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18. Analysis of RC circuits. Charging and discharging processes

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PHY204 Lecture 18 [rln18]

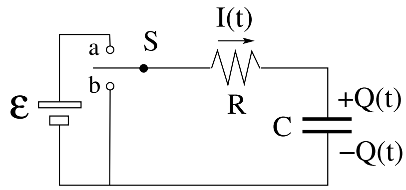
RC Circuit: Fundamentals

Specifications:

- $\mathcal{E}$ (emf)
- R (resistance)
- C (capacitance)

Switch S:

- a: charging
- b: discharging



Time-dependent quantities:

- $Q(t)$: instantaneous charge on capacitor
- $I(t) = \frac{dQ}{dt}$: instantaneous current
- $V_R(t) = I(t)R$: instantaneous voltage across resistor
- $V_C(t) = \frac{Q(t)}{C}$: instantaneous voltage across capacitor

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This lecture is devoted to RC circuits, which contain resistors and capacitors in addition to EMF sources. The slide on this page introduces the prototypical RC circuit. It has a switch with two settings.

Setting a connects the EMF source to a loop with a resistor and a capacitor. In this setting, a current through the resistor charges up the capacitor. Setting b disconnects the EMF source from the loop. The consequence is that the capacitor will be discharged by a current through the resistor.

In the RC circuit, explicit time-dependence is omnipresent. The current in the loop and the voltages across the resistor and capacitor are all functions of time and they are functionally related as shown on the slide.

The current direction of choice is cw, indicated by an arrow on the slide. This choice implies that charge $+Q(t)$ flows onto the upper plate and charge $-Q(t)$ onto the lower plate.

With this choice, the charging process (setting a) is described by a positive current $I(t)$ and an increasing charge $Q(t)$, whereas the discharging process (setting b) is described by a negative $I(t)$ and a decreasing $Q(t)$. Note that if $I(t)$ is positive, $Q(t)$ can be positive or negative and vice versa.

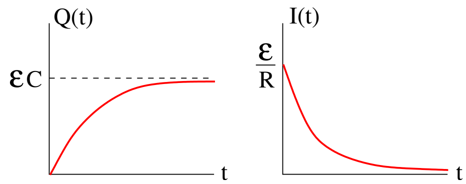
The resistor rule and the EMF rule from lecture 16 still hold. Here we add the capacitor rule: If we cross a capacitor in the declared current direction, i.e. from the plate with charge $+Q(t)$ to the plate with charge $-Q(t)$, we go down in potential, $\Delta V = -Q(t)/C$.

RC Circuit: Charging the Capacitor

- Loop rule: $\mathcal{E} - IR - \frac{Q}{C} = 0$
- Differential equation: $R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E} \quad \Rightarrow \quad \frac{dQ}{dt} = \frac{\mathcal{E}C - Q}{RC}$

$$\int_0^Q \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC} \quad \Rightarrow \quad -\ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = \frac{t}{RC} \quad \Rightarrow \quad \frac{\mathcal{E}C - Q}{\mathcal{E}C} = e^{-t/RC}$$

- Charge on capacitor: $Q(t) = \mathcal{E}C\left[1 - e^{-t/RC}\right]$
- Current through resistor: $I(t) \equiv \frac{dQ}{dt} = \frac{\mathcal{E}}{R}\,e^{-t/RC}$



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Consider the charging process with the switch closed to setting a. We have a one-loop circuit and invoke the loop rule for the analysis (see first item). We go around clockwise, first across the EMF source, then across the resistor, and finally across the capacitor, to return to the starting point.

The three terms represent the voltages across the three device in the loop. The voltage across the resistor depends on the current $I(t)$. The voltage across the capacitor depends on the charge $Q(t)$ it carries. Current and charge are functionally related. The former is the derivative of the latter.

We recognize that the loop rule is a differential equation for the function $Q(t)$. It can be solved, as shown in the second item, by separation of variables. The lower integration boundaries represent the initial conditions: at time zero there is no charge on the capacitor yet.

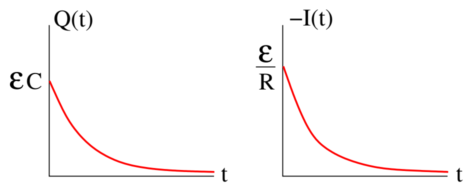
The explicit solution for $Q(t)$ is stated in the third item and plotted at the bottom of the slide. It graphically describes the buildup of charge on the capacitor. The final value $\mathcal{E}C$ for the charge on the capacitor is reached asymptotically as $t \to \infty$.

Also shown, analytically and graphically, is the rate dQ/dt at which charge builds up on the capacitor. This is equal to the current $I(t)$ through the resistor. The current is largest initially, then tapers off to zero as $t \to \infty$.

With a change in scale the graph on the left represents the voltage $V_C(t) = Q(t)/C$ across the capacitor and the graph on the right the voltage $V_R(t) = RI(t)$ across the resistor. The former increases as the charge increases. The latter decreases and approaches zero with the current.

RC Circuit: Discharging the Capacitor

- Loop rule: $IR + \frac{Q}{C} = 0$
- Differential equation: $R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad \Rightarrow \quad \frac{dQ}{dt} = -\frac{Q}{RC}$
  $\Rightarrow \int_{\mathcal{E}C}^{Q} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC} \quad \Rightarrow \quad \ln\left(\frac{Q}{\mathcal{E}C}\right) = -\frac{t}{RC} \quad \Rightarrow \quad \frac{Q}{\mathcal{E}C} = e^{-t/RC}$
- Charge on capacitor: $Q(t) = \mathcal{E}Ce^{-t/RC}$
- Current through resistor: $I(t) \equiv \frac{dQ}{dt} = -\frac{\mathcal{E}}{R}\, e^{-t/RC}$



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Throwing the switch from setting a to setting b means disconnecting the EMF source from the loop. The loop rule now involves two terms only.

We go around the loop counterclockwise first across the resistor (against the declared current direction) and then across the capacitor from the plate with charge $-Q(t)$ to the plate with charge $+Q(t)$.

The loop rule, recognized as differential equation for $Q(t)$. We solve it in the second item, just as we did on the previous page. Note the different initial condition in the first integral. The clock is reset to $t = 0$ when we throw the switch. At that instant, the capacitor is still fully charged at $\mathcal{E}C$.

The explicit solution for $Q(t)$ and its derivative, $I(t)$, during discharging are shown analytically and graphically on the slide.

During the discharging process, the voltage which drives the current through the resistor is provided by the capacitor. It is a gradually diminishing voltage $Q(t)/C$, pushing a gradually diminishing current $I(t)$.

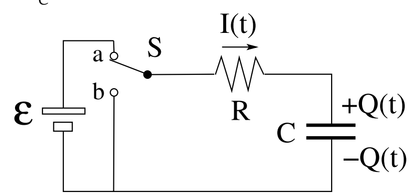
The charge $Q(t)$ on the capacitor is still positive during discharging, but the current $I(t)$ is now negative A negative cw current is a positive ccw current.

RC Circuit: Energy Transfer While Charging

Loop rule: $IR + \frac{Q}{C} = \mathcal{E}$ (I is positive)

- $I\mathcal{E}$: rate at which emf source delivers energy
- $IV_R = I^2R$: rate at which energy is dissipated in resistor
- $IV_C = \frac{IQ}{C}$: rate at which energy is stored in capacitor

Balance of energy transfer: $I^2R + \frac{IQ}{C} = I\mathcal{E}$



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On this page and the next we examine the energy transfer between devices during the charging and discharging processes.

The slide here restates the loop rule during the charging process. Each term represents the voltage across one of the three devices that are connected in the loop when the switch has setting a.

When we multiply each term with the same factor I, the equation remains valid but now has a different interpretation. All terms have the SI unit Watt [W=J/s], representing power, i.e. transfer of energy per time unit. The meaning of all three terms is spelled out in the three items on the slide.

Initially, when the current is large, the power transfer is large. A fraction of what the EMF source delivers is being dissipated in the resistor and the rest is being stored on the capacitor. As the current tapers off, the energy transfer comes to a halt.

The energy transferred to the resistor is converted into heat or radiation and irretrievable. However, the energy transferred to the capacitor is retrievable. It is the potential energy,

$$U = \frac{Q^2}{2C} = \frac{1}{2}C\mathcal{E}^2,$$

rendered in familiar expressions from earlier. We also recall where that energy sits, namely in the electric field between the capacitor plates.

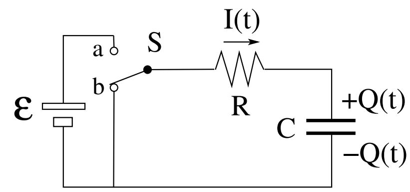
Energy is never lost, always converted from one form into another. Some conversions are reversible, others less so. There are very efficient ways to convert electric energy into mechanical energy and vice versa as we shall discuss later. It takes a heat engine (with limited efficiency) to convert thermal energy into mechanical energy or electrical energy.

RC Circuit: Energy Transfer While Discharging

Loop rule: $IR + \frac{Q}{C} = 0$ (I is negative)

- $IV_R = I^2R$: rate at which energy is dissipated in resistor
- $IV_C = \frac{IQ}{C}$: rate at which capacitor releases energy

Balance of energy transfer: $I^2R + \frac{IQ}{C} = 0$



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In the setting b of the switch, the EMF source has been disconnected. The capacitor is fully charged at $\mathcal{E}C$ when the discharging process begins. The energy stored on the capacitor initially is

$$U = \frac{1}{2}C\mathcal{E}^2.$$

There are now only two terms in the loop equation. When multiplied by the instantaneous current $I(t)$, each term again represents a power transfer. The meanings are spelled out in the two items on the slide.

Energy is now being released from the capacitor and dissipated in the resistor. During the transfer the electric field between the capacitor plates gradually collapses and the resistor heats up. Electric energy is being converted into thermal energy.

The rate at which the resistor dissipates energy depends on R. However, the total energy dissipated in the resistor during the discharging process is independent of R,

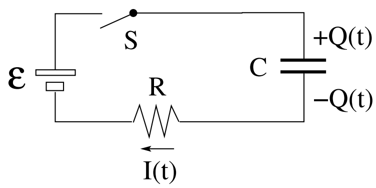
$$\int_0^\infty dt\, I^2(t)R = \frac{\mathcal{E}^2}{R}\int_0^\infty dt\, e^{-2t/RC} = \frac{1}{2}C\mathcal{E}^2.$$

It is the energy initially stored in the electric field of the capacitor.

RC Circuit: Some Physical Properties

Specification of RC circuit by 3 device properties:

- $\mathcal{E}$ [V] (emf)
- R [Ω] (resistance)
- C [F] (capacitance)



Physical properties of RC circuit during charging process determined by 3 combinations of the device properties:

- $\mathcal{E}/R = I(t=0)$: rate at which charge flows onto capacitor initially
- $\mathcal{E}C = Q(t=\infty)$: total charge placed on capacitor ultimately
- $RC = \tau$: time it takes to place 63% of the charge onto the capacitor
  $[1 - e^{-1} = 0.632\ldots]$

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How do we characterize an RC circuit?

One way to characterize it is by stating its specifications: EMF $\mathcal{E}$, resistance R, and capacitance C.

An alternative is to characterize it by key physical properties such as are captured in the three combinations of device specifications, $\mathcal{E}/R$, $\mathcal{E}C$, and RC, with interpretations stated on the slide.

This characterization tells us how to adjust the device properties if we aim for particular values of any of the characteristic properties.

Key points to remember in the analysis of RC circuits:

- The voltage across a capacitor with no charge is zero. An empty capacitor is an invisible device.
- There is no steady current through a capacitor. A capacitor blocks direct current once it is fully charged.

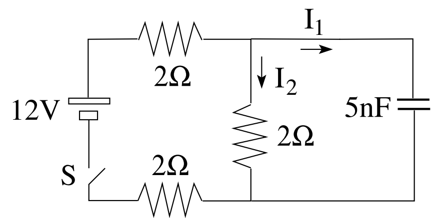
At time $t = 0$ the capacitor in this circuit is discharged and the switch is being closed.

Find the current I_1

(a) at $t = 0$,
(b) at $t = \infty$.

Find the current I_2

(c) at $t = 0$,
(d) at $t = \infty$.



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There are several different ways to analyze this situation. One safe way is to systematically employ the junction rule and the loop rule. In simple situations such this, there are short cuts that lead to the answers more quickly. Learning to read circuits pays off big.

(a) & (c) $t = 0$: At this instant, there is no charge on the capacitor, which implies that there is zero voltage across it.

We recognize that the capacitor is in parallel with the middle resistor. Hence there is zero voltage across that resistor as well. A resistor with zero voltage across it has zero current through it: $I_2 = 0$.

We conclude that the current I_1 instantaneously flows through the outer loop containing the EMF source, the (discharged) capacitor, and two resistors in series. The current driven by the EMF is $I_1 = 12\text{V}/(2\Omega + 2\Omega) = 3\text{A}$.

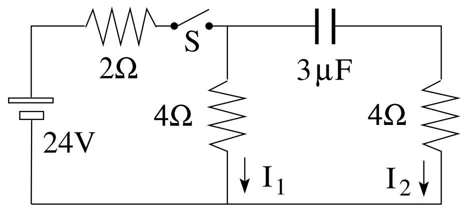
(b) & (d) $t = \infty$: After a long time, the capacitor is charged and blocks any further current through it: $I_1 = 0$.

The three resistors are now effectively in series with the current I_2 through all of them in a single loop. We infer that $I_2 = 12\text{V}/(2\Omega + 2\Omega + 2\Omega) = 2\text{A}$.

What is the charge on the capacitor after a long time? We recall that the capacitor is in parallel with the middle resistor. For this reason it has the same voltage across: $(2\Omega)(2\text{A}) = 4\text{V}$. Hence the charge on the capacitor is $Q = (5\text{nF})(4\text{V}) = 20\text{nC}$.

In the RC circuit shown, the switch S has been open for a long time.

(a) Find the currents I_1 and I_2 immediately after the switch has been closed.
(b) Find the currents I_1 and I_2 a very long time later.



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The questions are the same as on the previous page, but the circuit is different. Reading it properly produces quick answers.

(a) At $t = 0$ the capacitor is uncharged with zero voltage across. It is instantaneously invisible as a device, which has the consequence that the two 4Ω-resistors are instantaneously in parallel. We can replace them by a 2Ω-resistor, which then is in series with the other 2Ω-resistor.

The current driven by the EMF source through these two resistors in series is $I_1 + I_2 = 24\text{V}/(2\Omega + 2\Omega) = 6\text{A}$. That current splits into equal parts through the two parallel 4Ω-resistors. Hence we have $I_1 = I_2 = 3\text{A}$.

(b) At $t = \infty$ the capacitor blocks any further current through its branch, implying that $I_2 = 0$. That leaves a one-loop circuit with a current I_1 flowing through the battery and two resistors in series. We infer that $I_1 = 24\text{V}/(2\Omega + 4\Omega) = 4\text{A}$.

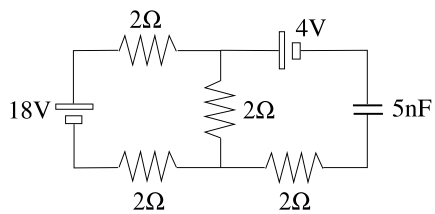
Bonus question: Find the charge on the capacitor after a long time. We can answer the question if we know the voltage across the capacitor.

We know that $I_2 = 0$ after a long time. In consequence, the voltage across the 4Ω-resistor on the far right is zero. Therefore, the capacitor is effectively in parallel with the 4Ω-resistor in the middle. They have the same voltage across. That voltage is $(4\Omega)(4\text{A}) = 16\text{V}$. Hence the charge on the capacitor is $Q = (3\mu\text{F})(16\text{V}) = 48\mu\text{C}$.

RC Circuit: Application (1)

This circuit has been running for a very long time.

(a) Find the current through the 18V battery.
(b) Find the total power dissipated in the resistors.
(c) Find the charge stored on the capacitor.



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When all transients have faded away, the circuit is either at equilibrium or in a steady state. In this case it means that the current through the branch on the right is blocked by the charged capacitor. There is a steady current through the loop on the left.

(a) The three 2Ω-resistor in the loop on the left are effectively in series. The current driven by the EMF is $I = 18\text{V}/(2\Omega + 2\Omega + 2\Omega) = 3\text{A}$.

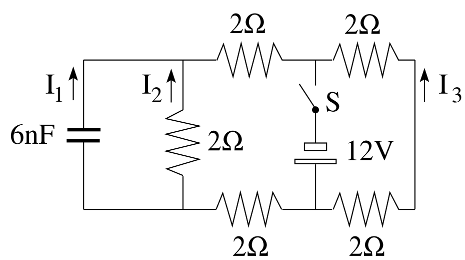
(b) No power is being dissipated in the branch on the right, where no current flows. Hence we have $P = (2\Omega + 2\Omega + 2\Omega)(3\text{A})^2 = 54\text{W}$.

(c) To answer this question we use the loop rule around the loop on the right, starting at bottom right and going ccw:

$$+\frac{Q}{5\text{nF}} + 4\text{V} - \underbrace{(2\Omega)(3\text{A})}_{6\text{V}} - \underbrace{(2\Omega)(0\text{A})}_{0} = 0 \quad \Rightarrow \quad Q = (2\text{V})(5\text{nF}) = 20\text{nC}.$$

In this 3-loop RC circuit, the switch S is closed at time $t = 0$.

(a) Find the currents I_1, I_2, I_3 just after the switch has been closed.
(b) Find the currents I_1, I_2, I_3 a very long time later.



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To the inexperienced eye this circuit looks more complex than it really is. What happens to the left of the switch is independent of what happens to the right of it.

On the right, a steady current I_3 through two 2Ω-resistors in series is established at once. We have $I_3 = 12\text{V}/(2\Omega + 2\Omega) = 3\text{A}$ immediately after the switch has been closed and also a long time later.

On the left we have, in essence, the same circuit as the one analyzed on page 7, except for a different capacitance.

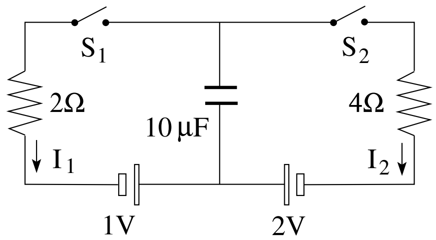
At time $t = 0$ the capacitor is uncharged, implying zero voltage across it. In consequence the voltage across the 2Ω-resistor parallel to the capacitor is zero as well, implying zero current $I_2 = 0$ through it. Hence we have a current $I_1 = 12\text{V}/(2\Omega + 2\Omega) = 3\text{A}$ as reasoned on page 7.

At $t = \infty$ the current through the capacitor is blocked, $I_1 = 0$, and the three resistors on the left are effectively in series: $I_2 = 12\text{V}/(2\Omega + 2\Omega + 2\Omega) = 2\text{A}$.

RC Circuit: Application (6)

In the RC circuit shown, both switches are initially open and the capacitor is discharged.

(a) Close switch S_1 and find the currents I_1 and I_2 immediately afterwards.
(b) Find the currents $I_1 I_2$ and the charge Q on the capacitor a very long time later.
(c) Now close switch S_2 also and find the currents I_1 and I_2 immediately afterwards.
(d) Find the currents I_1, I_2 and the charge Q on the capacitor a very long time later.



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When we close switch S_1 there is only action in the loop on the left. The battery on the left will charge up the capacitor in the middle by driving a ccw current I_1 through the resistor on the left.

Initially, the capacitor is still discharged. The current is $I_1 = 1\text{V}/2\Omega = 0.5\text{A}$. After a long time, the charging process is complete and the current has ground to a halt: $I_1 = 0$. The charge on the capacitor is $Q = (10\mu\text{F})(1\text{V}) = 10\mu\text{C}$.

When we close switch S_2, the charge on the capacitor is instantaneously the same and the voltage across it as well. The loop equation on the right starting at the bottom right corner and going cw the yields,

$$+2\text{V} - 1\text{V} - (4\Omega)I_2 = 0 \quad \Rightarrow \; I_2 = 0.25\text{A}.$$

After another long time, the charge on the capacitor will have settled down to a different value and again block any current through the branch in the middle. We then have effectively a one loop circuit with current $I_2 = -I_1$.

We determine that current by invoking the loop rule for that big loop, again starting at the bottom right corner and going cw:

$$+2\text{V} - 1\text{V} + (2\Omega)I_1 \underbrace{-(4\Omega)I_2}_{+(4\Omega)I_1} = 0 \quad \Rightarrow \; I_1 = -I_2 = -\frac{1}{6}\text{A}.$$

We can use the loop equation on the right again to calculate the final charge on the capacitor:

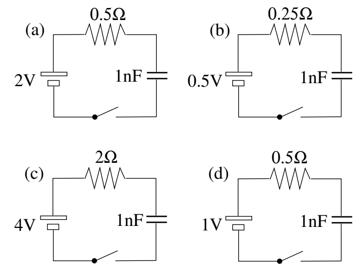
$$+2\text{V} - \frac{Q}{10\mu\text{F}} - (4\Omega)\left(\frac{1}{6}\text{A}\right) = 0 \quad \Rightarrow \; Q = \frac{40}{3}\mu\text{C}.$$

RC Circuit: Application (2)

The switches are closed at $t = 0$. This begins the charging process in each RC circuit.

Name the circuit in which...

(i) the charge flows into the capacitor at the highest rate initially,
(ii) the capacitor has the most charge ultimately,
(iii) the capacitor is 63% full in the shortest time.



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This is the quiz for lecture 18.

The characterizations discussed on page 6 are helpful resources.