omega't) + \frac{R}{2L\Omega'} \sinh(\Omega't) \right] \quad \text{with} \quad \Omega' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}
\]
Name the \( LC \) circuit with the highest and the lowest angular frequency of oscillation.

(1)

(2)

(3)

(4)
At time $t = 0$ a charge $Q = 2C$ is on each capacitor and all currents are zero.

(a) What is the energy stored in the circuit?
(b) At what time $t_1$ are the capacitors discharged for the first time?
(c) What is the current through each inductor at time $t_1$?
In these $LC$ circuits all capacitors have equal capacitance $C$ and all inductors have equal inductance $L$. Sort the circuits into groups that are equivalent.
Oscillator with Two Modes

Electromagnetic:

mode #1: \( L \frac{dI}{dt} + \frac{Q}{C} + \frac{Q}{C} + L \frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt} \)

\[ \Rightarrow \frac{dI}{dt} = -\frac{Q}{LC} \quad \Rightarrow \quad \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}} \]

mode #2: \( L \frac{dI}{dt} + \frac{Q}{C} + \frac{2Q}{C} = 0, \quad I = \frac{dQ}{dt} \)

\[ \Rightarrow \frac{dI}{dt} = -\frac{3Q}{LC} \quad \Rightarrow \quad \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \sqrt{\frac{3}{LC}} \]

Mechanical:

mode #1: \( \omega = \sqrt{\frac{k}{m}} \)

mode #2: \( \omega = \sqrt{\frac{3k}{m}} \)
In the circuit shown the capacitor is without charge. When the switch is closed to position \( a \)... 
(a) find the initial rate \( \frac{dI}{dt} \) at which the current increases from zero, 
(b) find the charge \( Q \) on the capacitor after a long time. 
Then, when the switch is thrown from \( a \) to \( b \)... 
(c) find the time \( t_1 \) it takes the capacitor to fully discharge, 
(d) find the maximum current \( I_{max} \) in the process of discharging.
RLC Circuit: Application (2)

In the circuit shown the capacitor is without charge and the switch is in position \( a \).

(i) When the switch is moved to position \( b \) we have an \( RL \) circuit with the current building up gradually: \( I(t) = (\mathcal{E}/R)[1 - e^{-t/\tau}] \).

Find the time constant \( \tau \) and the current \( I_{max} \) after a long time.

(ii) Then we reset the clock and move the switch from \( b \) to \( c \) with no interruption of the current through the inductor. We now have an \( LC \) circuit: \( I(t) = I_{max} \cos(\omega t) \).

Find the angular frequency of oscillation \( \omega \) and the maximum charge \( Q_{max} \) that goes onto the capacitor periodically.

![RLC Circuit Diagram]
In the circuit shown the capacitor is without charge and the switch is in position \textit{a}.

(i) When the switch is moved to position \textit{b} we have an \textit{RC} circuit with the capacitor being charged up gradually: $Q(t) = \varepsilon C [1 - e^{-t/\tau}]$.

Find the time constant $\tau$ and the charge $Q_{\text{max}}$ after a long time.

(ii) Then we reset the clock and move the switch from \textit{b} to \textit{c}.

We now have a \textit{LC} circuit: $Q(t) = Q_{\text{max}} \cos(\omega t)$.

Find the angular frequency of oscillation $\omega$ and the maximum current $I_{\text{max}}$ that flows through the inductor periodically.