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19. Deterministic Chaos

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Abstract
Part nineteen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc19]

Dissipative Dynamical Systems [mln101]

Nature of phase flow:

- *Conservative systems*: Incompressible flow (Liouville theorem). Hamiltonian systems have even phase-space dimensionality.
- *Dissipative systems*: Progressive contraction of phase flow to some attractor. Phenomenological character of equations of motion allow for odd dimensionalities.

2D dissipative flow: Two kinds of attractors.

- Point attractors (2D → 0D) [mln73] [msl10].
- Limit cycles (2D → 1D) [mln74].

3D dissipative flow: Four kinds of attractors.

- Point attractors (3D → 0D) [msl16].
- Limit cycles (3D → 1D) [msl17].
- Toroidal attractors (3D → 2D) [msl18].
- Strange attractors (3D → fract.D) [msl19].

Rössler band: Example of strange attractor.

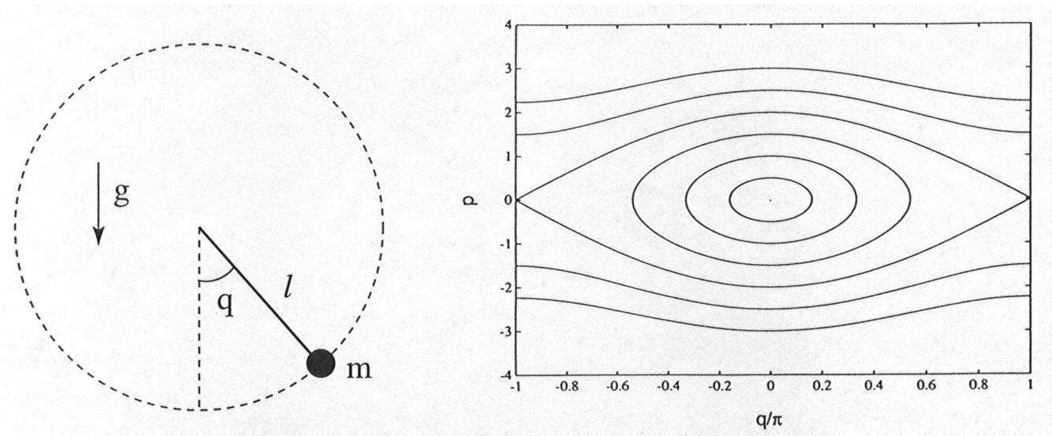
- Equations of motion:
  $\dot{x} = -y - z, \quad \dot{y} = x + \frac{y}{5}, \quad \dot{z} = \frac{1}{5} + z(x - 5.7).$
- Flow on attractor: stretching, folding, and squeezing.

Integrabiliy as a Universal Property [mln98]

In autonomous Hamiltonian systems with one degree of freedom, integrability is always guaranteed. The evidence has previously been demonstrated in the contexts of Lagrangian and Hamiltonian mechanics.

Prototypical example: plane pendulum.

Hamiltonian: $H(q,p) = \dfrac{p^2}{2m} + mgl(1-\cos q).$



Two-dimensional (2D) phase space.

$E =$ const. on sets of lines (1D).

Trajectories confined to $E =$ const.

Canonical transformation to action-angle coordinates:

$$\begin{aligned}
(q,p) \quad &\Leftrightarrow \quad (\vartheta, J) \\
H(q,p) \quad &\Leftrightarrow \quad K(J) \\
\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \quad &\Leftrightarrow \quad \dot{\vartheta} = \frac{\partial K}{\partial J} \equiv \omega(J),\ \dot{J} = 0 \\
q(t),\ p(t) \quad &\Leftrightarrow \quad \vartheta(t) = \omega(J)t + \vartheta_0,\ J = \text{const.}
\end{aligned}$$

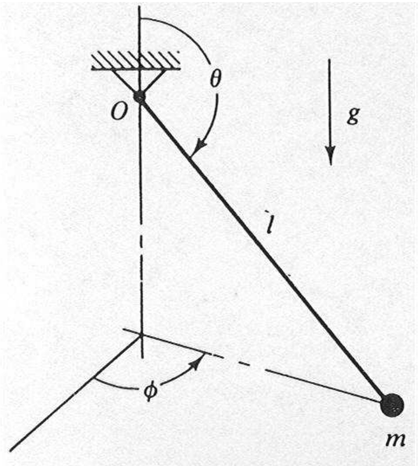
Integrability as a Contingent Property [mln99]

In autonomous Hamiltonian systems with two (or more) degrees of freedom, integrability is the exception rather than the rule.

Prototypical examples of integrable and nonintegrable systems emerge from modifications of the plane pendulum [mln98] by adding a degree of freedom.

Integrable system:

spherical pendulum



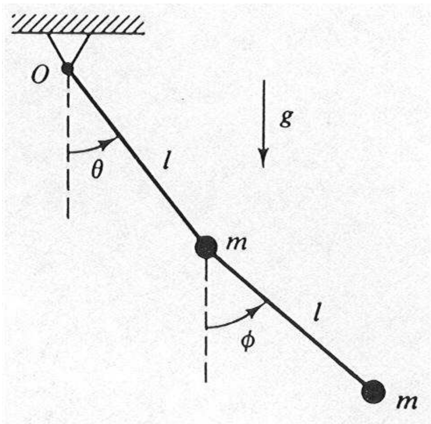
4D phase space: $(\theta, \phi, p_\theta, p_\phi)$

3D hypersurface: $E =$ const.
3D hypersurface: $p_\phi =$ const.

Trajectories are nonintersecting on 2D invariant torus $E =$ const. $\cap$ $p_\phi =$ const.

Nonintegrable system:

plane double pendulum



4D phase space: $(\theta, \phi, p_\theta, p_\phi)$

3D hypersurface: $E =$ const.
No second invariant subspace.

Trajectories are nonintersecting on 3D energy hypersurface.

Types of trajectories:

Integrable system:

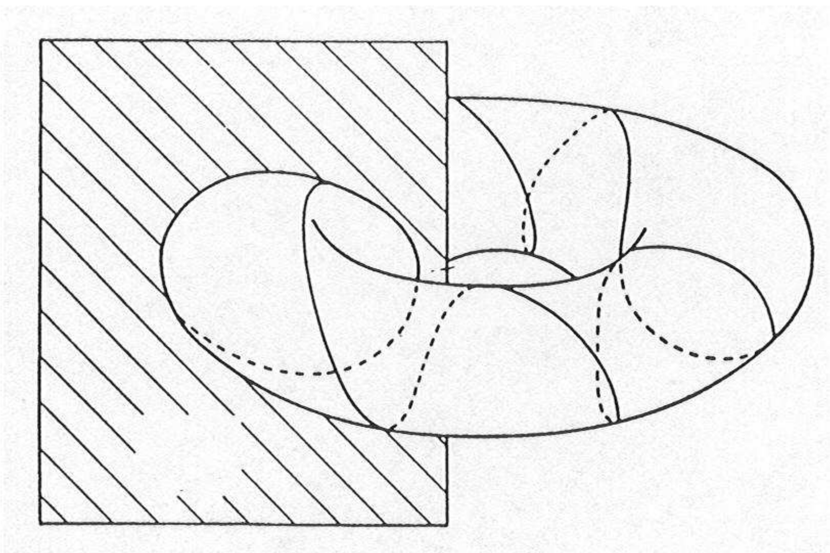
- periodic,
- quasiperiodic.

Nonintegrable system:

- periodic,
- quasiperiodic,
- chaotic.

Poincaré Surface of Section [mln100]

- Calculate trajectory: $\theta(t),\ \phi(t),\ p_\theta(t),\ p_\phi(t)$.
- Select points with $p_\phi = 0,\ \dot{p}_\phi > 0$.
- Project these points onto a plane in (θ, ϕ, p_θ)-space.



Invariant torus specified by two actions J_1, J_2.

Position of phase point on a given torus specified by two angle coordinates: $\vartheta_1(t) = \omega_1(J_1, J_2)t + \vartheta_1^0,\ \vartheta_2(t) = \omega_2(J_1, J_2)t + \vartheta_2^0$.

Periodic trajectories: rational ω_1/ω_2.
Quasiperiodic trajectories: irrational ω_1/ω_2.

Dynamical System with 2 Degrees of Freedom [msl15]

Newton's equation of motion: $m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} \equiv (\dot{x}_1, \dot{x}_2)$

Velocity vector field in 4D phase space: $(\dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2)$.

Solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ describes trajectory in 4D phase space.
All trajectories are tangential to velocity vector field and nonintersecting.

Orbits are projections of trajectories onto (x_1, x_2)-plane.

Conservative force $\mathbf{F}(\mathbf{x})$: $\oint d\mathbf{s} \cdot \mathbf{F} = 0$ for all closed paths in (x_1, x_2)-plane.

Potential energy: $V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{s} \cdot \mathbf{F}$.

In conservative system, first integral of the motion guaranteed to exist:
$E(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + V(x_1, x_2)$.

Existence of second integral of the motion $K(x_1, x_2, \dot{x}_1, \dot{x}_2) = \text{const}$ guarantees *integrability* of dynamical system.

In *nonintegrable* systems, any trajectory in 4D phase space is confined to a 3D hypersurface $E = \text{const}$. In *integrable* systems, any trajectory is confined to the intersection of $E = \text{const}$ and $K = \text{const}$ in 4D phase space. The resulting 2D manifold has the topology of a *torus*.

Poincaré surface of section: Plot only those points of a trajectory where it crosses a particular hyperplane (e.g. $x_1 = 0$) in a particular direction (e.g. with $\dot{x}_1 > 0$).

On the Poincaré cut, lines represent quasiperiodic trajectories (on invariant tori) and fixed points represent periodic trajectories.

In *integrable* systems, the Poincaré maps of all trajectories are confined to lines. In *nonintegrable* systems, the Poincaré maps of quasiperiodic trajectories are confined to lines whereas the Poincaré maps of chaotic trajectories spread into 2D regions.

Nearby quasiperiodic trajectories move apart *linearly* in time.
Nearby chaotic trajectories move apart *exponentially* in time.

Introduction to Hamiltonian Chaos [mln108]

This series of papers is designed as a tutorial for beginning graduate students. Each article includes problems for further study.

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Computers in Physics 4 (1990), 549-553.
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