18. Hamilton-Jacobi Theory

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Abstract
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18. Hamilton-Jacobi Theory

• Hamilton’s principal function [mln96]
• Hamilton’s characteristic function [mln97]
• Hamilton-Jacobi equation for the harmonic oscillator [mex97]
• Hamilton’s principal function for central force problem [mex98]
• Hamilton’s characteristic function for central force problem [mex99]
• Particle in time-dependent field [mex201]
• Hamilton-Jacobi theory for projectile motion [mex202]
Hamilton’s Principal Function

We seek a canonical transformation \( H(q,p,t) \rightarrow K(Q,P) \equiv 0 \).
Here \( q \) stands for \( q_1, \ldots, q_n \) etc.

Canonical equations:
\[
\begin{align*}
\dot{q}_i &= \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \rightarrow \quad \dot{Q}_i &= \frac{\partial K}{\partial P_i} = 0, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0.
\end{align*}
\]

Hamilton’s principal function: \( S(q,P,t) \).

- \( S \) is the \( F_2 \)-type generating function of this canonical transformation.
- \( S \) depends on \( n+1 \) variables \( q_1, \ldots, q_n, t \) and \( n \) parameters \( P_1, \ldots, P_n \).
- \( p_j = \frac{\partial S}{\partial q_j}, \quad Q_j = \frac{\partial S}{\partial P_j}, \quad K - H = \frac{\partial S}{\partial t} \).

Hamilton-Jacobi equation: \( H \left( q_1, \ldots, q_n; \frac{\partial S}{\partial q_n}, \ldots, \frac{\partial S}{\partial q_1}; t \right) + \frac{\partial S}{\partial t} = 0 \).

- First-order partial differential equation for \( S(q,P,t) \).
- Integration constants \( P_1, \ldots, P_n \) plus additive constant.
- For given solution: \( p_j(q,P,t) = \frac{\partial S}{\partial q_j}, \quad Q_j(p,P,t) = \frac{\partial S}{\partial P_j} \) = const.
- \( \Rightarrow q_j(Q,P,t), \quad p_j(Q,P,t) \) (transformation relations).
- The constants \( Q_j, P_j \) are functions of the initial values \( q_j^{(0)}, p_j^{(0)} \).

Physical significance of Hamilton’s principal function:
\[
\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j} \dot{q}_j + \frac{\partial S}{\partial t} = \sum_j p_j \dot{q}_j - H = L.
\]
Hamilton’s Characteristic Function

Two distinct ways of solving the Hamilton-Jacobi equation become available when the Hamiltonian does not explicitly depend on time.

If \( H(q, p) = E = \text{const} \), then \( \frac{\partial S}{\partial t} = -E = \text{const} \).

Set \( S(q, P, t) = W(q, P) - Et \).

Hamilton’s characteristic function: \( W(q_1, \ldots, q_n; P_1, \ldots, P_n) \).

Method #1:
- Solve the Hamilton-Jacobi equation \( H \left( q, \frac{\partial S}{\partial q} \right) + \frac{\partial S}{\partial t} = 0 \).
- Proceed as in [mln96] but use \( S(q, P, t) = W(q, P) - Et \).
- One of the integration constants is reserved: \( P_1 = E \).

Method #2:
- Solve the Hamilton-Jacobi equation \( H \left( q, \frac{\partial W}{\partial q} \right) - E = 0 \).
- \( W(q, P) \) is a \( F_2 \)-type generating function of a canonical transformation to action-angle coordinates with \( P_1 = K(P) = E \).
- Canonical Equations: \( \dot{Q}_j = \frac{\partial K}{\partial P_j} = \delta_{j1}, \quad \dot{P}_j = -\frac{\partial K}{\partial Q_j} = 0 \).
- Solution: \( P_j = \text{const}, \quad Q_j = t\delta_{j1} + Q_j^{(0)} \).
- Transformation to original canonical coordinates:
  \[
  Q_j = \frac{\partial}{\partial P_j} W(q, P), \quad p_j = \frac{\partial}{\partial q_j} W(q, P).
  \]
  \[
  \Rightarrow \quad q_j = q_j \left( Q_j^{(0)}, P, t \right), \quad p_j = p_j \left( Q_j^{(0)}, P, t \right).
  \]
[mex97] Hamilton-Jacobi equation for the harmonic oscillator

Determine the time evolution of the canonical coordinates \( q(t), p(t) \) for the harmonic oscillator, \( H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 \), by solving the Hamilton-Jacobi equation along two different avenues.

(a) Use the ansatz \( S(q, E, t) = W(q, E) - Et \) for Hamilton’s principal function. Solve the Hamilton-Jacobi equation for \( S(q, E, t) \). Use \( Q = \frac{\partial S}{\partial E} \) to derive \( q(t) \) and \( \frac{\partial S}{\partial q} \) to derive \( p(t) \).

(b) Solve the Hamilton-Jacobi equation for Hamilton’s principal function \( W(q, E) \). Use \( Q = \frac{\partial W}{\partial E} \) to derive \( q(E, Q) \) and \( \frac{\partial W}{\partial q} \) to derive \( p(E, Q) \). Substitute these results into \( H(q, p) \) to obtain the transformed Hamiltonian \( K(E) = E \). Solve the canonical equations for the transformed canonical coordinates \( Q, E \) and substitute them into \( q(E, Q) \) and \( p(E, Q) \) to obtain \( q(t), p(t) \).

Solution:
Consider the one-body central-force problem specified by the Hamiltonian

\[ H(r, p, \ell) = \frac{1}{2m} \left( p^2 + \frac{\ell^2}{r^2} \right) + V(r), \]

where \( p \equiv p_r \) and \( \ell \equiv p_\vartheta \) are the canonical momenta conjugate to \( r \) and \( \vartheta \), respectively. Solve the Hamilton-Jacobi equation for Hamilton’s principal function. (a) Use the ansatz \( S(r, \vartheta, \ell, E, t) = W_1(r, \ell, E) + \ell \vartheta - Et \) for the principal function and determine \( W_1(r, \ell, E) \). (b) Infer from \( \partial S/\partial E = R = \text{const} \) the time evolution of the radial motion \( r(t, E, \ell, r_0) \). (c) Infer from \( \partial S/\partial \ell = \Theta = \text{const} \) the orbital relation \( \vartheta(r, E, \ell, r_0, \vartheta_0) \), which, in combination with \( r(t, E, \ell, r_0) \), determines the time evolution of the angular motion.

Solution:
Consider the one-body central-force problem specified by the Hamiltonian

\[ H(r, p, \ell) = \frac{1}{2m} \left( p^2 + \frac{\ell^2}{r^2} \right) + V(r), \]

where \( p \equiv p_r \) and \( \ell \equiv p_\theta \) are the canonical momenta conjugate to \( r \) and \( \vartheta \), respectively. Solve the Hamilton-Jacobi equation for Hamilton’s characteristic function. (a) Use the ansatz \( W(r, \vartheta, \ell, E) = W_1(r, \ell, E) + \ell \vartheta \) for the characteristic function and determine \( W_1(r, \ell, E) \). (b) The characteristic function \( W(r, \vartheta, \ell, E) \) is the generating function of a canonical transformation \((r, \vartheta) \to (R, \Theta)\), which transforms the Hamiltonian as follows: \( H(r, p, \ell) = K(E, \ell) = E \). Solve the canonical equations for \( R, \Theta \). (c) Infer from \( \partial W / \partial E = R = \text{const} \) the time evolution of the radial motion \( r(t, E, \ell, r_0) \). (c) Infer from \( \partial W / \partial \ell = \Theta = \text{const} \) the orbital relation \( \vartheta(r, E, \ell, r_0, \vartheta_0) \), which, in combination with \( r(t, E, \ell, r_0) \), determines the time evolution of the angular motion.

Solution:
Consider the dynamical system described by the time-dependent Hamiltonian

\[ H(q, p, t) = \frac{p^2}{2m} - mAtq, \]

where \( A \) is a constant. (a) Find Hamilton’s principal function \( S(q, P, t) \) as the solution of the Hamilton-Jacobi equation. (b) Derive the solutions \( q(t), p(t) \) from \( S(q, P, t) \) for initial conditions \( q(0) = 0, p(0) = mv_0 \).

**Solution:**
Hamilton-Jacobi theory for projectile motion

Consider a particle of mass $m$ moving in a uniform vertical gravitational field:

$$H = \frac{1}{2m} (p_x^2 + p_y^2) + mgy.$$

(a) Find Hamilton’s principal function $S(x, y, P_1, P_2, t)$ as the solution of the Hamilton-Jacobi equation. (b) Derive the solutions $x(t), y(t)$ from $S(x, y, P_1, P_2, t)$ for initial conditions $x(0) = y(0) = 0, \dot{x}(0) = \dot{x}_0, \dot{y}(0) = \dot{y}_0$.

Solution: