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Some Results on Graph Representations and Closure Systems

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SOME RESULTS ON GRAPH REPRESENTATIONS AND CLOSURE SYSTEMS

BY

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ABSTRACT

Representations of graphs are a way of encoding the structure of a graph by using other discrete structures. The object of a representation of a graph is to encode its structure efficiently. Typically a graph can be encoded by an $n \times n$ adjacency matrix. It is possible, however, to encode graphs much more efficiently using other representation schemes.

This work considers tree representations of graphs. A tree representation of a target graph G is an assignment of subtrees of a host tree to the vertices of G in such a way that if $uv \in E(G)$, then the subtree assigned to the vertex u and the subtree assigned to the vertex v have at least t nodes in common. This study considers tree representations such that the host tree comes from the family of subdivided n-stars. The largest such representable asteroidal set is constructed, and a lower bound on the length of the longest cycle representable on this family of host tree is also discovered.

Next we move to a different area of study. The study of closure systems and closure spaces is a relatively new direction in mathematics. Jamison writes in his new text that 'the notion of closure is pervasive throughout mathematics'. Surely, closure and closed sets can be discussed in almost any mathematical setting. It has been shown by Pfaltz in 1995 that for any finite ground set S, with $|S| = n$ such that $n \geq 10$, there are n^n unique closure operators on S.

In topology, the separation properties provide criteria for categorizing topological spaces. While not all closure spaces are topological spaces, we may still explore whether the separation properties hold under certain conditions. This work defines a class of closure operators on the integers and investigates the conditions for which the resulting closure space satisfies the different definitions of separability.

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To my fiancée, Kerry Flinn, thank you for all of your support.

Lastly, thank you to the remaining faculty and graduate students at the University of Rhode Island for many great mathematical discussions.

DEDICATION

To my parents

Ron and Gloria,

my brother

Brandon,

and to my grandmother

Jane Losier (1918 - 2010)

PREFACE

This dissertation contains a mixture of background material, published work, and work that is in preparation for publication. In accordance with the guidelines set forth by the Graduate School at the University of Rhode Island, the dissertation is written in manuscript format.

The study of graph representations is an active area of research within graph theory. Multiple mathematicians, including Eaton, Erdős, Füredi, Golumbic, Grable, Jamison, Kostochka, Rödl, Trotter, and more have done work in this area. Much of the work done in this dissertation discusses problems in the theory of graph representations.

A representation of a graph consists of three objects, 1) a host set $\mathcal{S}$, 2) an assignment function f, and 3) a conflict rule g. The host set may be any collection of objects, ranging from an arbitrary set of elements, to trees, to subsets of the plane, and anything in between. The assignment function assigns a subset of the host set to each vertex of a target graph (that is, a graph which we desire to represent). Finally, the conflict rule compares these assigned subsets to determine whether or not two vertices should be adjacent. If, given a host set $\mathcal{S}$ and conflict rule g, there is a suitable assignment function such that the graph G is induced by the conflict rule, then we say that G is $(\mathcal{S}; g)$-representable. Graph representation problems have interesting applications, for example, in the field of computing. In this dissertation, specific representation schemes are explored, and previously unknown limitations of these schemes are revealed.

Chapter 1 of this document will provide a short introduction into the basic graph theory and the notation commonly used throughout the dissertation. The next four chapters concern results on graph representations. Chapter 2 serves as an introduction to graph representations, providing motivating examples and a bit of history. While the contents of this dissertation will focus specifically on tree representations of graphs, in the second chapter we will discuss the traditional set representations, tree representations, and one of the main differences between the two.

In the third chapter we discuss a result discovered by J.R. Walter in his 1972 dissertation. Here we verify that Walter's result holds in the discrete case. We also use this chapter as motivation for chapters 4 and 5 which can be viewed as extensions of his work.

In chapters 4 and 5 we discuss representations of asteroidal sets and of cycles, respectively, on a family of host trees called subdivided stars. Chapter 4 contains an original result which has been accepted for publication in the Journal of Combinatorial Mathematics and Combinatorial Computing; while Chapter 5 contains work towards generalizing a known result about tree representations of cycles.

Chapters 6 and 7 discuss work I have done in a different area of mathematics. The notion of closure and closed sets can be discussed in almost all areas of mathematics. For a ground set S, closure can be defined in many different ways. In fact, it has been shown by Pfaltz, in 1995, that for a ground set S of size n, with $n \geq 10$, there are n^n unique closure operators on S.

Chapter 6 contains an introduction to closure systems and closure spaces. Four examples of closure operators on different ground sets are given and briefly explored. Finally, Chapter 7 discusses original results about closure spaces resulting from defining a certain class of closure operators on the integers. We will examine these closure spaces specifically with respect to separability. The separation properties give mathematicians useful ways to classify spaces. Traditionally these properties have been used to classify topological spaces, but closure spaces are also a natural place to discuss separability.

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CHAPTER 1

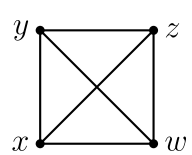
Some Background in Graph Theory

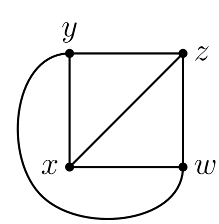
1.1 Introduction

Before we begin discussing the main topics of this dissertation, it will be necessary to have a bit of background in graph theory. This first chapter will provide the basic background necessary for reading the next four chapters of this work.

1.2 What Is A Graph?

In graph theory, a graph G is a pair of sets. That is, $G = (V, E)$, where V is a set of elements called vertices and E is a set containing pairs of vertices, called edges. Typically a graph can be drawn with each vertex denoted by a dot and an edge xy by an arc connecting the dots corresponding to x and y. While there are many ways to arrange the vertices and edges of a given graph, the particular drawing does not change the underlying structure of the graph. It can be beneficial to note, however, that some drawings are more desirable than others. In particular one may try to draw a given graph in such a way that the number of edge crossings is minimal. For example, consider the graph $G = (V, E)$, where $V = \{x, y, z, w\}$, and $E = \{xy, xz, xw, yz, yw, zw\}$. The two drawings below both depict G, however, the first drawing has one edge crossing, while the second has no edge crossings.





Again, both of the drawings depict the graph G, but the graph on the right shows that G is planar. That is, G can be drawn in the plane with no edge crossings.

A graph $G = (V, E)$ is said to be simple if it contains no multiple edges, that is to say that the edge xy occurs at most once in E; there are no loops, meaning that there are no edges of the form xx in E; and edges do not have a direction associated with them, so the edge xy and the edge yx are no different. In graph theory it is also possible to consider multigraphs, where loops and multi-edges are permitted, as well as di-graphs, where edges are given direction. In this dissertation, when we refer to a graph we will always mean a simple graph.

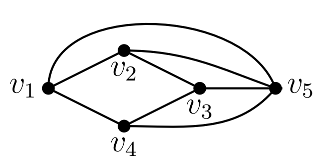
1.3 More on Vertices and Edges

We say that two vertices x and y in a graph G are adjacent if xy is an edge in G. The neighborhood of a vertex x in a graph G, denoted by $N_G(x)$ (or $N(x)$ for short) is the set of all other vertices in G which are adjacent to x. We consider the closed neighborhood of a vertex x, denoted by $N[x]$ to be $N[x] = N(x) \cup \{x\}$. We also say that an edge is incident to a vertex x if x is one of the endpoints of that edge. The degree of a vertex x in a graph G, denoted $deg_G(x)$ (or $deg(x)$ for short) is the number of edges incident to x in the graph G.

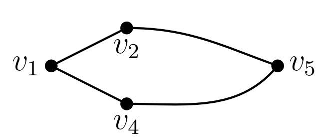
1.4 Subgraphs

We say that a graph $H = (V', E')$ is a subgraph of the graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In the event that H is a subgraph of G, we often write $H \subset G$. Furthermore, a subgraph H of a graph G is said to be an induced subgraph of G if for any pair x and y of vertices in H such that xy is an edge in G, xy is also an edge in H. If G is a graph with vertex set V and U is a non-empty subset of V, then we denote by $G[U]$ the induced subgraph of G with vertex set U. See below for an example of a graph, one of it's subgraphs, and one of it's induced subgraphs.

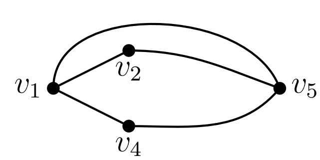
A Graph



A Subgraph



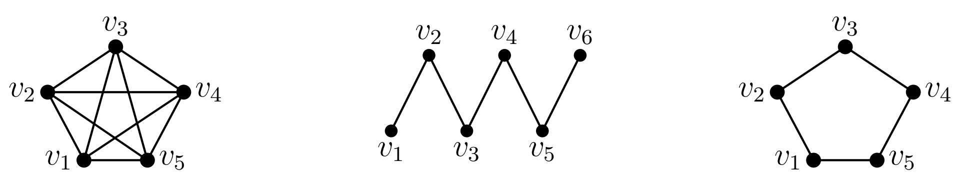
An Induced Subgraph



1.5 Some Special Classes of Graphs

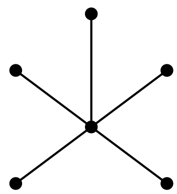
Throughout the dissertation we will refer to and make use of some special classes of graphs. A graph is said to be complete if that graph contains an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n and has $\binom{n}{2}$ edges. A path is a graph whose vertex set is $V = \{v_1, v_2, ..., v_n\}$ and whose edge set is $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. Note that the length of a path is typically measured by the number of edges it contains; and the path of length n is denoted by P_n. A cycle is a graph whose vertex set is $V = \{v_1, v_2, ..., v_n\}$ and whose edge set is $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. The cycle of length n is denoted by C_n. See below for examples of a complete graph, a path, and a cycle.

K_5 P_5 C_5

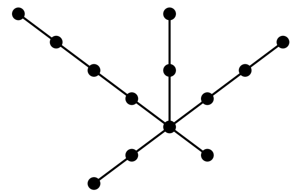


Another class of graphs, called trees, are connected graphs that contain no cycles. Trees are graphs which will be important objects throughout chapters 2, 3, 4, and 5 of this dissertation. Any tree will have at least two vertices of degree one; and these degree one vertices are typically called the leaves of the tree. A star is a special type of tree with one central vertex connected to all of the other vertices. Note that this forces all of the vertices except the central vertex to be leaves, since any additional edge would create a cycle. The star with n leaves is denoted by $K_{1,n}$. Finally, we will be interested in looking at subdivisions of stars. A subdivision of a graph is created when a single edge is replaced with a path. See below for an example of a star and a subdivision of that star.

$\underline{K_{1,5}}$



$\underline{\text{A Subdivision of } K_{1,5}}$



Note that in the example above, our star has undergone multiple subdivisions, and that not all of the subdivisions replaced edges with paths of the same length. In any case, we call such a graph a subdivision of $K_{1,5}$. In general, within this dissertation, we label any graph which is a subdivision of $K_{1,n}$ by $\mathcal{K}_{1,n}$. These graphs in particular will be extremely important in chapters 3, 4, and 5 of this dissertation.

Note that this chapter is by no means a comprehensive overview of basic graph theory. It does, however, provide the reader with the basic graph theoretic knowledge required to proceed with reading this dissertation. New definitions will be presented throughout the dissertation as they are needed. Any common graph theoretic terms the reader may come across which have not been identified in this chapter may be found in [1].

List of References

[1] D. West, *Introduction to Graph Theory.* Upper Saddle River, NJ: Prentice Hall, 2001.

CHAPTER 2

Introduction to Graph Representations

2.1 Introduction

The scheduling-conflict problem can be modeled by what is commonly called a conflict graph. Consider, for example, that four classes: MATH, ENGLISH, HISTORY, and ECONOMICS wish to schedule final exams. Further consider that there are five students who are taking more than one of these classes. Call the students A, B, C, D, and E. The exams should be scheduled such that no student has two or more exams during the same time slot. A conflict graph can help determine the minimum number of time slots necessary for the exams.

Let the enrollments of the five students be as follows:

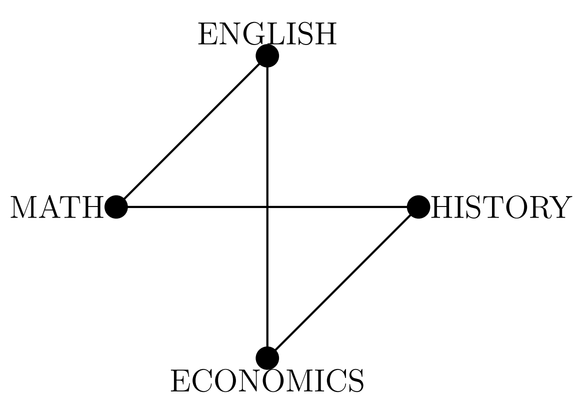
MATH: A,C,D

ENGLISH: D,E

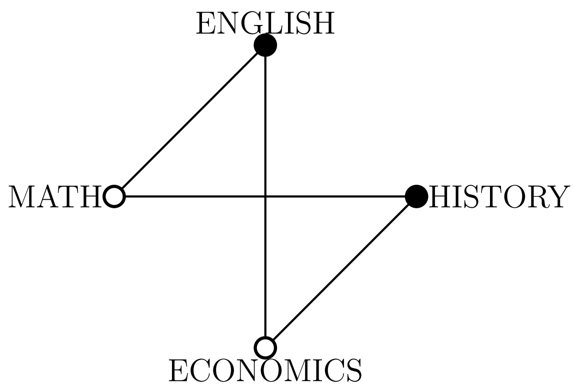
HISTORY: B,C

ECONOMICS: B,E

The information can be displayed graphically in the following conflict graph.



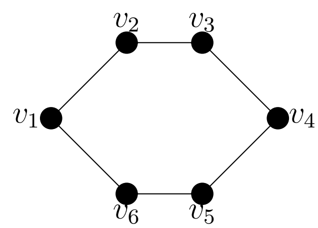
Notice that student D causes a conflict in scheduling MATH and ENGLISH during the same time, and that this is denoted by an edge in the graph. The other adjecencies are defined via similar conflicts. Note that a proper coloring of a graph is a coloring of the vertices such that no two adjacent vertices are given the same color. Now, the number of colors required to properly color of the conflict graph will give the number of required time slots.



Notice that MATH and ECONOMICS (corresponding to the white vertices) may be scheduled together, while ENGLISH and HISTORY (corresponding to the black vertices) may be scheduled together. This shows, that in order to schedule the exams as desired, we must use two different exam periods.

Notice that in the previous example, a conflict existed whenever the class rosters had a non-empty intersection. However, a conflict may be defined in any way that one wishes. It may be the case that the dean of the school has determined that too many exam slots are used, and that it would be best to declare a conflict between two classes only if three or more students are enrolled in both. In this case, the tolerance for conflict has been increased from 1 to 3, and all of the exams from the previous example could be given concurrently.

One may also consider the reverse problem. That is, one can declare a set of elements, called a host set, and also define a conflict rule with the goal of discovering which graphs can be realized as a result of assigning subsets of the host set to vertices of a target graph and analyzing the conflicts that arise. This venture isn't completely superficial. Consider the problem of communicating the structure of the following graph to a computer.

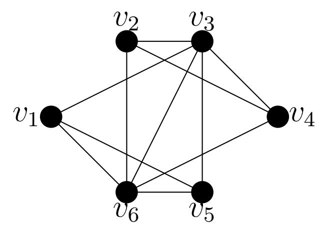


The representation scheme commonly discussed in introductory courses in graph theory is the adjacency matrix. The adjacency matrix for such a graph is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

where a 1 in the (i, j)th entry of the matrix communicates that the edge $v_i v_j$ exists in the graph. Such a representation is costly in terms of computing resources, though. In general, the adjacency matrix for a graph on n vertices is an $n \times n$ matrix. This means that communicating the graph structure of a graph on n vertices via this method requires defining an $n \times n$ array. Are there more efficient ways to communicate the graph structure to a computer? If so, what are their limitations? These are two questions which are interesting to explore. We can use graph representations, and the idea of the conflict graph to explore these questions. For example, consider the 6-vertex graph which was given as the previous example. Let $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ and assign subsets of $\mathcal{S}$ to the vertices via the assignment function $f(v_i) = \{i \mod 6, i + 1 \mod 6\}$. Drawing the conflict graph with tolerance 1 for the this set assignment yields the graph from the previous example.

Notice that we only have to statically define the host set $\mathcal{S}$, define a function which assigns the subsets to each of the vertices, and then to define a function which determines when a conflict exists. Also, once such a host set and conflict rule are communicated to the computer, we may modify the assignment function in order to construct new graphs. Consider the conflict graph defined by the host set and conflict rule above, but where the set assignments are given by the function $h(v_i) = \{i \mod 6, 3(i+1 \mod 2)\}$.



The adjacency matrix for this graph is communicated by defining an entirely new 6×6 array. This shows that defining the graph by the graph representation is more versatile than the definition via the adjacency matrix.

2.2 Two Types of Graph Representations

While there are certainly many types of graph representation schemes (see [1] for a comprehensive list), we will limit ourselves to considering two types. We first consider what is commonly referred to as a set representation of a graph. We then look to a second representation scheme, and our main object of study, tree representations of graphs.

2.2.1 Set Representations

Graph representations have been a topic of interest to many graph theorists since Szpilrajn-Marczewski introduced the notion of a set representation of a graph [2] in 1945. Given a graph $G = (V, E)$, a representation of G consists of the following collection of objects: (1) a set $\mathcal{S}$, (2) a function $f : V \to \mathscr{P}(\mathcal{S})$ (the power set of $\mathcal{S}$), and (3) a function $g : f(V) \times f(V) \to \{0, 1\}$ so that $g(f(v_1), f(v_2)) = 1$ iff $v_1v_2 \in E$. We call $\mathcal{S}$ the *host set*, f the *assignment function*, and g the *conflict* *rule*. A graph G is representable under a given host set $\mathcal{S}$ and conflict rule g if there exists a suitable assignment function f, in which case we say that G is $(\mathcal{S}; g)$-representable.

The following theorem of Erdős, Goodman, and Pósa [3] is one of the most basic and important results in the theory of graph representations.

Theorem 2.2.1 *For any finite graph $G = (V, E)$, there exists a host set $\mathcal{S}$ of sufficient size such that G is $(\mathcal{S}; 1)$-representable.*

Proof. Given a finite graph G, let $\mathcal{S} = \{1, 2, ..., |E|\}$, where $|E|$ denotes the number of edges in G. Label each edge $uv \in E$ with a distinct element from $\mathcal{S}$, call it $\ell(uv)$. Now, assign to each vertex the set consisting of all of the elements assigned to the edges which that vertex is incident to. Note that this assigns each element of $\mathcal{S}$ uniquely to two vertices. That is, if $uv \in E$, then the sets assigned to vertex u and vertex v have the element $\ell(uv)$ in common. If $uv \notin E$, then the sets assigned to the vertex u and the vertex v have an empty intersection. This, by definition, is an $(\mathcal{S}; 1)$-representation of G. □

Noting that the graph representation resulting from the above construction is unlikely to be the most efficient representation, those mathematicians pursuing the study of graph representations began the search for the most efficient graph representation scheme. That is mathematicians search for the answer to the question: among all possible conflict rules, which is the one which minimizes the size of the host set required to represent any graph on n vertices.

This new direction has resulted in the exploration of many different types of graph representations, including (but not limited to) interval representations, circular arc representations, box representations, and tree representations. In many cases the exploration of these representation schemes has led to new classes of graphs, such as interval graphs, circular arc graphs, visibility graphs, and more. For a more comprehensive description of the types of graph representations which have been explored, please consult [1].

2.2.2 Tree Representations

Tree representations of graphs are an interesting variation on the set representations that we have discussed so far. The host is a tree, giving more structure than merely a set of elements; objects assigned to vertices of a represented graph are subtrees of the host tree; and an edge exists between two vertices if and only if their assigned subtrees intersect in t or more nodes, where t is a prescribed conflict-tolerance.

It can be seen that tree representations and set representations have significant differences. For example, consider Theorem 2.2.1 about set representations. Could there possibly be a similar theorem about tree representations? After a few moments the reader may notice a problem. In 1974 Gavril proved that the class of graphs having tree representations with tolerance 1 consisted exactly of those graphs whose induced cycles are all 3-cycles [4].

Increasing the conflict tolerance allows for less restrictive classes of graphs to be representable: for example longer cycles can be represented if the conflict tolerance is raised to at least 3. Increasing this conflict tolerance does increase the difficulty of the tree representation problem, though. It has been both fruitful and interesting to study the classes of graphs representable with higher conflict tolerance, but on fixed families of host tree. Eaton and Faubert [5] studied graphs representable on caterpillars, where a caterpillar is defined as a tree whose vertices all lie on or adjacent to its longest path. Eaton and Barbato [6] studied cycles representable on subdivisions of the 3-star. Chapters 2, 3, and 4 explore further those graphs representable on subdivisions of stars.

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[1] R. Jamison, "Towards a comprehensive theory of conflict-tolerance graphs," *Discrete Applied Mathematics*, vol. 18, pp. 47–56, 2012.

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CHAPTER 3

Discretization of a Known Result

A Note to the Reader

When Lekkerkerker and Boland discovered that the class of graphs representable on an interval are exactly the chordal, non-asteroidal graphs [1], they also proved that the path representable graphs were exactly the interval representable graphs. A path is the discrete version of an interval. In his dissertation, J.R. Walter classified the graphs representable on a union of three intervals joined at a common endpoint. This result only appeared in Walter's dissertation and was never peer reviewed. Unlike Lekkerkerker and Boland, Walter did not verify that his result would still apply on the discrete analog of his tree. Walter's tree can be discretized as a subdivision of the 3-star. The inclusion of this chapter serves two purposes. Firstly verification that Walter's result still holds in the discrete case is needed, and secondly, Walter's result provides motivation for the work done in chapters 4 and 5.

A Discretization of a Result of J. R. Walter

by

J.R. Walter, Communicated by A.J. Gilbert

3.1 Introduction

Walter's dissertation considered tree representations of graphs. Recall from Chapter 2 that tree representations are a variation on the traditional graph representation. The host is a tree, giving more structure than merely a set of elements; objects assigned to vertices of a represented graph are subtrees of the host tree; and an edge exists between two vertices if and only if their assigned subtrees have a non-empty intersection. An important distinction between set representations and tree representations is that, while all graphs are $(\mathcal{S}; 1)$-representable for large enough $\mathcal{S}$, there exist graphs which are not $(\mathcal{T}; 1)$-representable, for any host tree $\mathcal{T}$. For example, graphs containing cycles of length four or greater are always forbidden in tree representations where the conflict-tolerance t is set at 1.

James Walter considered the fixed host tree consisting of three unit intervals joined at a common endpoint. We will discretize Walter's result by considering the host tree $\mathcal{K}_{1,3}$. That is, we consider the host tree to be a subdivision of the 3-star.

Throughout this chapter it will be useful to have specific notation for objects such as the target graph, the host trees, and collections of subtrees which induce a representation of the target graph. We will use the following conventions on notation to help us distinguish these objects.

1. host trees and host sets will be denoted by calligraphic letters ($\mathcal{T}, \mathcal{H}, \mathcal{S}$, etc.).

2. the standard font will be used to denote the target graph, its subgraphs, and subtrees of the host tree (G, H, T, etc.).

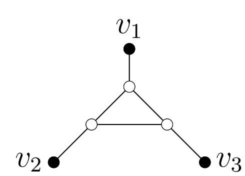
3. A representation of a graph (that is, the collection of assigned subtrees) will be denoted by script letters ($\mathscr{T}, \mathscr{S}$, etc.).

4. Under a representation $\mathscr{T}$, of a target graph G, we use T_v to denote the subtree from $\mathscr{T}$ assigned to the vertex v of the target graph.

3.2 The Main Result

The remainder of this chapter will be dedicated to proving and verifying Walter's result in the discrete case. We begin by giving a definition and follow with the result and its proof.

Definition 3.2.1 *An* ***asteroidal set*** *in a graph G is $A \subset V(G)$ so that $\forall v_1, v_2, v_3 \in A$, and $\forall i, j, k \in \{1, 2, 3\}$ with i, j, and k distinct, there exists a path between v_i and v_j which does not intersect $N(v_k)$. If $|A| = m$, then A is said to be an m-asteroid of G. Furthermore, if A is the maximum size asteroidal set contained in G, then G is said to be m-asteroidal.*



$\{v_1, v_2, v_3\}$ is an asteroidal set in the graph

Theorem 3.2.2 *A graph G is $(\mathcal{K}_{1,3}; 1)$-representable if and only if G is chordal, at most three asteroidal, and has the property that if u_1, u_2 and v_1, v_2 are vertices of two asteroidal triples of G, then any path from u_1 to u_2 must be adjacent to any path from v_1 to v_2 [2].*

Proof. Assume that G is $(\mathcal{K}_{1,3}; 1)$-representable.

Observation 3.2.3 *G is chordal. That is, all of the induced cycles of G are 3-cycles.*

Proof. Since $\mathcal{K}_{1,3}$ is a tree, then by the result of Gavril [3], G must be chordal.▼

Observation 3.2.4 *G is at most 3-asteroidal.*

Proof. Let v_i, v_j, and v_k be vertices contained in an asteroidal set of G. It is easy to see that v_i, v_j, and v_k must be assigned subtrees contained on different branches of $\mathcal{K}_{1,3}$. Assume otherwise, then at least one of the assigned subtrees must contain the branch node (the unique node of degree 3) of $\mathcal{K}_{1,3}$, otherwise we would have an asteroidal triple represented on an interval, violating the 1962 result of Lekkerkerker and Boland [1].

Assume T_{v_i} contains the branch node of $\mathcal{K}_{1,3}$. If T_{v_j} and T_{v_k} are represented by subtrees contained on different branches of $\mathcal{K}_{1,3}$, then any path from v_j to v_k in G must contain a vertex whose assigned subtree contains the branch node of G. This vertex must be adjacent to v_i, a contradiction that v_i, v_j, and v_k are contained in an asteroidal set of G. So, T_{v_j} and T_{v_k} must be contained on the same branch of $\mathcal{K}_{1,3}$. This, however, means that one of T_{v_j} and T_{v_k} must be 'closer' to the branch node of $\mathcal{K}_{1,3}$. Say that T_{v_j} is closer. Any path from T_{v_i} to T_{v_k} must contain a vertex whose assigned subtree intersects T_{v_j}. That is, any path from v_i to v_k contains a vertex adjacent to v_j, again a contradiction. Thus, T_{v_i}, T_{v_j}, and T_{v_k} must all lie on different branches of $\mathcal{K}_{1,3}$. Since there are only three branches, there can be only three such vertices. ▼

Observation 3.2.5 *If v_1, v_2 and u_1, u_2 are vertices contained in asteroidal triples of G, then any path from v_1 to v_2 in G, must be adjacent to any path from u_1 to u_2 in G.*

Proof. Note that T_{v_1} and T_{v_2} must lie on different branches of $\mathcal{K}_{1,3}$, as must T_{u_1} and T_{u_2}. Now, any path in G from v_1 to v_2 must contain a vertex v^*, whose assigned subtree contains the branch node of $\mathcal{K}_{1,3}$. Similarly, any path in G from

u_1 to u_2 must contain a vertex u^*, whose assigned subtree also contains the branch node of $\mathcal{K}_{1,3}$. That is, v^*u^* is an edge in G, and the two paths are adjacent. ▼

The three observations show that if G is $(\mathcal{K}_{1,3}; 1)$-representable, then G is chordal, at most 3-asteroidal, and has property that if v_1, v_2 and u_1, u_2 are pairs of vertices from asteroidal triples of G, then any path from v_1 to v_2 in G must be adjacent to any path from u_1 to u_2 in G. ▽

We now assume that G is chordal, at most 3-asteroidal, and has the property that if v_1, v_2 and u_1, u_2 are pairs of vertices from asteroidal triples on G, then any path in G from v_1 to v_2 is adjacent to any path in G from u_1 to u_2. From now on we will refer to the third property as the 'adjacency of asteroidal paths property'. We proceed by cases, giving definitions at the point where they are first needed.

Definition 3.2.6 *A vertex, v, is said to be* ***simplicial*** *if $N[v]$ is a complete subgraph of G. Furthermore, v is said to be* ***strongly simplicial*** *if $G - N[v]$ is connected, and* ***weakly simplicial*** *otherwise.*

Definition 3.2.7 *If G has a pair of simplicial vertices u, v such that either $N[u] = N[v]$ or $N(u) = N(v)$, then the graph $G' = G - u$ is said to obtained from G via an* ***elementary reduction****. If G has no such pair of vertices, then G is said to be* ***reduced****.*

Observation 3.2.8 *If G' is the reduced graph corresponding to the graph G, and G' has a representation on a tree, $\mathcal{T}$, then G has a representation on a subdivision of $\mathcal{T}$.*

Indeed, this must be true. Let G' have a representation, $\mathscr{T}'$ on the tree $\mathcal{T}$. To obtain a representation $\mathscr{T}$ of G on $\mathcal{T}$, for any pair of simplicial vertices u and v of G such that $N[u] = N[v]$, one can assign $T'_u = T_u = T_v$. For any pair of simplicial vertices u and v of G such that $N(u) = N(v)$, then since u is simplicial,

$\cap_{v \in N(u)} T'_v \neq \emptyset$. Using subdivision if necessary, assign T_u and T_v to be disjoint subtrees contained in the intersection. Replacing T'_u with T_u, and keeping all other subtrees the same, gives a representation of G on $\mathcal{T}$.

Definition 3.2.9 *A chordal graph G is called **extremal** if every simplicial vertex of G is strongly simplicial.*

Case 1: Assume that G is extremal

Observation 3.2.10 *Let G be an extremal, reduced, chordal graph, then G is n-asteroidal iff G has exactly n simplicial vertices.*

Assume that G has n simplicial vertices $v_1, v_2, ..., v_n$, all of which are strongly simplicial. Consider a subset $\{v_i, v_j, v_k\}$ of the simplicial vertices. Since $G - N[v_p]$ is connected for each $p \in \{i, j, k\}$, there exist a path in $G - N[v_p]$ between the remaining pair of simplicial vertices. This path is also a path in G, and is clearly not adjacent to v_p. Since this can be done for any choice of three simplicial vertices, we must have that $\{v_1, v_2, ..., v_n\}$ satisfies the definition of being an n-asteroid of G.

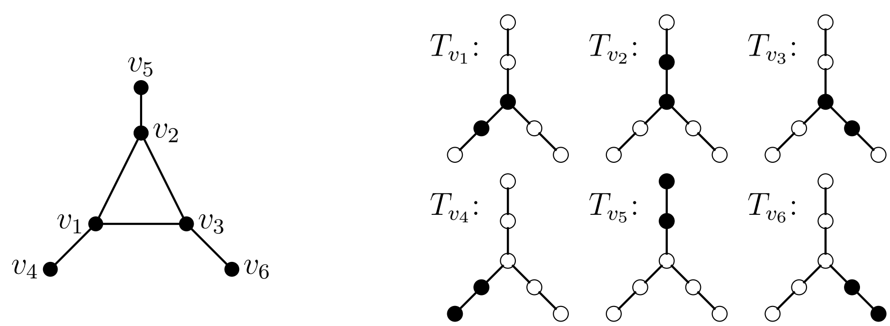
Assume that G is asteroidal, with n-asteroid $\{v_1, v_2, ..., v_n\}$. Assume that $\{v_1, v_2, ..., v_n\}$ is chosen to contain the largest number of simplicial vertices, but that at least one is not simplicial. Say v_1 is not a simplicial vertex. For each distinct pair (i, j) with i and j from the set $\{2, 3, ..., n\}$, let $P^1_{i,j}$ be the collection of all paths in G from v_i to v_j, not adjacent to v_1. It is obvious that $P = \cup_{i,j} P^1_{i,j}$ is a connected subgraph of G, and by a result contained in the Lekkerkerker and Boland paper [1], each non-empty component of $G - N[P]$ contains a simplicial vertex of G. In particular, we look at the component containing v_1, to find a simplicial vertex v^*. Now, there is a path in this component from v^* to v_1, which is clearly not adjacent to any of the other vertices contained in our asteroidal set.

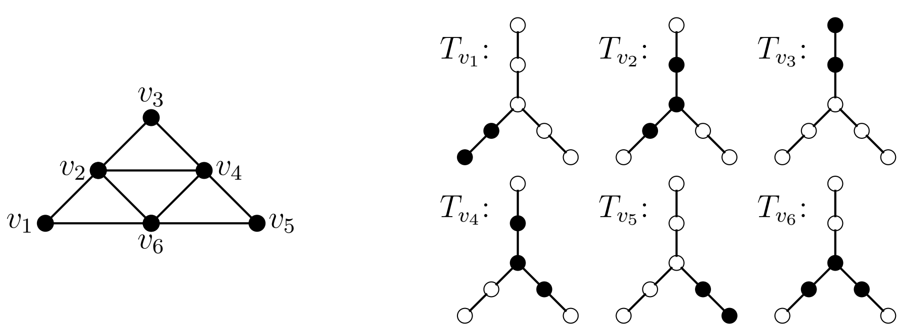
Appending this path to the existing asteroidal paths corresponding to v_1 gives the necessary paths to show that $\{v^*, v_2, ..., v_n\}$ is an n-asteroid consisting of more simplicial vertices than the original, contradicting our assumption that the original triple contained the most simplicial vertices. ∇

Note that the extremal condition was not required to show that if our graph was n-asteroidal, then each of the asteroidal vertices can be assumed to be simplicial. We will move forward with the extremal case, and then consider the remaining cases with asteroidal triples consisting of simplicial vertices.

We this observation in mind, we prove that such a chordal, extremal graph is representable on $\mathcal{K}_{1,3}$ by induction.

The two 3-asteroidal, chordal graphs on six vertices are easily representable on $\mathcal{K}_{1,3}$. See the following representations, where the target graph is shown on the left and the corresponding assigned subtrees are shown on the right as the subtrees induced on the black nodes of each copy of a $\mathcal{K}_{1,3}$.





We now assume G has n vertices, with $n > 6$, and that every 3-asteroidal, chordal, extremal graph on fewer vertices has a $(\mathcal{K}_{1,3}; 1)$ representation.

By what was shown in Observation 3.2.10, there is a unique asteroidal triple of G, consisting of simplicial vertices, v_1, v_2, and v_3.

Let $G' = G - v_1$. If G' is not asteroidal, then it is representable on a path $\mathcal{P}$ because of Lekkerkerker and Boland's result [1]. Furthermore, since $N[v_1]$ is complete, then $\cap_{v \in N(v_1)} T_v \neq \emptyset$, so we let x be a node in this intersection. Let $\mathcal{T} = \mathcal{P} \cup xx^*$, where xx^* denotes an edge between node x and a new node, x^*, not originally on $\mathcal{P}$. Now, note that $\mathcal{T}$ is a subdivision of $K_{1,3}$. We obtain a representation of G on $\mathcal{T}$ from the representation of $G - v_1$ on $\mathcal{P}$ by appending the edge xx^* to the subtrees assigned to vertices in $N(v_1)$, assigning $T_{v_1} = \{x^*\}$, and leaving all other subtrees the same.

Thus we will assume that $G - v_1$ remains 3-asteroidal, with asteroidal triple $\{u_1, u_2, u_3\}$ consisting of simplicial vertices. Note that it must be the case that $u_1 \in N(v_1)$, while $u_2 = v_2$ and $u_3 = v_3$. Since otherwise we would be contradicting that G was 3-asteroidal. This means that $\{u_1, v_2, v_3\}$ is an asteroidal triple of simplicial vertices of G'. It also means that v_2 and v_3 are contained in the same component of $G' - N[u_1]$. Furthermore, this must be the only component of $G' - N[u_1]$, since the guaranteed existence of a simplicial point in any other non-empty component of $G' - N[u_1]$ would imply that G' has more than 3 simplicial vertices, contradicting that G is 3-asteroidal. By definition then, u_1 is a strongly simplicial vertex in G'. If u is any other simplicial vertex of G', then $u \in N_{G'}(u_1)$, implying that $N_G(u) = N_{G'}(u_1)$, so u is also strongly simplicial. Thus, G' is extremal, and has a representation on $\mathcal{K}_{1,3}$.

Observe that this representation can be chosen in such a way that all of the subtrees assigned to members of $N(v_1)$ contain a common leaf, p, of $\mathcal{K}_{1,3}$. Indeed, $\cup_{v \in G' - N_{G'}[u_1]} T_v$ is a connected subset of $\mathcal{K}_{1,3}$, so the component of $\mathcal{K}_{1,3} - \cup_{v \in G' - N_{G'}[u_1]} T_v$ containing T_{u_1} contains a leaf, p. For all $v \in N_{G'}[u_1]$, append the path from some vertex in $\cap_{v\in N_{G'}[u_1]} T_v$ to p. Assign $T_{v_1} = \{p\}$ to obtain a representation of G on $\mathcal{K}_{1,3}$. $\bigtriangledown$

Definition 3.2.11 *A chordal graph, G, is said to be* ***quasi-extremal*** *if every asteroidal triple of simplicial vertices consists only of strongly simplicial vertices.*

Case 2: G is quasi-extremal

Assume that G is reduced, and has a weakly simplicial vertex v. Let $\{v_1, v_2, v_3\}$ be an asteroidal triple of simplicial vertices of G.

Let K_1 be the component of $G - N[v]$ containing v_1. Note that K_1 cannot contain v_2 or v_3. Say K_1 contained v_2. In that case, there is a path from v_1 to v_2 in K_1, therefore avoiding $N[v]$. Now, v_2 is strongly simplicial, so there is a path from v to v_1 in $G - N[v_2]$, and a similar path from v to v_2 avoiding $N[v_1]$. Thus $\{v, v_1, v_2\}$ is an asteroidal triple, but v is weakly simplicial, contradicting quasi-extremality.

We also show that $G_1 = G - K_1$ can be represented on a path, $\mathcal{P}$. Since v_1, v_2, and v_3 must all lie in different components of $G - N[v]$, we know that v is not strongly simplicial in $G - K_1$. Assuming that $G - K_1$ is not representable on a path, then $G - K_1$ must contain an asteroidal triple $\{u_1, u_2, u_3\}$.

Definition 3.2.12 *Let K be a component of $G - N[v]$, where v is a weakly simplicial vertex of G. We define $A(K) = \cup_{u\in K}(N(u) \cap N(v))$, as the* ***attachment vertices*** *of K; and define $B(K) = \cap_{u\in K}(N(u) \cap N(v))$, as the* ***common attachment vertices*** *of K.*

Note that $u_1 \notin A(K_1)$, since otherwise $\{v_1, u_2, u_3\}$ would form an asteroidal triple of simplicial vertices of G, u_2 and u_3 would have to lie in different components of $G - N[v]$; but $N_{G-K_1}[v] \subset N_{G-K_1}[u_1]$, so we could have no path from u_2 to u_3 not adjacent to u_1, contradicting that $\{u_1, u_2, u_3\}$ is an asteroidal triple. Since

$u_1 \notin A(K_1)$, we must have that u_1 is a simplicial vertex of G. By the quasi-extremality assumption, each of u_1, u_2, and u_3 are strongly simplicial in G. Letting $v_1 = u_4$ gives $\{u_1, u_2, u_3, u_4\}$, a set of four strongly simplicial vertices of G, and thus is a 4-asteroid in G, contradicting that G was 3-asteroidal. So G must be representable on the path $\mathcal{P}_1$.

Also, we note that $G_2 = K_1 \cup A(K_1) \cup \{v\}$ is representable on a path $\mathcal{P}_2$, since G_2 is a connected, vertex-induced subgraph of $G - K_2$, where K_2 is the component of $G - N[v]$ containing v_2. It is clear that v is strongly simplicial in G_2, so all of the vertices of $N[v]$ may be assigned subtrees containing a common leaf, p, of the path that G_2 is represented on. We verify this assertion with the following claim.

Claim 3.2.13 *Let G be a graph representable on an path, $\mathcal{P}$, and let v be a simplicial vertex of G. Then there is a representation of G on $\mathcal{P}$ in such a way that the subtrees assigned to all the vertices in $N[v]$ all contain a common leaf of $\mathcal{P}$, if and only if there does not exist two vertices u_1 and u_2 in G such that a path between u_1 and v which does not intersect $N[u_2]$, and similarly a path between u_2 and v which does not intersect $N[u_1]$.*

Proof. We will consider cases:

1. Consider first, that v is strongly simplicial. Let G have a representation, $\mathscr{T}$ on the host path $\mathcal{P}$. Now, consider $X = \mathcal{P} - \left(\cup_{u \in N[v]} T_u\right)$. Now, X must be a connected subset of $\mathcal{P}$. This, more importantly, means that there is a leaf, p of $\mathcal{P}$ in X. Since all of the subtrees defined on X are part of the same clique, we may extend the subtree assignments of all of these vertices such that their assigned subtrees contain the leaf p.

2. We now consider that v is weakly simplicial. Then we must have that $G - N[v]$ has more than one component. We call them $K_1, K_2, ..., K_m$. Let

$H_1, H_2, ...H_m$, where $H_i = \cup_{v \in K_i} T_v$. Note that it is clear that for $i \neq j$, $H_i \cap H_j = \emptyset$. Supposed that the original K_i's are chosen in such a way that H_i lies to the left of H_{i+1}. Assume that $H_1, H_2, ..., H_k$ lie to the left of T_v in our representation, and that $H_{k+1}, H_{k+2} + ... + H_m$ lie to the right of T_v.

a) Suppose $A(K_{k+1} \subset B(K_i)$ for each $i \leq k$. Now, we can extend subtrees assigned to points in $A(K_{k+1})$ to the left of H_1. Now we can attach the representation of H_{k+1} to the left of H_1, which contadicts our assumption that the most components possible where represented to the left.

b) Supposed there exists and $i \in [k]$ so that $A(K_{k+1}) \not\subset B(K_i)$. Let j be the largest such index less than k for which $A(K_{k+1}) \not\subset B(K_j)$.

   I. Assume $j = k$

      i. If $A(K_k) \subset B(K_{k+1})$, then, similar to what was done in Case 1., we can insert H_{k+1} between H_k and T_v. This contradicts the original assumption.

      ii. If $A(K_k) \not\subset B(K_{k+1})$ Let $w_1 \in A(K_k) \setminus B(K_{k+1})$ and $w_2 \in A(K_{k+1}) \setminus B(K_k)$. Now, we must have a vertex $u_1 \in K_k$ such that u_1 is not adjacent to w_2 and $u_2 \in K_{k+1}$ so that u_2 is not adjacent to w_1. Now in $K_k \cup \{w_1\}$, we take a path from u_1 to w_1 that is not adjacent to w_2, and add to this path the edge $w_1 v_1$, giving us a path from u_1 to v_1 which avoids $N[u_2]$. Similarly we can find a path from u_2 to v_1 which avoids $N[u_1]$.

   II. Assume $j \neq k$. By the hypothesis on j we must have that $A(K_{k+1}) \subset B(K_{j+1})$. Now, if $A(K_j) \subset A(K_{k+1})$, we can again insert a representation of K_{k+1} between H_j and H_{j+1}, contradicting the hypothesis on k. On the other hand, if $A(K_j) \not\subset B(K_{k+1})$, then the construction from the previous sections gives the desired u_1 and u_2. $\diamond$

Now that we have representations of two different graphs on two different host trees, it may be beneficial to provide some way to distinguish them. Let T_v^1 denote the subtree of $\mathcal{P}_1$ assigned to the vertex v of G_1, and T_v^2 denote the subtree of $\mathcal{P}_2$ assigned to vertex v of G_2.

We note that in our path representation of G_1, it must be that $\cap_{w \in N_{G_1}[v]} T_w^1$ is non-empty. Let x be a vertex in this intersection. Notice that $\mathcal{P}_1 \cup xp \cup \mathcal{P}_2$ is a subdivision of $K_{1,3}$. To obtain a representation of G on $\mathcal{K}_{1,3}$ let $T_v = T_v^1$ if $v \in G_1 \setminus N[v_1]$, $T_v = T_v^2$ if $v \in G_2 \setminus N[v_1]$, and $T_v = T_v^1 \cup xp \cup T_p^2$ if $v \in N[v_1]$. ∇

Definition 3.2.14 *Let v_i be a simplicial vertex of G and let K be a component of $G - N[v_i]$. We call $v \in K$ a* ***full-neighbor*** *of $N(v_i)$ if v is adjacent to every vertex of $N(v_i)$.*

Case 3: G is neither extremal, nor quasi-extremal

Subcase 1: No component of $G - N[v_1]$ contains a full neighbor of $N(v_1)$.

By induction, obtain a representation of $G - v_1$ on $\mathcal{K}_{1,3}$. Since $N(v_1)$ is complete, then $\cap_{v \in N(v_1)} T_v \neq \emptyset$. Also, since there is no full-neighbor of $N(v_1)$, this non-empty intersection cannot intersect with a subtree assigned to a vertex outside of $N(v_1)$. Assign $T_{v_1} = \cap_{v \in N(v_i)} T_v$, and we have a representation of G on $\mathcal{K}_{1,3}$. $\blacktriangledown$

Subcase 2: $G - N[v_1]$ contains a component, K, consisting entirely of full-neighbors of $N(v_1)$.

If K_0 is the component of $G - N[v_1]$ containing v_2 and v_3, then $K \neq K_0$, since neither v_2 nor v_3 can be full-neighbors of $N(v_1)$.

If v is any vertex of K, then $\{v, v_2, v_3\}$ forms an asteroidal triple of G. Suppose $G - v_1$ has a representation on $\mathcal{K}_{1,3}$. Let $H = \cup_{v \in K} T_v$, then H is completely contained on the branch of $\mathcal{K}_{1,3}$ which does not contain T_{v_2} or T_{v_3}. Let $\delta = \cap_{v \in N(v_1)} T_v$. Note that $\delta \cap H \neq \emptyset$, and $\delta \cap H$ can be assumed not to be exactly δ, by making use of subdivision if necessary. Let $T_{v_1} = \delta \setminus H$, and we have a representation of G on $\mathcal{K}_{1,3}$. ▼

<u>Subcase 3:</u> One of the components K_i, with $i \neq 0$, contains a full-neighbor of $N(v_1)$ as well as a vertex which is not a full neighbor of $N(v_1)$.

Assume the component is K_1. Let $G_1 = G - v_1$, and $G_2 = K_1 \cup N[v_1]$. Note that v_1 is strongly simplicial in G_2, and that G_2 cannot be asteroidal since any path from v_2 to v_3 in K_0 would not be adjacent to the asteroidal paths between the vertices in G_2, contradicting the adjacency of asteroidal paths condition. Since G_2 is not asteroidal, it has a representation, $\mathscr{T}^2$ on a path, $\mathcal{P}$. Furthermore, since each vertex of $B(K_1)$ is adjacent to all of the other vertices in G_2, we assume that every vertex in $B(K_1)$ is assigned all of $\mathcal{P}$.

By induction, we can also assume that G_1 is representable on $\mathcal{K}_{1,3}$, with representation $\mathscr{T}^1$. Since K_1 contains a full-neighbor of $N(v_1)$ and K_1 also contains a simplicial vertex of G, call it v_4, the set $\{v_2, v_3, v_4\}$ forms an asteroidal triple of G_1, and as a result, the corresponding subtrees lie on different branches of $\mathcal{K}_{1,3}$. Since there is a path from v_2 to v_3 missing $N[v_1]$, this path also misses $N[v]$ for all $v \in K_1$, because there is no way to cross into K_1 from K_0 without going through $N[v_1]$. This forces $\delta = \cap_{v \in N(v_1)} T_v^1$ onto a branch different from $T_{v_2}^1$ and $T_{v_3}^1$.

Now consider γ, the connected portion of $\mathcal{K}_{1,3}$ containing all of the subtrees assigned to vertices in K_1, and containing no nodes of $\mathcal{K}_{1,3}$ contained in subtrees assigned to vertices in the other components. Note that the branch node of $\mathcal{K}_{1,3}$ is not in γ since that node must be contained in a subtree assigned to a vertex in K_0, since K_0 contains both v_2 and v_3.

This implies that γ is properly contained on the branch of $\mathcal{K}_{1,3}$, not containing $T^1_{v_2}$ or $T^1_{v_3}$. So we have that δ and γ are properly contained on the same branch of $\mathcal{K}_{1,3}$, and that both γ and δ are paths. Note that $\delta \cap \gamma \neq \emptyset$, since K_1 contains a full neighbor of $N(v_1)$. Now we can obtain a representation of G on $\mathcal{K}_{1,3}$ by replacing γ by $\mathcal{P}$ (the path on which G_2 is represented), so that the leaf p, contained in $\cap_{v \in N[v_1]} T^2_v$, is in δ. For each $v \in G_1 \cap G_2$ let $T_v = T^1_v \cup T^2_v$, for $v \in G_1 \setminus G_2$ let $T_v = T^1_v$, and for $v \in G_2 \setminus G_1$ let $T_v = T^2_v$. This assignment of subtrees gives a representation of G on $\mathcal{K}_{1,3}$. ▼

Note 3.2.15 *Let $\{v_1, v_2, v_3\}$ be an asteroidal set in G. If v_i is weakly simplicial, then we label the components $K^i_0, K^i_1, K^i_2, ..., K^i_{n_i}$ of $G - N[v_i]$, such that K^i_0 is the component which contains the other two vertices of the asteroidal triple.*

Subcase 4: Suppose that over all possible choices of asteroidal triples $\{v_1, v_2, v_3\}$ of G, with v_1 weakly simplicial, the component K^1_0 of $G - N[v_1]$, which contains v_2 and v_3 is the only component of G containing a full neighbor of $N(v_1)$.

Note that v_2 and v_3 cannot be full neighbors of $N(v_1)$, so K^1_0 contains vertices that are full neighbors of $N(v_1)$ as well as those that are not.

Observation 3.2.16 *No component other than K^1_0 may contain two vertices of an asteroidal triple of G. Otherwise, we are in violation of the intersection of asteroidal paths property.*

We must consider four possibilities in proving Subcase 4.

Possibility 1: There is a choice of asteroidal triple of simplicial vertices, $\{v_1, v_2, v_3\}$, such that v_1 is weakly simplicial, but v_2 and v_3 are strongly simplicial.

Let $H = N[v_1] \cup (\cup^{n_1}_{i=1} K^1_i)$. We will show that H is representable on a path, $\mathcal{P}$ in such a way that all subtrees assigned to vertices in $N[v_1]$ contain a common leaf of $\mathcal{P}$. Next we will construct a representation of G on $\mathcal{K}_{1,3}$.

If H is not as desired, then either H is asteroidal, or not all vertices of $N[v_1]$ can be represented by subtrees containing a common leaf of $\mathcal{P}$.

If the latter is true, then by Claim 3.2.13 there must exist vertices u_1' and u_2' of H such that there exists a path in H from u_1' to v_1 not adjacent to u_2', and similarly a path from u_2' to v_1 not adjacent to u_1'. Note that since v_1 is weakly simplicial in H, all components, particularly the two containing u_1' and u_2', contain simplicial vertices. Therefore, we replace u_1' and u_2' each with u_1 and u_2, the simplicial vertex in the corresponding component. The existence of u_1 and u_2 gives a 4-asteroid, $\{u_1, u_2, v_2, v_3\}$, contradicting that G was at most 3-asteroidal.

It must be the case, that H remains 3-asteroidal. Let $\{u_1, u_2, u_3\}$ be a 3-asteroid of H. Note that at most one of these vertices, say u_1, is from $N(v_1)$. If u_2 and u_3 are in the same component, then $\{v_1, u_2, u_3\}$ is an asteroidal triple of G, and we have contradicted the adjacency of asteroidal paths condition, since the path from v_2 to v_3 is completely contained in K_0^1, and the path from u_2 to u_3 is completely contained within another component. Therefore, u_2 and u_3 are in different components of $H - N[v_1]$, and, as a result, $u_1 \notin N(v_1)$.

Since u_2 and u_3 are in different components of $H - N[v_1]$, any path between u_2 and u_3 must go through $N(v_1)$. Also, since K_0^1 is the only component of $G - N[v_1]$ containing a full-neighbor, we know that u_2 and u_3 are not full-neighbors of $N(v_1)$. Recall that v_2 and v_3 are part of an asteroidal triple with v_1, so neither of these are full neighbors of $N(v_1)$. These observations show that $\{v_2, v_3, u_2, u_3\}$ form a 4-asteroid in G. Indeed, the path in H from u_2 to u_3 is not adjacent to v_2 nor to v_3. The path from v_2 to v_3 in K_0 is adjacent to neither u_2 nor u_3. Since v_2 and v_3 are strongly simplicial in G, there is a path from u_2 to v_2 not adjacent to v_3 or to u_3, and similarly there is a corresponding path avoiding v_2, and there are corresponding paths for u_3. Again, we have contradicted that G was at most

3-asteroidal.

Now we know that H is representable on a path, $\mathcal{P}$, such that all of the subtrees assigned to vertices of $N[v_1]$ contain the common leaf, p.

Now let $G^* = G - \cup_{i=1}^{n_i} K_i^1$. By induction, we obtain a representation of G^* on $\mathcal{K}_{1,3}$. Furthermore, since v_1 is strongly simplicial in G^*, we can assume that all representatives of vertices from $N[v_1]$ in G^* contain a common leaf, p^* of $\mathcal{K}_{1,3}$. Identifying the nodes p and p^* gives a representation of G on $\mathcal{K}_{1,3}$. ▼

Possibility 2: There is a choice of asteroidal triple of G such that v_1 and v_2 are weakly simplicial, but v_3 is strongly simplicial.

Assume $\{v_1, v_2, v_3\}$ is chosen such that $|K_0^1|$ is maximal, and so that, with v_1 already chosen, $|K_0^2|$ is also maximal. Under this assumption we can show that $A(K_k^i) \subset N(v_j)$ for $k \geq 1$ and with i and j distinct elements of $\{1, 2\}$. If this is not the case, say $A(K_k^1) \not\subset N(v_2)$, then there is an attachment point z of K_k^1 not in $N(v_2)$. Let v_4 be any simplicial vertex of K_k^1. Now, since v_3 is a strongly simplicial vertex of G, there is a path from v_2 to v_4 in G, not adjacent to v_3. Also, since $\{v_1, v_2, v_3\}$ is an asteroidal triple of G, and there is a path from v_4 to z in K_1^1, then appending the path from v_4 to v_1 (through z) to the asteroidal paths involving v_1, show that $\{v_2, v_3, v_4\}$ is an asteroidal triple in G. If v_4 were strongly simplicial, we could apply possibility 1 to obtain the desired representation of G on $\mathcal{K}_{1,3}$. Otherwise v_4 is weakly simplicial, but the component K_0^4 of $G - N[v_4]$ must contain $K_0^1 \cup v_1$, contradicting the maximality of $|K_0^1|$.

It must be the case that $A(K_k^1) \subset N(v_2)$ and a similar argument shows $A(K_k^2) \subset N(v_1)$. This implies that $\left\{K_j^1 | j = 1, 2, ..., n_1\right\} = \left\{K_j^2 | j = 1, 2, ..., n_2\right\}$, since K_j^2 being a component of $G - A(K_j^2)$ implies that K_j^2 is a component of $G - N[v_1]$.

We now show that $H = N[v_1] \cup \left(\cup_{j=1}^{n_1} K_j^1\right)$ cannot be asteroidal. Suppose that $\{u_1, u_2, u_3\}$ is an asteroidal triple of H (and therefore of G). Now, we must have, by Observation 3.2.16, that no two of these vertices lie in the same component of $H - N[v_1]$. Also, none of these vertices is in $N(v_1)$, since any path between the other two would be adjacent to this vertex, As a result, all of these vertices lie in different components of $H - N[v_1]$.

Since v_3 is strongly simplicial in G, there is a path from u_i to u_j not adjacent to v_3. Also, since K_0^1 contains a full-neighbor z, of $N(v_1)$, and any path from u_i to u_j not adjacent to u_k contains a vertex of $N(v_1)$, one can construct the necessary paths to show that $\{u_1, u_2, u_3, v_3\}$ forms a 4-asteroid of G, contradicting that G was at most 3-asteroidal. Thus, H is non-asteroidal, and has a representation on a path, $\mathcal{P}$.

Let H_1 consist of all the components, K_i^1, which are represented to the left of T_{v_1} in the representation on $\mathcal{P}$. Similarly, let H_2 be those components represented to the right of T_{v_1}. Observe that $H_1^* = H_1 \cup N[v_1]$ is representable on a path, $\mathcal{P}_1$ so that all vertices in $N[v_1]$ are assigned subtrees containing a common leaf, p_1^* of $\mathcal{P}_1$. Now, since $A(K_j^1) \subset N[v_1] \cap N[v_2]$, the subgraph, $H_2^* = H_2 \cup N[v_2]$ can be represented on another path, $\mathcal{P}_2$, so that all vertices of $N[v_2]$ are assigned representatives containing a common leaf, p_2^*, of $\mathcal{P}_2$. Again, we have made use of Claim 3.2.13.

Now, since $G^* = G - \cup_{i=1}^{n_1} K_j^1$ is a proper, vertex-induced, 3-asteroidal subgraph of G, then G^* has a representation on $\mathcal{K}_{1,3}$. Moreover, since $\left\{K_j^1 | j = 1, 2, ..., n_1\right\} = \left\{K_j^2 | j = 1, 2, ..., n_2\right\}$, then v_1, v_2, v_3 are strongly simplicial vertices of G^*. Since v_1, v_2, and v_3 are all strongly simplicial, we may assume that the representation is such that if p_1, p_2, and p_3 are the leaves of $\mathcal{K}_{1,3}$, then each vertex of $N[v_i]$ is assigned a subtree which contains p_i. Most importantly, the subtrees assigned to vertices in $N[v_1] \cap N[v_2]$ contain both p_1 and p_2. Now, identifying p_1^* with p_1 as well as p_2^* with p_2 gives a representation of G on $\mathcal{K}_{1,3}$. ▼

Possibility 3: G has the property that for each choice of asteroidal triple such that v_1 is weakly simplicial, then v_2 and v_3 must also be weakly simplicial.

Choose $\{v_1, v_2, v_3\}$ such that $|K_0^1|$ is maximum, and so that $|K_0^2|$ is maximum with the selection of v_1 already made, and finally so that $|K_0^3|$ is maximum with the selections of v_1 and v_2 already made.

Under these assumptions, we show that for each pair, (i, j), with $i \in \{1, 2, 3\}$ and $j \in \{1, 2, ..., n_i\}$, there is a $j \in \{1, 2, 3\}$ with $j \neq i$, so that $A(K_k^i) \subset N[v_j]$. Assuming the opposite, we let $A(K_k^1) \not\subset N[v_2]$ and $A(K_k^1) \not\subset N[v_3]$. Also, let v be a simplicial vertex of G in K_k^1, let $w_1 \in A(K_k^1) - N[v_3]$ and $w_2 \in A(K_k^1) - N[v_2]$. Note that w_1 and w_2 may not be distinct. Now, there is a path from v to w_1 in $K_k^1 \cup \{w_1\}$, and a path from w_1 to v_2 not adjacent to v_3. The union of these paths is a path from v to v_2, not adjacent to v_3. Similarly, we have a path from v to v_3, not adjacent to v_2. Finally, any path from v_2 to v_3 which is not adjacent to v_1 is clearly not adjacent to v. So, we have $\{v, v_2, v_3\}$ is a new asteroidal triple in G, with v_3 weakly simplicial, so by our assumptions v must be weakly simplicial. However, $K_0^1 \cup \{v_1\}$ is contained in the component of $G - N[v]$ which contains v_2 and v_3, contradicting the maximality of $|K_0^1|$. Therefore, we have either $A(K_k^1) \subset N[v_2]$ or $A(K_k^1) \subset N[v_3]$. Similarly, the maximality of $|K_0^2|$ guarantees that for each $k \in \{1, 2, ..., n_2\}$, then $A(K_k^2) \subset N[v_1]$ or $A(K_k^2) \subset N[v_3]$. Finally the maximality of $|K_0^3|$ induces the analogous implication that, for each $k \in \{1, 2, ..., n_3\}$, then $A(K_0^3) \subset N[v_1]$ or $A(K_0^3) \subset N[v_2]$. This gives us that each K_j^i is also a K_n^m for j and m both at least 1.

We now define the following subgraphs of G

$$J_0 = \left\{K_j^i | A(K_j^i) \subset N[v_1] \cap N[v_2] \cap N[v_3]\right\}$$

$$J_1 = \left\{K_j^i | A(K_j^i) \subset N[v_1] \cap N[v_2]\right\} \quad J_2 = \left\{K_j^i | A(K_j^i) \subset N[v_1] \cap N[v_3]\right\}$$

$$J_3 = \left\{K_j^i | A(K_j^i) \subset N[v_2] \cap N[v_3]\right\} \quad J = \cup_{i=0}^{3} J_i$$

Note that J_0 consists of components common to $G-N[v_1]$, $G-N[v_2]$, and $G-N[v_3]$. J_1 consists of the components common to $G-N[v_1]$ and $G-N[v_2]$, and so on. Also, note that J contains all of the components of $G-N[v_i]$ for $i=1,2,3$.

We will show that $J_1 \cup N[v_1]$ and $J_2 \cup N[v_1]$ are both representable on paths such that, in each case, each vertex of $N[v_1]$ is represented by a subtree containing a common leaf. Also, $J_3 \cup N[v_2]$ is representable on a path such that the vertices of $N[v_2]$ are represented by subtrees containing a common leaf of that path.

Suppose that this is not the case and that $J_3 \cup N[v_2]$ does not have such a representation. We know that $J_3 \cup N[v_2]$ cannot be asteroidal, using the same argument that was used for H in Possibility 1. Therefore, $J_3 \cup N[v_2]$ is indeed representable on a path, so if $J_2 \cup N[v_2]$ doesn't have the desired property, then by Claim 3.2.13 we must have two simplicial vertices u_1 and u_2 so that there is a path in $J_3 \cup N[v_2]$ from u_1 to v_2 not adjacent to u_2 and another path from u_2 to v_2 not adjacent to u_1. Since such paths exist, and since $\{v_1, v_2, v_3\}$ is an asteroidal triple of G, we must have that $\{v_1, u_1, u_2\}$ is an asteroidal triple of simplicial vertices in G. From Observation 3.2.16, we have that u_1 and u_2 must be in different components of J_3. Furthermore, since v_1 is weakly simplicial, u_1 and u_2 must be as well.

Since K_0^2 is the only component of $G-N[v_2]$ containing a full-neighbor of $N(v_2)$, we know that u_1 and u_2 are not full-neighbors of $N(v_2)$. Therefore, the component of $G-N[u_1]$ containing v_1 and u_2 must contain $K_0^2 \cup \{u_2\}$, contradicting the maximality of $|K_0^2|$. This gives us that $J_3 \cup N[v_2]$ is indeed representable on a path such that all vertices of $N[v_2]$ are assigned representatives containing a common leaf, p. Similar arguments work for $J_1 \cup N[v_1]$, $J_1 \cup N[v_2]$, and $J_2 \cup N[v_1]$. Only two cases now remain.

To show the remaining cases, we let $A(J_i) = \cup_{k_m^k \in J_i} A(K_m^k)$. Notice that $J_2 \cup A(J_2) \cup \{v_3\}$ is isomorphic to $J_2 \cup A(J_2) \cup \{v_1\}$, and that these graphs are reductions of

$J_2 \cup N[v_3]$ and $J_2 \cup N[v_1]$, respectively. Since $J_2 \cup A(J_2) \cup \{v_3\}$ is representable on a path, we know that $J_2 \cup N[v_3]$ is representable on a path with a representation such that each vertex of $N[v_3]$ has an assigned subtree containing a leaf of the path. Similarly, $J_3 \cup N[v_3]$ is representable on a path such that each vertex of $N[v_3]$ contains a leaf of that path.

Note that, as a consequence of this, no two vertices of any J_i may be part of the same asteroidal triple of G, since otherwise we would contradict the adjacency of asteroidal paths condition on G. It can also be seen that if $u_i \in J_i$, then $\{u_1, u_2, u_3\}$ forms an asteroidal triple of G, regardless of the selection of u_1, u_2, or u_3. To check this, let $u_1 \in K_1^1 \in J_1$, $u_2 \in K_1^3 \in J_2$, and $u_3 \in K_1^2 \in J_3$. By the choices of the J_i's, we have the existence of $z_1 \in A(K_1^1) - N[v_3]$, $z_2 \in A(K_1^3) - N[v_2]$, and $z_3 \in A(K_1^2) - N[v_1]$. There is a path from u_1 to z_1 to z_2 to u_2, not adjacent to u_3. Similarly, the other asteroidal paths exist, and we have shown that $\{u_1, u_2, u_3\}$ is an asteroidal triple of G.

Option 1: J contains no asteroidal triple of G.

It is clear that at least one of J_1, J_2, or J_3 must be empty. Suppose $J_3 = \emptyset$, and let $J^* = J \cup N[v_1]$. Suppose J^* is asteroidal, with asteroidal triple $\{u_1, u_2, u_3\}$. Since J contains no asteroidal triple of G, one of the vertices, say u_3 is in $N[v_1]$. Also, since no component of J may contain both u_1 and u_2, each path from u_1 to u_2 must pass through $N(v_1)$, and is therefore adjacent to u_3. Thus, J^* is not asteroidal, and can be represented on a path, $\mathcal{P}$.

In such a representation, subtrees assigned to vertices of J_1 and those assigned to vertices of J_2 must lie on opposite sides of the tree assigned to v_1. If neither J_1 nor J_2 are empty, we assume the representation is chosen so that subtrees assigned to J_2 lie to the left of T_{v_1}.

Let H_1 consist of components of J which are represented to the right of T_{v_1}, while H_2 consists of those represented to the left. Now, $H_1^* = H_1 \cup N[v_1]$ is representable on a path, $\mathcal{P}_1$, such that each member of $N[v_1]$ is represented by a subtree containing a common leaf, p_1^* of $\mathcal{P}_1$. Similarly for H_2 and $N[v_3]$ on the path $\mathcal{P}_2$, since the attachment points of each component are also in $N[v_3]$. Now $G - J$ is a proper subgraph of G, with v_1, v_2, v_3, strongly simplicial vertices of $G - J$. This gives that $G - J$ has a representation on $\mathcal{K}_{1,3}$, so that if p_1, p_2, p_3 are the leaves of $\mathcal{K}_{1,3}$, then each tree assigned to a vertex of $N[v_i]$ contains p_i. Adding the edge $p_1p_1^*$ attaches $\mathcal{P}_1$ to the end of this branch of $\mathcal{K}_{1,3}$. Alter the subtree representative of v_1 by taking the union of the two subtrees together with the edge $p_1p_1^*$. Similarly we attach a representation of H_2 to $\mathcal{K}_{1,3}$ by adding the edge $p_3p_3^*$ and adjusting T_{v_3} accordingly. The result is a representation of G on $\mathcal{K}_{1,3}$.

<u>Option 2:</u> Suppose J does contain an asteroidal triple of G.

Each component of J contains a simplicial vertex of G, so J must contain an asteroidal triple of simplicial vertices. Let $\{u_1, u_2, u_3\}$ be such an asteroidal triple. Suppose that K_i denotes the component of J containing u_i. Since J_2 cannot contain more than one vertex of any asteroidal triple of G, we can assume that K_1 is a component of $G - N[v_1]$. This implies that u_1, u_2, and u_3 are strongly simplicial in G, since if u_1 were weakly simplicial, then the component of $G - N[u_1]$ containing u_2 and u_3 contains $K_0^1 \cup \{v_1\}$, contradicting the maximality of $|K_0^1|$. Since u_1 is strongly simplicial, u_2 and u_3 must also be strongly simplicial by the assumption in this possibility.

Note that $J - K_i$ cannot contain an asteroidal triple of G, otherwise we would have another asteroidal triple of strongly simplicial vertices of G. This gives us more than three non-adjacent strongly simplicial vertices, contradicting that G was at most 3-asteroidal.

Now, either all of the K_i's are components of J_0, or at least one of them is a component of J_j for $j \in \{1, 2, 3\}$, otherwise one can discover a larger asteroidal set in G.

Sub-Option 1: $K_3 \in J_3$

As before, v_1, v_2, and v_3 are strongly simplicial vertices of $G - J$, and $G - J$ has a representation on $\mathcal{K}_{1,3}$ so that subtrees assigned to vertices in $N[v_i]$ are assigned subtrees containing p_i.

Since $J - J_3$ is a subset of $J - K_3$, we have that $J - J_3$ contains no asteroidal triple of G. This gives that $(J - J_3) \cup N[v_1]$ is representable on a path, and as in Option 1, we can form H_1 and H_2, and attach their path representations onto $\mathcal{K}_{1,3}$ at p_1 and p_2 respectively to represent $G - J_3$ on $\mathcal{K}_{1,3}$.

Now, $J_3 \cup N[v_2]$ is representable on a a path $\mathcal{P}$ so that each vertex of $N[v_2]$ contains a common leaf, p^* of $\mathcal{P}$. We may attach the representation of $J_3 \cup N[v_2]$ to $\mathcal{K}_{1,3}$, by adding the edge $p^* p_2$, and altering the subtree assigned to T_{v_2} to contain this edge, obtaining a representation of G on $\mathcal{K}_{1,3}$.

Sub-Option 2: For each asteroidal triple of G, none of the vertices lie in J_1, J_2, or J_3. Again, at least one of the J_j's must be empty. For simplicity, assume it is J_3. Imitating the construction in sub-option 1, with $(J - K_3) \cup N[v_1]$ and $K_3 \cup N[v_2]$ in place of $(J - J_3) \cup N[v_1]$ and $J_3 \cup N[v_2]$, respectively, gives the representation.

▼

That is, we have shown that a graph satisfying the hypotheses, however which is neither extremal, nor quasi-extremal, does have a $(\mathcal{K}_{1,3}, 1)$-representation. ▽

We have provided an exhaustive proof that as long as a graph, G, satisfies the hypotheses, then we can construct a representation of G on $\mathcal{K}_{1,3}$. □

After wading through the details in this proof, one can see how difficult completely classifying the graphs representable on a certain class of host tree can be. Walter's result remains the only case of the following conjecture which has been verified.

Conjecture 3.2.17 *A graph, G, is* $(\mathcal{K}_{1,n};1)$*-representable iff it is chordal, at most* n*-asteroidal, and has the adjacency of asteroidal paths property.*

Since it can be seen from Walter's result that two obstacles in representing graphs by subtrees of $\mathcal{K}_{1,3}$ are asteroidal sets and cycles, it would be natural to consider these structures while trying to extend the result. This is exactly what we will do in chapters 4 and 5.

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CHAPTER 4

Tree Representations of Asteroidal Sets

Representing Asteroidal Sets on Subdivisions of Stars

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abstract. Consider a simple undirected graph $G = (V, E)$. A family of subtrees, $\{T_v\}_{v \in V}$, of a tree $\mathcal{T}$ is called a $(\mathcal{T}; t)$-representation of G provided $uv \in E$ if and only if $|T_u \cap T_v| \geq t$. In this paper we consider $(\mathcal{T}; t)$-representations for graphs containing large asteroidal sets, where $\mathcal{T}$ is a subdivision of the n-star $K_{1,n}$. An asteroidal set in a graph G is a subset A of the vertex set such that for all 3-element subsets of A, there exists a path in G between any two of these vertices which avoids the neighborhood of the third vertex. We construct a representation of an asteroidal set of size $n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$ and show that no graph containing a larger asteroidal set can be represented.

4.1 Introduction

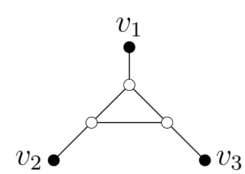
The study of graph representations is an active research area in graph theory. Given a graph $G = (V, E)$, a representation of G is the following collection of objects: (1) a set S, (2) a function $f : V \to \mathscr{P}(S)$ (the power set of S), and (3) a function $g : f(V) \times f(V) \to \{0, 1\}$ so that $g(f(v_1), f(v_2)) = 1$ iff $v_1 v_2 \in E$. We call S the *host set*, f the *assignment function*, and g the *conflict rule*. A graph G is representable under a given host set S and conflict rule g if there exists a suitable assignment function f, in which case we say that G is $(S; g)$-representable.

Much is known about graph representations when the conflict rule depends on the size of the intersection between assigned subsets. Such intersection representations have been studied extensively. A comprehensive list of authors can be found in [1]. Given conflict-tolerance $t = 1$, all graphs are $(S; 1)$-representable for large enough S. A central objective in the theory of graph representations is to find the smallest host set on which certain classes of graphs may be represented with respect to some given conflict rule. For example, the cycle C_n can be represented on a host set of size n. Indeed, assigning the set $\{i, i + 1 \mod (n)\}$ to vertex v_i gives a set representation of C_n, however there is no possible assignment of subsets from a smaller host set that will induce C_n with $t = 1$.

In this paper we consider tree representations of graphs. Tree representations are a variation on the traditional graph representation. The host is a tree, giving more structure than merely a set of elements; objects assigned to vertices of a represented graph are subtrees of the host tree; and an edge exists between two vertices if and only if their assigned subtrees intersect in t or more nodes, where t is a prescribed conflict-tolerance. An important distinction between set representations and tree representations is that given a host tree, not all graphs have a representation where the conflict-tolerance is $t = 1$, even if we allow subdivision of the tree. For example, cycles of length four or greater are always forbidden.

The classes of graphs representable using different host trees differ significantly (see [2, 3, 4, 5, 6, 7]), and are thus interesting to study. A well known and interesting example is the result of Lekkerkerker and Boland [8]. We first give a definition and then state their result.

Definition 4.1.1 *An asteroidal set in a graph G is $A \subset V(G)$ so that $\forall v_1, v_2, v_3 \in A$, and $\forall i, j, k \in \{1, 2, 3\}$ with $i, j,$ and k distinct, there exists a path between v_i and v_j which does not intersect $N(v_k)$. If $|A| = m$, then A is said to be an m-asteroid of G. Furthermore, if A is the maximum size asteroidal set contained in G, then G is said to be m-asteroidal.*



$\{v_1, v_2, v_3\}$ is an asteroidal set in the graph

Theorem 4.1.2 *A graph is representable on an interval if and only if it is chordal and non-asteroidal.*

For convenience of notation, consider $\mathcal{K}_{1,n}$ to be a subdivision of the tree $K_{1,n}$. In 1972 James Walter wrote his dissertation [9] on graphs representable on $\mathcal{K}_{1,3}$ with tolerance $t = 1$. This can be thought of as a generalization of the result of Lekkerkerker and Boland, since $\mathcal{K}_{1,3}$ is a logical extension from studying paths. Walter discovered that graphs containing certain induced cycles and certain asteroidal configurations were not representable under this host tree and conflict rule. Walter's result provides motivation for the work done in this paper, and is stated as follows.

Theorem 4.1.3 *A graph G is $(\mathcal{K}_{1,3}; 1)$-representable iff G is chordal, at most 3-asteroidal, and for any two pairs v_1, v_2 and u_1, u_2 of vertices contained in asteroidal triples of G, any path connecting v_1 and v_2 must be adjacent to any path connecting u_1 and u_2.*

Given this result, it seems that cycles and asteroidal sets are interesting structures to study while considering tree representations of graphs. It was Jamison who stated that it would be interesting to explore what happens if the conflict-tolerance is greater than one. In 2001, Eaton and Barbato [2] studied representations of cycles on $\mathcal{K}_{1,3}$ with arbitrary tolerance t. They described all cycles representable on $\mathcal{K}_{1,3}$ with conflict tolerance t, showing that arbitrarily large cycles can be represented on $\mathcal{K}_{1,3}$ at the cost of increasing the tolerance. Their theorem is restated below.

Theorem 4.1.4 *For $t = 3, 4,$ and 5 the maximum n such that C_n is $(\mathcal{K}_{1,3}; t)$-representable is $3t-3$. For $t \geq 6$ the largest such n satisfies the following inequality*

$$\frac{1}{4}t^2 + t + \frac{3}{4} \leq n \leq \frac{1}{4}t^2 + \frac{3}{2}t - \frac{3}{4}$$

A related result on cycles is due to Eaton and Faubert [4]. The two considered tree representations where the host tree is a caterpillar. That is, a tree in which every node is either on, or adjacent to, its longest path. Again, we provide a definition and then state their result.

Definition 4.1.5 *We say that $G \in cat[h, t]$ if there exists a caterpillar with maximum degree h such that G is representable on this caterpillar with tolerance t.*

Theorem 4.1.6 *If $n \leq (h-2)(t-1)+2$ with $h \geq 3$ and $t \geq 2$, then $C_n \in cat[h, t]$.*

The two went on to show that $C_n \in cat[4, 3]$ for all values of n. They also completely classified the graphs which are in $cat[2, t]$ and $cat[3, 1]$ [3].

Going back to the result of Eaton and Barbato, note that for $k \geq 6$, the cycle C_k contains an asteroidal set of size $\lfloor \frac{k}{2} \rfloor$. That is, their result on cycles implies that graphs containing arbitrarily large asteroidal sets are representable on $\mathcal{K}_{1,3}$. We explore this result further, and construct the largest asteroidal set representable on $\mathcal{K}_{1,n}$ with arbitrary conflict tolerance $t > 1$ and $n \geq 3$. The remainder of this paper will be devoted to proving the following main theorem and discussing a few open problems.

Theorem 4.1.7 *For $t > 1$ and $n \geq 3$, an asteroidal configuration of size $n + \sum_{k=2}^{n} \binom{n}{k} \binom{t-2}{k-1}$ which is $(\mathcal{K}_{1,n}; t)$-representable exists. Furthermore, any graph containing a larger asteroidal set is not $(\mathcal{K}_{1,n}; t)$-representable.*

Note that any terms not defined in this introduction, but used throughout the paper, can be found in [10].

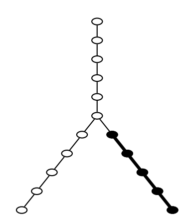
4.2 Construction of the Graph Containing the Asteroidal Set

4.2.1 Vertices and their Assigned Subtrees

For convenience, we will refer to the graph we are constructing as the *target graph*. Recall that our host tree is $\mathcal{K}_{1,n}$. It shall be sufficient to assume that each branch of the host tree contains t nodes. For clarity, the word *node* will be used to indicate a vertex of the host tree, distinguishing these from vertices of the target graph.

We call the unique node of degree n the branching node of the host tree, and the branches of the host tree are labeled with the integers $0, 1, ..., n-1$. Furthermore, within this paper, any reference to the size of a subtree or of an intersection should be interpreted in terms of number of nodes. It should be noted that each subtree defined below will correspond to a vertex in the target graph. The vertex set of the target graph will be described in terms of four disjoint subsets $\mathcal{V}, \mathcal{W}, \mathcal{P}$, and $\mathcal{Q}$. Note that the subscripts of the vertices in $\mathcal{V}$ and $\mathcal{P}$ will give information as to which branches their corresponding subtrees exist on, and should be interpreted modulo n. The desired asteroidal set will be a subset of the vertex set of the target graph. The description of each set of vertices below is accompanied by an image, giving an example of the subtree representative of one of the vertices from that set. The sample host tree $\mathcal{K}_{1,3}$ is shown; the tolerance used in the examples is $t = 5$; and the selected subtrees are denoted by thick edges and filled nodes.

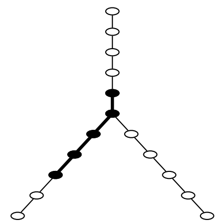
1. Let $\mathcal{V}$ be a collection of vertices represented by distinct subtrees, each of size exactly t, and contain the leaf of a branch of $\mathcal{K}_{1,n}$. These subtrees are paths on the exterior portion of each branch of $\mathcal{K}_{1,n}$. Since there are n branches of $\mathcal{K}_{1,n}$, we will have n vertices in $\mathcal{V}$. We will refer to these vertices later as $v_0, ..., v_{n-1}$, where the subscripts denote the branch on which the corresponding assigned subtree exists.



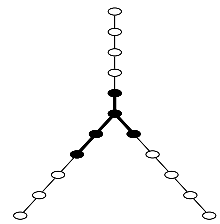
The subtree assigned to one of the vertices in $\mathcal{V}$

2. For $2 \leq k \leq n$, we define $\mathcal{W}_k$ to be the set of vertices represented by distinct subtrees which are of size exactly t and exist non-trivially on exactly k branches of $\mathcal{K}_{1,n}$. There are $\binom{n}{k}\binom{t-2}{k-1}$ such subtrees, and therefore the same number of corresponding vertices. We take $\mathcal{W}$ to be the union, over all values of k, of the sets $\mathcal{W}_k$. Clearly $\{\mathcal{W}_k\}_{k=2}^{n}$ forms a partition of $\mathcal{W}$, and therefore

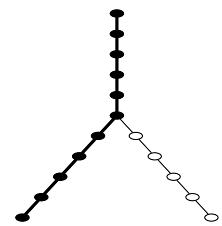
$|\mathcal{W}| = \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$. Note that we assume the convention that $\binom{a}{b} = 0$ for $b > a$.



The subtree assigned to one of the vertices in $\mathcal{W}_2$



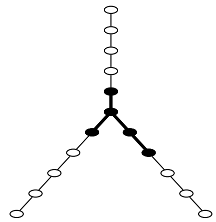
The subtree assigned to one of the vertices in $\mathcal{W}_3$



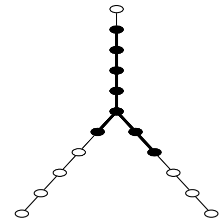
The subtree assigned to one of the vertices in $\mathcal{P}$

3. Construct $\mathcal{P}$, the collection of all vertices representable by subtrees existing non-trivially on exactly two consecutive branches of $\mathcal{K}_{1,n}$, and which extend out to the leaf node on each of these branches. We define vertex p_i, whose assigned subtree we denote by T_{p_i} and contains the entirety of branches i and $i+1$ for $0 \leq i \leq n-1$.

4. Let $\mathcal{Q}$ be a collection of vertices, each assigned a subtree which is an extension of a subtree assigned to a vertex in $\mathcal{W}$. That is, for each vertex in $\mathcal{W}$ we create $k(w)$ (recall that k is the number of legs of $\mathcal{K}_{1,n}$ on which T_w, the subtree assigned to w, lives non-trivially) new vertices in $\mathcal{Q}$. Each of these vertices is assigned a subtree which has been created by elongating, out to the second to last node, exactly one of the pre-existing legs of the subtree assigned to the corresponding vertex w. Consider the following example where we show T_w, the subtree assigned to one of the vertices from $\mathcal{W}$ and one of its extensions, a subtree assigned to a vertex in $\mathcal{Q}$.



T_w, the subtree assigned to the vertex $w \in \mathcal{W}$



One of the three extensions of T_w, giving the subtree assigned to one of vertices in $\mathcal{Q}$

4.2.2 Basic Claims About Adjacencies

Consider the graph $G = (\mathcal{V} \cup \mathcal{W} \cup \mathcal{P} \cup \mathcal{Q}\ ,\ E)$, where E is the edge set defined by the conflict tolerance relationship. Given the assignment of subtrees of $\mathcal{K}_{1,n}$ from the previous section, we have the following claims and observations about E. Note that we will use the notation T_x to refer to the subtree assigned to vertex x of G.

Observation 4.2.1 *$\mathcal{V}$ is an independent set.*

Observation 4.2.2 *There are no edges between the vertices of $\mathcal{Q}$ and $\mathcal{V}$.*

Observation 4.2.3 $N(v_i) = \{p_{i-1}, p_i\}$

Claim 4.2.4 *$\mathcal{W}$ is an independent set.*

Proof. Note that for each $w \in \mathcal{W}$, we have that $|T_w| = t$. Let w_i, w_j be distinct vertices in $\mathcal{W}$. Then their assigned subtrees are distinct. That is, there exists at least one node of $\mathcal{K}_{1,n}$ which is a node of $T_{w_i} \cup T_{w_j}$, but which is not a node of $T_{w_i} \cap T_{w_j}$. This gives that $|T_{w_i} \cap T_{w_j}| < |T_{w_i}| = t$. Therefore, given any two vertices from $\mathcal{W}$, their assigned subtrees have an intersection of fewer than t nodes, and thus the vertices cannot be adjacent. In short, $\mathcal{W}$ is an independent set. ∎

Observation 4.2.5 *$\mathcal{V} \cup \mathcal{W}$ is an independent set.*

Claim 4.2.6 *Vertices p_i and p_j are adjacent if and only if $j \in \{i-1, i+1\}$.*

Proof. If $j \neq i-1$ or $i+1$ then p_i and p_j are assigned subtrees which exist on distinct pairs of branches of $\mathcal{K}_{1,n}$. That is, their assigned subtrees intersect in exactly a single node (the branching node). Since $t > 1$, we have that p_i and p_j are not adjacent. ■

Claim 4.2.7 *Let $w \in \mathcal{W}$, then $N(w) \cap \mathcal{P} \neq \emptyset$ iff T_w exists non-trivially only on two consecutive branches of $\mathcal{K}_{1,n}$. Furthermore, if $|N(w) \cap \mathcal{P}| \neq 0$, then $|N(w) \cap \mathcal{P}|$ $= 1$.*

Proof. Consider $p \in \mathcal{P}$ and $w \in \mathcal{W}$ with $pw \in E$. Then $|T_p \cap T_w| \geq t$. However, since $|T_w| = t$, this implies that $T_p \cap T_w = T_w$. Now, since T_w exists on at least two branches of $\mathcal{K}_{1,n}$, and we know that $T_w \subset T_p$, then T_w exists non-trivially on the same two branches of $\mathcal{K}_{1,n}$ as T_p (which are indeed consecutive).

If T_w exists non-trivially only on two consecutive branches of $\mathcal{K}_{1,n}$ (call them branch i and $i+1$), then $T_w \subset T_{p_i}$. Therefore, $|T_w \cap T_{p_i}| = t$, implying that $p_i \in N(w)$. ■

Claim 4.2.8 *If q is a vertex from $\mathcal{Q}$, then q and p_j are adjacent iff T_q was obtained from some T_w by elongating branch j or branch $j+1$.*

Proof. If T_q exists non-trivially on exactly two branches of $\mathcal{K}_{1,n}$, then the result follows directly from the previous claim. We therefore assume that T_q exists non-trivially on at least 3 branches of $\mathcal{K}_{1,n}$. Let $qp_j \in E$. Assume that T_q was obtained from T_w by an elongation of branch i, where $i \notin \{j, j+1\}$. Now, since T_{p_j} exists non-trivially only on branches j and $j+1$ of $\mathcal{K}_{1,n}$, we have $a_j + a_{j+1} \geq t-1$, where a_j and a_{j+1} are the lengths of the legs of T_q on branch j and $j+1$ of $\mathcal{K}_{1,n}$. Furthermore, since $i \neq j$ and $i \neq j+1$, we must have that the previous statement is also true about T_w. This, however, gives a contradiction since $|T_w| = t$ and T_w lives non-trivially on at least 3 branches of $\mathcal{K}_{1,n}$. ■

Claim 4.2.9 *For $w_1, w_2 \in \mathcal{W}$, there exists $q_1 \in \mathcal{Q}$, with the property that $q_1 \in N(w_1) \setminus N(w_2)$.*

Proof. Let $w_1, w_2 \in \mathcal{W}$ with $w_1 \neq w_2$. Assume that $N(w_1) \subset N(w_2)$. Now, consider $\mathcal{Q}_{w_1}$ the set of all $q \in \mathcal{Q}$ so that T_q can be obtained via an elongation of one of the legs of T_{w_1}. Then, since $N(w_1) \subset N(w_2)$, we know $\mathcal{Q}_{w_1} \subset N(w_2)$. That is, $\forall q \in \mathcal{Q}_{w_1}, |T_{w_2} \cap T_q| \geq t$. However, $|T_{w_2}| = t$, so this implies $T_{w_2} \cap T_q = T_{w_2}$, $\forall q \in \mathcal{Q}_{w_1}$. This gives us that $T_{w_2} \subset \cap_{q \in \mathcal{Q}_{w_1}} T_q$. However, we know that $\cap_{q \in \mathcal{Q}_{w_1}}$ T_q is exactly T_{w_1}. Therefore, we have that $T_{w_2} \subset T_{w_1}$; but, since $|T_{w_1}| = |T_{w_2}|$, we know $T_{w_2} = T_{w_1}$. Thus $w_1 = w_2$, a contradiction. This gives that there must exist a $q_1 \in N(w_1) \setminus N(w_2)$. ■

4.2.3 Verification of the Asteroidal Properties of $\mathcal{V} \cup \mathcal{W}$

Theorem 4.2.10 *$\mathcal{V} \cup \mathcal{W}$ is an m-asteroidal set of the graph G which has vertex set $\mathcal{V} \cup \mathcal{W} \cup \mathcal{P} \cup \mathcal{Q}$ and edge set E, where $m = n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$.*

Proof. We recall that an asteroidal set A is an independent set which has the property that for any selection of 3 vertices, $a_1, a_2, a_3 \in A$ there is a path connecting any two of them, which avoids the neighborhood of the third.

Firstly, notice that $C = v_0 p_0 v_1 ..., v_{n-1} p_{n-1} v_0$ is a cycle in G. This cycle will be used extensively in the verification of the properties of the asteroidal set. Also worth noting is that, for convenience of notation we will use $P(u, v)$ to denote a path in G connecting the vertices u and v instead of the more commonly used (uv)-path. The following claims 2.11 through 2.16 verify that this definition is satisfied on $\mathcal{V} \cup \mathcal{W}$.

Claim 4.2.11 *For $v_1, v_2, v_3 \in \mathcal{V}$, $\forall i, j, k \in \{1, 2, 3\}$, with i, j, and k distinct, there exists a path, $P(v_i, v_j)$, so that $P(v_i, v_j) \cap N(v_k) = \emptyset$.*

Proof. Let $v_1, v_2, v_3 \in \mathcal{V}$. Now, we construct a path between v_i and v_j which avoids $N(v_k) = \{p_{k-1}, p_k\}$. Then the cycle C provides two paths from v_i to v_j, one of which must avoid the sequence $p_k v_k p_{k+1}$. ▼

Claim 4.2.12 *For $v_1, v_2 \in \mathcal{V}$, $w \in \mathcal{W}$, there exists a path, $P(v_1, v_2)$, so that $P(v_1, v_2) \cap N(w) = \emptyset$.*

Proof. The cycle C connects the vertices v_1 and v_2 via two paths. Now, recall that $N(w)$ contains at most one of the vertices on this cycle. If $N(w) \cap \mathcal{P} = \emptyset$, then we take either portion of the cycle as our path. If $N(w) \cap \mathcal{P}$ is non-empty, then there must be exactly one vertex $p \in N(w) \cap \mathcal{P}$, so we traverse the portion of the cycle from v_1 to v_2 in the direction which avoids p. This gives us the desired path. ▼

Claim 4.2.13 *For $v_1, v_2 \in \mathcal{V}$, $w \in \mathcal{W}$, $\forall i, j \in \{1, 2\}$ with i and j distinct, there exists a path, $P(v_i, w)$, so that $P(v_i, w) \cap N(v_j) = \emptyset$.*

Proof. We begin from w. Let $\mathcal{Q}_w$ denote the collection of all $q \in \mathcal{Q}$, so that T_q was obtained from T_w by elongating one of its legs. Now, notice that $|N(w) \cap \mathcal{Q}_w| \geq 2$, so there exists $q_1, q_2 \in N(w) \cap \mathcal{Q}_w$. Then, recall $\forall q \in \mathcal{Q}$, there exists two vertices $p_a, p_b \in N(q) \cap \mathcal{P}$. Since q_1 and q_2 are distinct, there are at least three distinct vertices $p_a, p_b, p_c \in N(q_1) \cup N(q_2)$. Also, recall that $N(v_j) \cap \mathcal{P} = \{p_{j-1}, p_j\}$. Therefore, there exists $p \in \{p_a, p_b, p_c\}$ with $p \notin \{p_{j-1}, p_j\}$. We take the path from w to this vertex p. We again traverse the path from vertex p to v_i which uses the portion of the cycle C avoiding the sequence $p_{j-1} v_j p_j$. ▼

Claim 4.2.14 *For $w_1, w_2, w_3 \in \mathcal{W}$, $\forall i, j, k \in \{1, 2, 3\}$, with i, j, and k distinct, there exists a path, $P(w_i, w_j)$, so that $P(w_i, w_j) \cap N(w_k) = \emptyset$.*

Proof. Recall the existence of $q_i \in N(w_i)$ so that $q_i \notin N(w_k)$, and $q_j \in N(w_j)$ so that $q_j \notin N(w_k)$. We traverse from w_i to q_i and from w_j to q_j. If $q_i = q_j$, then we have the desired path already, so assume they are not equal. Now, q_i and q_j may be adjacent, however, we are unsure, so we travel on. Recall that $N(w_k)$ may contain at most one vertex from $\mathcal{P}$. Also, recall that each vertex from $\mathcal{Q}$ is adjacent to exactly two vertices from $\mathcal{P}$. Therefore we can extend from q_i to at least one of its neighbors p_i and from q_j to at least one of its neighbors p_j without crossing into $N(w_k)$. Again, if $p_i = p_j$ we have the desired path, so we assume this is not the case. Now, we are on the cycle C. If $N(w_k) \cap \mathcal{P} = \emptyset$, then we connect p_i to p_j via either portion of the cycle. Otherwise $N(w) \cap \mathcal{P}$ is a single vertex, p, and we connect p_i to p_j by a path along the cycle in the direction which avoids p. In either case we have completed a path connecting the vertices w_i and w_j which avoids $N(w_k)$. ▼

Claim 4.2.15 *For $w_1, w_2 \in \mathcal{W}$, $v \in \mathcal{V}$, there exists a path, $P(w_1, w_2)$, so that $P(w_1, w_2) \cap N(v) = \emptyset$.*

Proof. If $N(w_1) \cap N(w_2) \neq \emptyset$ then we can draw a path from w_1 to w_2 via their common neighbor. Recall that there are no edges between the partitions $\mathcal{V}$ and $\mathcal{Q}$, so this path satisfies the requirements. Otherwise, $N(w_1) \cap N(w_2) = \emptyset$. Then, similar to the proof of claim 2.13, there exist paths from w_1 to p_a and from w_2 to p_b where $p_a, p_b \notin N(v)$. Now, we can use the cycle C to connect the vertices p_a and p_b with a path that does not intersect $N(v)$. ▼

Claim 4.2.16 *For $w_1, w_2 \in \mathcal{W}$, $v \in \mathcal{V}$, $\forall i, j \in \{1, 2\}$ with i and j distinct, there exists a path, $P(w_i, v)$, so that $P(w_i, v) \cap N(w_j) = \emptyset$.*

Proof. We begin from w_i. Recall that there exists $q \in N(w_i)$, so that $q \notin N(w_j)$. We move from w_i to q. Also, recall that $N(w_j)$ contains at most one vertex in the partition $\mathcal{P}$, and that q is adjacent to exactly two vertices from $\mathcal{P}$. That is, we can extend from q to one of its neighbors in $\mathcal{P}$ without crossing into $N(w_j)$. Now we find ourselves again on the cycle C. If $N(w_j) \cap \mathcal{P} = \emptyset$, we travel either half of the cycle from p to v. Otherwise, $N(w_j) \cap \mathcal{P}$ consists of a single vertex, and we travel the half of the cycle connecting p and v which avoids this vertex. The union of the two selected paths provides a single path from w_i to v with the desired property.

▼

We have shown that $\mathcal{V} \cup \mathcal{W}$ satisfies the definition of an asteroidal set. Recall that $|\mathcal{V}| = n$ and $|\mathcal{W}| = \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$, and that $\mathcal{V} \cap \mathcal{W} = \emptyset$. Therefore we have exhibited an asteroidal set of size $n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$. ■

4.3 No Larger Asteroid is Representable Under the Restrictions of n and t

In the current section we show that the construction of a larger asteroidal set under the current restrictions of n and t is impossible. Note that showing this will prove the following theorem:

Theorem 4.3.1 *If G is a $(\mathcal{K}_{1,n}; t)$-representable graph then G is at most $\left(n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}\right)$-asteroidal.*

Proof. Firstly, observe that an asteroidal set must be an independent set. Furthermore, it is easily seen that at most $\sum_{k=1}^{n} \binom{n}{k}\binom{t-2}{k-1}$ vertices may be assigned subtrees which contain the branching node of $\mathcal{K}_{1,n}$ such that no two subtrees intersect in t or more nodes. Now, note that we have almost 'saturated' the branching node of the host tree with subtrees while constructing the asteroidal set of size $n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$.

Let $\mathcal{M}$ be the set of vertices which are assigned subtrees containing the branching node and extending non-trivially only on a single branch of $\mathcal{K}_{1,n}$. We need only to consider whether the vertices of $\mathcal{M}$ and $\mathcal{V}$ can coexist in the same asteroidal set.

Claim 4.3.2 *Let m_i be a vertex with assigned subtree, T_{m_i}, existing non-trivially only on branch i of $\mathcal{K}_{1,n}$ and which contains the branching node of $\mathcal{K}_{1,n}$. Then, any path from v_i to another asteroidal vertex a, must be adjacent to m_i.*

Proof. Note that by Lekkerkerker and Boland's result, a must have a non-trivial intersection with a branch of $\mathcal{K}_{1,n}$ other than branch i. If not, then $\{v_i, m_i, a\}$ is an asteroidal triple which is represented on an interval, a direct violation of their theorem.

We consider any path beginning at v_i and ending at a. Along this path there must be a vertex u_ℓ, the last vertex along the path whose assigned subtree does not contain the branching node of $\mathcal{K}_{1,n}$. Now, we must have that $T_{u_{\ell+1}}$ contains the branching node of the host tree, and also has an intersection of size at least t with T_{u_ℓ}. Therefore, either $T_{m_i} \subset T_{u_{\ell+1}}$ or $T_{u_{\ell+1}} \cap T_{u_\ell} \subset T_{m_i}$. Recalling that $|T_{m_i}| \geq t$ gives that either $|T_{m_i} \cap T_{u_{\ell+1}}| \geq t$ or $|T_{m_i} \cap T_{u_\ell}| \geq t$, inducing either the edge $u_{\ell+1}m_i$ or $u_\ell m_i$. That is, any path connecting v_i with another asteroidal vertex must be adjacent to the vertex m_i. ▼

The previous claim directly implies that the vertices m_i and v_i cannot be part of the same asteroidal set. A similar agument can be used to show the same result about v_i and a vertex whose subtree is a proper sub-path of the i^{th} branch of the host tree. This implies that per branch we may only have one vertex whose assigned subtree exists non-trivially only on that branch, no matter the configuration of subtrees. Note that this property is satisfied in the configuration constructed in section 2. ■

Combining the result from section 2 with this result, we have shown the main result from the introduction, and have rediscovered the surprising corollary originally seen by Eaton and Barbato:

Corollary 4.3.3 *Graphs containing arbitrarily large asteroidal sets are repre-*

sentable on $\mathcal{K}_{1,3}$ [2].

Since the size of the largest cycle which is $(\mathcal{K}_{1,3}; t)$-representable is eventually quadratic in t, and the size of the largest representable asteroidal set grows exponentially, the following observation can be made directly from combining the main result here with Eaton and Barbato's result on cycles.

Observation 4.3.4 *If an asteroidal configuration of size m is $(\mathcal{K}_{1,3}; t)$-representable, it is not neccessary that every asteroidal configuration of size m or smaller has such a representation. In fact, the size gap can be made arbitrarily large.*

The following fairly obvious observation can be made, however.

Observation 4.3.5 *If an asteroidal configuration of size m is $(\mathcal{K}_{1,n}; t)$-representable, then there exists an asteroidal configuration of size $m - k$ for each $k \leq m - 3$ which is also $(\mathcal{K}_{1,n}; t)$-representable.*

4.4 Open Problems

There are still many interesting open problems in the theory of tree representations, as well as some questions stemming from the main result in this paper. Answers to the following questions would be interesting.

Problem 4.4.1 *For small fixed values of n and t, which asteroidal configurations are $(\mathcal{K}_{1,n}; t)$-representable?*

We can already see that the answer is non-trivial. The cycle gives an example showing that even though some large asteroid may be representable, not all smaller asteroidal configurations can be represented. The number of m-asteroidal configurations grows quickly, so given the current tools, it seems that analyzing relatively small cases is in order.

Conjecture 4.4.2 *A graph G is $(\mathcal{K}_{1,n}; 1)$-representable iff G is chordal, at most n asteroidal, and G satisfies the condition that given any two pairs of vertices from asteroidal sets in G, any path connecting the first pair must be adjacent to any path connecting the second pair (Walter [9]).*

The argument for necessity in Walter's conjecture is straight forward. The argument towards sufficiency, however, seems to require surgical detail, and has not yet been resolved.

Problem 4.4.3 *Exactly which graphs are $(\mathcal{K}_{1,n}; t)$-representable?*

A complete characterization of the class of $(\mathcal{K}_{1,n}; t)$-representable graphs is the ultimate goal here. The fact that Walter's 1972 conjecture still remains undecided may be a good indicator that a full solution is still out of reach, however.

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CHAPTER 5

Tree Representations of Cycles

Representing Cycles on Subdivisions of Stars

by

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abstract. Consider a simple undirected graph $G = (V, E)$. A family of subtrees, $\{T_v\}_{v \in V}$, of a tree $\mathcal{T}$ is called a $(\mathcal{T}, t)$-representation of G provided $uv \in E$ if and only if $|T_u \cap T_v| \geq t$. In this paper we consider $(\mathcal{T}, t)$-representations for cycles, where $\mathcal{T}$ is a $K_{1,n}$-subdivision, denoted by $\mathcal{K}_{1,n}$. We show that the maximum length cycle which is $(\mathcal{K}_{1,4}; t)$-representable is at least on the order of t^3. Furthermore, we conjecture that the length of the longest cycle which is $(\mathcal{K}_{1,n}; t)$-representable is $O(t^{n-1})$.

5.1 Introduction

The study of graph representations is an active research area in graph theory. Given a graph $G = (V, E)$, a representation of G is the following collection of objects: (1) a set S, (2) a function $f : V \to \mathcal{P}(S)$, and (3) a function $g : f(V) \times f(V) \to \{0, 1\}$ so that $g(f(v_1), f(v_2)) = 1$ iff $(v_1, v_2) \in E$. We call S the *host set*, f the *assignment function*, and g the *conflict rule*. We say that a graph G is representable under a given host set S and conflict rule g if there exists a suitable assignment function f.

Much is known about graph representations when the conflict rule depends on intersection between assigned subsets. Intersection Representations have been well-studied by many authors, see [1] for a comprehensive list.

Certain substructures within a graph can make that graph difficult or impossible represent with certain host sets and conflict rules. One common structure which has been given as a forbidden subgraph for certain conflict rules is the cycle. That is, giving restrictions for n and t prohibits representability of graphs containing induced cycles with length above a certain threshold dependent on n and t.

5.2 Preliminaries

In this paper we consider tree representations of graphs. A tree representation of a graph is a representation where the host set is a given tree $\mathcal{T}$, subsets assigned to vertices of a representable graph G must be subtrees of $\mathcal{T}$, and the conflict rule is a threshold intersection rule, where two vertices of G are adjacent if and only if their assigned subtrees intersect in at least some prescribed tolerance t. We will study the cycles which are $(\mathcal{K}_{1,n}; t)$-representable.

At this point we recall the 2001 result by Eaton and Barbato [2] which describes all cycles which are $(\mathcal{K}_{1,3}; t)$-representable. Their paper shows that arbitrarily large cycles can be represented on $\mathcal{K}_{1,3}$ at the cost of increasing the tolerance. We explore the same structure, but generalize the host tree to $\mathcal{K}_{1,n}$.

We begin by taking generalizations of two preliminary lemmas from the paper of Eaton and Barbato.

Lemma 5.2.1 *In a $(\mathcal{K}_{1,n}; t)$-representation of C_p for $p > 3$, each vertex must be assigned a subtree such that the degree of the branching node, x, is at least 2.*

Proof. The proof is by induction on p. We first show the case for $p = 4$. Assume that for at least one of the assigned subtrees, $T_{v_1}, T_{v_2}, T_{v_3}, T_{v_4}$, the degree of x (the branching node of $\mathcal{T}$) is at most one. Without loss of generality we shall assume that it is T_{v_1}. Let $P_1, ..., P_n$ denote the branches of $\mathcal{T}$. Again, without loss of generality, assume that $T_{v_1} \subset P_1$. Now, we also know that T_{v_2} and T_{v_4} must each intersect T_{v_1} in at least t nodes, but may not intersect one another in t or more nodes. This again forces at least one of T_{v_2} or T_{v_4} to be a subset of P_1. Now, T_{v_3} must intersect T_{v_2} and T_{v_4} in at least t nodes each, but may not intersect T_{v_1} in t or more nodes. However, since $\mathcal{K}_{1,n}$ is a tree, there is a unique path between T_{v_2} and T_{v_4}, and T_{v_1} contains this path. Similarly, for T_{v_3} to intersect T_{v_1} and T_{v_4} in at least t nodes each, it must contain this path. Now, it is easily seen that

$|T_{v_1} \cap T_{v_3}| \geq t$. Thus, such a constructions of a C_4 is impossible.

We now consider $p > 4$. Assume that C_p has a representation on $\mathcal{T}$ such that in at least one of the assigned subtrees, T_{v_1}, $deg_{T_{v_1}}(x) < 2$. Assume $T_{v_1} \subset P_1$ and again, T_{v_p} and T_{v_2} must each intersect T_{v_1} in at least t nodes, but may not intersect eachother in t or more nodes. This forces $T_{v_2} \subset P_1$ as well. Let $T_{v^*} = T_{v_1} \cup T_{v_2}$. Note that T_{v^*} is also a subset of P_1. Furthermore, note that no new adjacencies are created. Indeed, if both $(T_{v_1} \setminus T_{v_2}) \cap T_{v_k} \neq \emptyset$ and $(T_{v_2} \setminus T_{v_1}) \cap T_{v_k} \neq \emptyset$, then $T_{v_1} \cap T_{v_2} \subset T_{v_k}$. However, since $|T_{v_1} \cap T_{v_2}| \geq t$, we know this cannot be the case because we would have v_k adjacent to both v_2 and v_1. Therefore, if $|T_{v^*} \cap T_{v_k}| \geq t$ then $|T_{v_k} \cap T_{v_1}| \geq t$ or $|T_{v_k} \cap T_{v_2}| \geq t$. Thus, the merging of the two subtrees creates no new adjacencies. Therefore, we have a representation of C_{p-1} with a subtree in which the branching node has degree less than 2. This gives that if C_p is representable, then C_{p-1} is also representable. It must be the case then that for no $p > 4$ does C_p have such a representation. Otherwise, we would ultimately arrive at a contradiction stating that C_4 has such a representation, which we proved was impossible. □

Lemma 5.2.2 *If C_p has a $(\mathcal{K}_{1,n}; t)$-representation, then there is a $(\mathcal{K}_{1,n}; t)$-representation of C_p so that the following conditions hold:*

1. $|T_u \cap T_v| = t$ *for all subtrees assigned to adjacent vertices u and v*

2. $|T_v| = t + 1$ *for all assigned subtrees*

Proof. Let p be the maximum value such that C_p is $(\mathcal{K}_{1,n}; t)$-representable. To prove 1), suppose that $|T_{v_1} \cap T_{v_2}| \geq t + 1$ and show that we can reduce $|T_{v_1} \cap T_{v_2}|$ and still have a $(\mathcal{K}_{1,n}; t)$-representation of C_p. We first remove any nodes of T_{v_2} which are not contained in T_{v_1} or T_{v_3} since they are extraneous. Note that this process may not disconnect T_{v_2} since the branching node, x, is contained in every assigned subtree. Let x_b^a denote the node on branch b, and which is of distance a from x, then for any $k \in [n]$ and $i \in \{1, 2, ..\}$ if for some $v \in C_p$, $x_k^i \notin T_v$, then for all $j > i$, we have $x_k^j \notin T_v$. Thus, any nodes removed from T_{v_2} must form disjoint subpaths of T_{v_2}, each containing a leaf of T_{v_2} and not containing x. Now, observe that there must be at least two nodes of $T_{v_1} \cap T_{v_2}$ which are not contained in T_{v_3} since otherwise $|T_{v_1} \cap T_{v_2}| \geq t+1$ would imply also that $|T_{v_1} \cap T_{v_3}| \geq t$. Without loss of generality, assume that at least one of these nodes is on branch one, and denote it x_1^i. Hence, $x_1^i \in (T_{v_1} \cap T_{v_2}) \setminus T_{v_3}$. Now, since x is a node in all of the representatives, we know that $x_1^j \notin T_{v_3}$ for any $j > i$. This means that there is a leaf of $T_{v_1} \cap T_{v_2}$, call it x_1^k, contained in $(T_{v_1} \cap T_{v_2}) \setminus T_{v_3}$. Observe that x_1^k is also a leaf in T_{v_2}, since we removed the nodes of T_{v_2} which were not contained in T_{v_1} or T_{v_3}. Now we remove x_1^k from T_{v_2}, thereby reducing $|T_{v_1} \cap T_{v_2}|$ by one, and we still have a $(K_{1,n}; t)$-representation for C_p as was claimed. To prove 2), suppose $|T_{v_1}| \geq t+2$ and we show that we can replace T_{v_1} with two subtrees, yielding a representation of a larger cycle. Firstly, we again remove the nodes of T_{v_1} which are not contained in T_{v_2} or T_{v_p} since they are extraneous. Once again, the process does not disconnect T_{v_1}. From 1) we have that $|T_{v_1} \cap T_{v_2}| = |T_{v_1} \cap T_{v_p}| = t$. There exists two vertices ℓ_1 and ℓ_2, such that ℓ_1 is a leaf of $T_{v_1} \cap T_{v_2}$ in $(T_{v_1} \cap T_{v_2}) \setminus T_{v_p}$ and ℓ_2 is a leaf in $\left(T_{v_1} \cap T_{v_p}\right) \setminus T_{v_2}$. Now we replace T_{v_1} with $T_{v_1} \setminus \ell_1$ and $T_{v_1} \setminus \ell_2$, which intersect eachother in at least t nodes, since $|T_{v_1}| \geq t+2$. This gives a $(\mathcal{K}_{1,n}; t)$-representation of C_{p+1}, a contradiction to the assumption that C_p was the longest cycle with such a representation. □

5.3 Developing Notation

We take $T_v = (x_1, x_2, ..., x_n)$ to describe the subtree extending out to the x_i^{th} node along branch i of the host tree. From this observation we can see that each subtree can be identified as a point in a finite subset of $\mathbb{N}^n$. With respect to the two lemmas cited in the previous section, we can see that we are interested in the subset consisting of points whose coordinate sum is exactly t. Note that there are exactly $\sum_{k=1}^{n} \binom{n}{k}\binom{t-2}{k-1}$ such points.

As a direct result of Lemmas 5.2.1 and 5.2.2, we obtain the following useful corollary.

Corollary 5.3.1 *Let X be the subset of $\mathbb{N}^n$ consisting only of points whose coordinate sum is t. All of the assigned subtrees of $\mathcal{K}_{1,n}$ used to represent any cycle of length at least four correspond to points in X.*

Since X is a subset of a vector space we will make use of the operations of scalar multiplication and vector addition where it makes notation convenient. We will use $\vec{e_i}$ to denote the i^{th} standard basis vector for $\mathbb{R}^n$.

We define edges on the pointset X as follows: $(\vec{x}, \vec{y}) \in E$ iff $\vec{x} = \vec{y} + \vec{e_i} - \vec{e_j}$ for $i, j \in [n]$ with $i \neq j$. We now consider the graph $\mathcal{H} = (X, E)$, in which we will search for the largest induced cycle.

Theorem 5.3.2 *The largest $(\mathcal{K}_{1,n}; t)$-representable cycle corresponds to the longest induced cycle of $\mathcal{H}$.*

Proof. This follows immediately by the construction of $\mathcal{H}$. □

5.3.1 The Structure of $\mathcal{H}$

We can now categorize the points of X according to certain common characteristics

Definition 5.3.3 *A point $\vec{x}$ of X is said to be* ***simplicial*** *if its closed neighborhood in $\mathcal{H}$ forms a complete subgraph of $\mathcal{H}$.*

Fact 5.3.4 *There are exactly n simplicial points of X. These points are of the form $t\vec{e_i}$ for $i \in [n]$.*

It is obvious that no simplicial point will be used as an assigned subtree in a representation of a cycle of length greater than three, since any two points adjacent to a simplicial point are also adjacent.

Definition 5.3.5 *A point of X possessing at least one zero coordinate is said to be a* ***boundary point****. A* ***type-k boundary point*** *is a boundary point having k zero entries.*

Boundary points are special since they do not have a 'full' neighborhood. There are $\sum_{k=1}^{n-1}\binom{n}{k}\binom{t-2}{k-1}$ boundary points in total; while there are $\binom{n}{k}\binom{t-2}{n-k-1}$ type-k boundary points for each k. Any type-k boundary point will have a neighborhood of size $2\binom{n-k}{2}+\binom{n-k}{1}\binom{k}{1}$.

Definition 5.3.6 *A point of X possessing no zero coordinates is said to be an* ***interior point****.*

Interior points have 'full neighborhoods'. That is, the neighborhood of an interior point contains $2\binom{n}{2}$ points.

Isolation Tubes

Note that there is a radius, r, from each originating point, $t\vec{e_i}$, such that, given two distinct originating points, $t\vec{e_i}$ and $t\vec{e_j}$, and distinct points x and y within a distance of r from $t\vec{e_i}$ and $t\vec{e_j}$, respectively, then x and y are not adjacent in $\mathcal{H}$.

Definition 5.3.7 *The* ***isolation tube*** *corresponding to the point $t\vec{e_i}$ consists of all the nodes which are of distance no more than $\lfloor \frac{t}{2} \rfloor - 1$ from $t\vec{e_i}$.*

Note 5.3.8 *We will refer to points of the form $(t-k)\vec{e_i}+\Omega_k$ as being on the k^{th}* ***level*** *of the isolation tube corresponding to $\vec{e_i}$, where Ω_k denotes some linear combination of the remaining basis vectors, the sum of whose coefficients is k.*

Theorem 5.3.9 *If $\vec{x}$ and $\vec{y}$ are points from distinct isolation tubes, then they are not adjacent in $\mathcal{H}$. That is, points from different isolation tubes are isolated from one another.*

Proof. Consider, t is even. Let $\vec{x} = (\frac{t}{2}+1)\vec{e_i} + \Omega_{\frac{t}{2}-1}$ and $\vec{y} = (\frac{t}{2}+1)\vec{e_j} + \Omega'_{\frac{t}{2}-1}$. Recalling the rules for adjacency in $\mathcal{H}$, we see that if there is any chance for $\vec{x}$ and $\vec{y}$ to be adjacent, then $\Omega_{\frac{t}{2}-1} = (\frac{t}{2}-1)\vec{e_j}$ and $\Omega'_{\frac{t}{2}-1} = (\frac{t}{2}-1)\vec{e_i}$. Thus, we have

$$\vec{x} = (\frac{t}{2}+1)\vec{e_i} + (\frac{t}{2}-1)\vec{e_j}$$
$$\vec{y} = (\frac{t}{2}-1)\vec{e_i} + (\frac{t}{2}+1)\vec{e_j}$$

Now, it is easily seen that $\vec{x}$ and $\vec{y}$ differ by 2 in two coordinates. Since adjacency requires exactly two coordinates which differ by 1, and all other coordinates equal, we have shown that there can be no edge connecting $\vec{x}$ and $\vec{y}$.

We now consider the case when t is odd:

$$\begin{aligned}\vec{x} &= \left(\lceil\frac{t}{2}\rceil + 1\right)\vec{e_i} + \Omega_{\lfloor\frac{t}{2}\rfloor-1}\\ &= \left(\frac{t+1}{2} + 1\right)\vec{e_i} + \Omega_{\lfloor\frac{t}{2}\rfloor-1}\\ &= \left(\frac{t+3}{2}\right)\vec{e_i} + \Omega_{\lfloor\frac{t}{2}\rfloor-1}\end{aligned}$$

and,

$$\begin{aligned}\vec{y} &= \left(\lceil\frac{t}{2}\rceil + 1\right)\vec{e_j} + \Omega'_{\lfloor\frac{t}{2}\rfloor-1}\\ &= \left(\frac{t+1}{2} + 1\right)\vec{e_j} + \Omega'_{\lfloor\frac{t}{2}\rfloor-1}\\ &= \left(\frac{t+3}{2}\right)\vec{e_j} + \Omega'_{\lfloor\frac{t}{2}\rfloor-1}\end{aligned}$$

Again, if we hope for $\vec{x}$ and $\vec{y}$ to be adjacent we must have that $\Omega_{\lfloor\frac{t}{2}\rfloor-1} = \left(\lfloor\frac{t}{2}\rfloor - 1\right)\vec{e_j}$ and $\Omega'_{\lfloor\frac{t}{2}\rfloor-1} = \left(\lfloor\frac{t}{2}\rfloor - 1\right)\vec{e_i}$. Thus,

$$\vec{x} = \left(\frac{t+3}{2}\right)\vec{e_i} + \left(\frac{t-3}{2}\right)\vec{e_j} = \left(\frac{t-1}{2} + 2\right)\vec{e_i} + \left(\frac{t-1}{2} - 1\right)\vec{e_j}$$

$$\vec{y} = \left(\frac{t-3}{2}\right)\vec{e_i} + \left(\frac{t+3}{2}\right)\vec{e_j} = \left(\frac{t-1}{2} - 1\right)\vec{e_i} + \left(\frac{t-1}{2} + 2\right)\vec{e_j}$$

Again, the difference in two of the non-zero coordinates is too large, and adjacency is not achieved. □

Thus, we have shown that the isolation tubes truly do isolate the points of $\mathcal{H}$. Furthermore, it is easily seen that increasing the distance from the originating point beyond this threshold will invite adjacencies. These observations result in the fact that the radius of each isolation tube truly is $\lfloor \frac{t}{2} \rfloor - 1$.

5.4 Finding the Largest Cycle in $\mathcal{H}$

Using what we have previously discussed about the isolation tubes, a good place to begin is by constructing the longest induced path within an isolation tube. Note that since all isolation tubes are isomorphic, if we are successful, we can obtain n such longest paths which can be joined to make a cycle, giving us a non-trivial lower bound on the number we search for.

5.4.1 More About the Isolation Tubes

Before we can find a longest path through an isolation tube, we must gain some more information about its 'geography'. Noting that the radius of the isolation tubes is $\lfloor \frac{t}{2} \rfloor - 1$ will allow us to have our discussion without using cases for t even and t odd. Also, since the isolation tubes are symmetric, we will consider the tube corresponding to $t\vec{e_1}$ as the representative case, and refer to $t\vec{e_1}$ as the originating point.

Claim 5.4.1 *Note that for $k \leq \lfloor \frac{t}{2} \rfloor - 1$, there are $\binom{k+n-2}{n-2}$ points at level k of any isolation tube. Furthermore, there are $\displaystyle\sum_{i=1}^{\lfloor \frac{t}{2} \rfloor - 1} \binom{i+n-2}{n-2}$ points total contained within any isolation tube.*

Proof. Note that points at distance k from $t\vec{e_1}$ must be of the form $\vec{x} = (t - k)\vec{e_1} + \Omega_k$. Since the sum of the coefficients must be exactly t, we know that there are $\binom{k+n-2}{n-2}$ ways to assign coefficients to the remaining $n-1$ basis vectors, since the first coefficient is fixed at $(t-k)$. Thus, there are $\binom{k+n-2}{n-2}$ points at level k of each isolation tube.

Clearly the levels partition the isolation tube, so summing over all the levels gives the total number of points in the isolation tube, as desired. □

Now that we know the total number of points within each isolation tube, it would be beneficial to know more about the neighborhood of each point. That is, how many neighbors does each point have, and exactly where are these neighbors?

Definition 5.4.2 *For a point $\vec{x}$ at level k of an isolation tube, we define the* ***back neighbors*** *of $\vec{x}$ to be neighbors of $\vec{x}$ at level $(k-1)$ of the isolation tube, the* ***cross neighbors*** *to be neighbors at level k, and the* ***forward neighbors*** *to be neighbors at level $k+1$.*

Claim 5.4.3 *Consider a point, $\vec{x}$, at level k, $(k \geq 1)$, of an isolation tube. The following must be true of $\vec{x}$,*

1. *$\vec{x}$ is a type-$(n-j)$ boundary point for some $j \in \{2, 3, ..., \min\{k, n\}\}$.*
2. *$\vec{x}$ has $(j-1)$ back neighbors, all occurring at the $(k-1)^{st}$ level.*
3. *$\vec{x}$ has $(j-1)(n-2)$ cross neighbors.*
4. *$\vec{x}$ has $n-1$ forward neighbors, all occuring at the $(k+1)^{st}$ level.*
5. *$\vec{x}$ has $jn - j$ total neighbors.*

Proof.

1. Since $\vec{x}$ is at level k of the isolation tube, we must have $\vec{x} = (t-k)\vec{e_1} + \Omega_k$. Again, since the coefficients must sum to t, we know that the sum of the remaining coefficients is k. Since k is at least 1, we must have that at least one of the other basis vectors has a non-zero coefficient, implying that $\vec{x}$ is at most a type-$(n-2)$ boundary point. We may have up to $\min\{k, (n-1)\}$ of the remaining coefficients being non-zero. Thus, $\vec{x}$ must be as claimed. ▼

2. By the way the edges of $\mathcal{H}$ are defined, it is clear that any back neighbors of $\vec{x}$ must be at the $(k-1)^{st}$ level of the isolation tube. By 1 we may assume that $\vec{x}$ is a type-$(n-j)$ boundary point. To move from the k^{th} level of the isolation tube to the $(k-1)^{st}$ level we must increase the coefficient of $\vec{e_1}$ by one. Thus, we may choose to decrease the coefficient of any of the $(j-1)$ other vectors with non-zero coefficients in order to make increasing the coefficient of $\vec{e_1}$ possible. That is, we have $(j-1)$ choices for back neighbor. ▼

3. Similar to 2, in order to move to a neighbor while remaining on level k, we must leave the coefficient of $\vec{e_1}$ the same, but decrease the coefficient of one of the vectors with a non-zero coefficient by one and increase the coefficient on any of the other $n-2$ vectors by one. Thus, we have $(j-1)$ other vectors with non-zero coefficients to choose from, and then $(n-2)$ viable vectors whose coefficient we wish to increase by one. By the fundamental principle of counting, this gives $(j-1)(n-2)$ cross neighbors. ▼

4. A forward neighbor must occur at level $(k+1)$ since it is impossible for a point to have a neighbor further than one level away. To move from level k to level $(k+1)$ we must decrease the coefficient of $\vec{e_1}$ by one and increase the coefficient of any other vector by one. There are certainly $(n-1)$ possibilities for this, as claimed. ▼

5. Summing over all of the back, cross, and forward neighbors gives $jn - j$ total neighbors, as desired. ▼

□

5.5 Construction of a Cycle for $n = 4$

In this section we construct a cycle in the case where $n = 4$, that is, we construct a long cycle having a $(\mathcal{K}_{1,4}; t)$-representation. As discussed in the previous section, our goal will be to construct long paths through the isolation tubes and then to connect the paths together in such a way that a single cycle is formed.

Consider the isolation tube corresponding to $\vec{e_1}$. We begin by selecting two points from level 1 of the isolation tube, namely $(t-1)\vec{e_1} + \vec{e_2}$ and $(t-1)\vec{e_1} + \vec{e_3}$. Proceed by selecting all points of the form $(t-k)\vec{e_1} + k\vec{e_2}$, for $k \leq \lfloor \frac{t}{2} \rfloor - 1$. Note that these points are along an induced path extending through all levels of the isolation tube. Furthermore, the selection of these points eliminates only points of the form $(t-k)\vec{e_1} + (k-1)\vec{e_2} + \vec{e_j}$ for $j \in \{3, 4\}$ from consideration as points along our long path. This is certainly a minimal elimination, as can be seen from the discussion about neighborhoods in the previous section.

We must now decide how to extend the path from $(t-1)\vec{e_1} + \vec{e_3}$ through the isolation tube, to level $\lfloor \frac{t}{2} \rfloor - 1$. Certainly we could take the same approach as we did with the other end of the path, but it is clear that for large enough t, the resulting path would not be of maximum length.

5.5.1 Methods for Choosing Points

Before we begin constructing the desired cycle, we define a few new terms and processes.

Definition 5.5.1 *Recall that a point of the form $(t-k)\vec{e_1} + \Omega_k$ is said to be at level k of the isolation tube. We will say that a point of the form $(t-k)\vec{e_1} + j\vec{e_2} + \Omega_{k-j}$*

*is on the j^{th} **stage** of the k^{th} level of the isolation tube.*

We define the following 'moves' as strategies for choosing points at certain levels throughout the isolation tube. It is best to think of these moves in terms of how they affect the coefficient of $\vec{e_2}$.

Definition 5.5.2 *An **up-one move** will be a move from the point $(t-k)\vec{e_1} + \Omega_k$ to $(t-(k+1))\vec{e_1} + \vec{e_2} + \Omega_k$. That is, such a move will jump down a level in the isolation tube to a point where the coefficient of $\vec{e_2}$ has been increased by one. Up-one moves will occur at odd levels of an isolation tube.*

Definition 5.5.3 *An **up-all move** will be series of moves contained on a single level, collecting a maximal number of points of the form $(t-k)\vec{e_1} + i\vec{e_2} + j\vec{e_3} + (k-i-j)\vec{e_4}$, for $i \in \{2, ..., k-2\}$, such that these points are along an induced path of $\mathcal{H}$. An up-all move will only occur on an even level of an isolation tube.*

Definition 5.5.4 *A **down-all move** will be a series of moves contained on a single level, collecting a maximal number of points of the form $(t-k)\vec{e_1} + i\vec{e_2} + j\vec{e_3} + (k-i-j)\vec{e_4}$, for $i \in \{0, 1, ..., k-2\}$, such that these points are along an induced path of $\mathcal{H}$. A down-all move will only occur at an even level of an isolation tube.*

Definition 5.5.5 *A **down-one move** will be a move from the point $(t-k)\vec{e_1} + j\vec{e_2} + \Omega_{k-j}$ to $(t-k)\vec{e_1} + (j-1)\vec{e_2} + \Omega_{k-j-1}$. That is, such a move remains on the same level, but will jump down a stage in the isolation tube. Down-one moves will occur at odd stages within a down-all move.*

5.5.2 How Many Points are Chosen?

At each level, other than levels one and two of the isolation tubes, we will perform exactly one of the moves defined in the previous section. We start by statically defining the points taken at levels one and two. These points are $(t-1)\vec{e_1}+\vec{e_3}$ at level one, and $(t-2)\vec{e_1}+\vec{e_3}+\vec{e_4}$ as well as $(t-2)\vec{e_1}+2\vec{e_4}$ at level two. We continue by repeating the four step process: *up-one*, *down-all*, *up-one*, *up-all*. We provide the following claims and observations about the number of points taken when performing each move.

Observation 5.5.6 *An up-one move takes a single point at the level which the move was performed.*

Definition 5.5.7 *We call a point* ***heavy*** *in a coordinate corresponding to $\vec{e_p}$ if that point is the leaf of a path, the coefficient of $\vec{e_p}$ does not determine a level or stage of an isolation tube, and the coefficient of $\vec{e_p}$ is the only non-zero coefficient other than those on the basis vectors determining level and stage.*

Claim 5.5.8 *An up-all move at level k takes $\frac{k^2}{4}-1$ points. Furthermore, if an up-all move were to begin from $(t-k)\vec{e_1}+2\vec{e_2}+(k-2)\vec{e_p}$ will end at the point $(t-k)\vec{e_1}+(k-2)\vec{e_2}+2\vec{e_p}$, for $p\in\{3,4\}$. That is, an up-all move will maintain 'heavy' coordinates.*

Proof. Let an up-all move occur at level k. Any up-all move must occur after an up-one move at the previous level. That is, the first point taken at level k will be of the form $(t-k)\vec{e_1}+2\vec{e_2}+(k-2)\vec{e_p}$, for $p\in\{3,4\}$.

Note that all odd stages will contribute only one point, think of this as a miniature up-one move. All even stages (let $j=2\ell$), except the final stage will contribute $\binom{k-2\ell+1}{2-1}-1=k-2\ell$ points for $\ell\in\{1,2,...,\frac{k-2}{2}-1\}$. The final stage will contribute $\binom{2+1}{2-1}=3$ points.

Note that it is impossible to take $k-2\ell+1$ points at all of the even levels since we would have two points adjacent to the point obtained from the miniature up-one move to the next stage. Indeed, the points $(t-k)\vec{e_1}+2\ell\vec{e_2}+\vec{e_q}+(k-2\ell-1)\vec{e_p}$

and $(t-k)\vec{e_1} + 2\ell\vec{e_2} + (k-2\ell)\vec{e_p}$ for $p, q \in \{3,4\}$ with p and q distinct, are both adjacent to $(t-k)\vec{e_1} + (2\ell+1)\vec{e_2} + (k-2\ell-1)\vec{e_p}$ in $\mathcal{H}$.

Note also that the selected points constitute an induced path in $\mathcal{H}$. Certainly there are only two points at each even stage which have a neighbor in the previous or next stage. Also, no selected point at an even stage has three or more neighbors which were also selected.

Consider neighbors of the points selected at an odd stage of level k. First we consider neighbors at the previous stage. Note that the point selected at stage $j-1$ (where $j-1$ is odd) must have been of the form $(t-k)\vec{e_1}+(j-1)\vec{e_2}+(k-(j-1))\vec{e_p}$ for $p \in \{3,4\}$. The back neighbors of this point at stage $j-2$ are of the form $(t-k)\vec{e_1}+(j-2)\vec{e_2}+(k-(j-2))\vec{e_p}$ and $(t-k)\vec{e_1}+(j-2)\vec{e_2}+(k-(j-3))\vec{e_p}+\vec{e_q}$ for $q \in \{3,4\}\backslash\{p\}$. However, the point $(t-k)\vec{e_1}+(j-2)\vec{e_2}+(k-(j-2))\vec{e_p}$ was not selected at stage $j-2$. Therefore our point has a unique selected neighbor at the previous stage. Now, consider the forward neighbors of $(t-k)\vec{e_1}+(j-1)\vec{e_2}+(k-(j-1))\vec{e_p}$. Clearly there can only be one neighbor, since the coefficient of $\vec{e_1}$ must remain constant, and the coefficient of $\vec{e_2}$ must be increased by one. The unique neighbor is $(t-k)\vec{e_1} + j\vec{e_2} + (k-j)\vec{e_p}$.

Consider two points at stage j of level k

$$\vec{x} = (t-k)\vec{e_1} + j\vec{e_2} + a\vec{e_3} + (k-(j+a))\vec{e_4}$$

$$\vec{y} = (t-k)\vec{e_1} + j\vec{e_2} + b\vec{e_3} + (k-(j+b))\vec{e_4}$$

such that $a, b \in \{1, 2, ..., k-j-1\}$. By the rules for adjacency in $\mathcal{H}$, $\vec{x}$ and $\vec{y}$ are adjacent iff $a = b+1$ or $a = b-1$. Note that both of these points were selected as part of our path, and the degree of each such point is two, as desired.

It is clear that the point $(t-k)\vec{e_1} + j\vec{e_2} + (k-j)\vec{e_p}$ can only have a single neighbor on stage j, that point is $(t-k)\vec{e_1} + j\vec{e_2} + (k-(j-1))\vec{e_p} + \vec{e_q}$. Thus, the degree of $(t-k)\vec{e_1} + j\vec{e_2} + (k-j)\vec{e_p}$ is 2 in the graph induced on the selected points. Also, since the point $(t-k)\vec{e_1} + j\vec{e_2} + (k-j)\vec{e_q}$ is not selected in stage j, the degree of $(t-k)\vec{e_1} + j\vec{e_2} + \vec{e_p} + (k-j-1)\vec{e_q}$ in the graph induced on the selected points must also be two.

Now that we have verified that an up-all move selects points forming an induced path, calculating the total points taken by summing the number of points taken at each stage, we obtain the desired result.

There is exactly one point taken at each odd stage other than 1 and $k-1$. Thus, contributing $\frac{k}{2}-2$ points. The even stages account for a total of

$$\begin{aligned} 3 + \sum_{i=2}^{\frac{k}{2}-1} 2i &= 3 + 2\left(\frac{\left(\frac{k}{2}-1\right)\left(\frac{k}{2}\right)}{2} - 1\right) \\ &= \left(\frac{k}{2}\right)^2 - \frac{k}{2} - 2 + 3 \\ &= \left(\frac{k}{2}\right)^2 - \frac{k}{2} + 1 \end{aligned}$$

points.

Adding the number of points from even stages and odd stages gives $\left(\frac{k^2}{4} - \frac{k}{2} + 1\right) + \left(\frac{k}{2} - 2\right) = \frac{k^2}{4} - 1$ total points, as claimed.

Furthermore, note that an up-all move will only occur at a level k which is equivalent to $2 \mod 4$. This means that $k-2 \equiv 0 \mod 4$. From this observation we can see that if an up-all move were to begin from $(t-k)\vec{e_1} + 2\vec{e_2} + (k-2)\vec{e_3}$, then such a move would end at the point $(t-k)\vec{e_1} + (k-2)\vec{e_2} + 2\vec{e_3}$. Note that this must be, since if $j \equiv 2 \mod 4$, we move from a 'heavy' $\vec{e_3}$ to a 'heavy' $\vec{e_4}$ through stage j, whereas if $j \equiv 0 \mod 4$, we move from a 'heavy' $\vec{e_4}$ back to a 'heavy' $\vec{e_3}$. Since $k-2 \equiv 0 \mod 4$, we must have that we end at $(t-k)\vec{e_1} + (k-2)\vec{e_2} + 2\vec{e_3}$. That is, an up-all move maintains the 'heavy' coordinate. ▼

Claim 5.5.9 *A down-all move will take $\frac{k^2}{4} + k$ total points. Furthermore, if a down-all move were to begin from $(t-k)\vec{e_1} + (k-2)\vec{e_2} + 2\vec{e_p}$, it will end at the* *point* $(t-k)\vec{e_1}+k\vec{e_q}$*, for* $p,q\in\{3,4\}$*. That is, a down-all move will maintain the 'heavy' coordinate.*

Proof. Let a down-all move occur at level k. Any down-all move must occur after an up-one move at the previous level. That is, the first point taken at level k will be of the form $(t-k)\vec{e_1}+(k-2)\vec{e_2}+2\vec{e_p}$, for $p\in\{3,4\}$.

Similarly to the up-all move, any even stages, except the first, will contribute $\binom{k-2\ell+1}{2-1}-1=k-2\ell$ points. The first stage will contribute 3 points. All of the odd stages, except for $k-1$ will contribute a single point.

Again, it is impossible to take $k-2\ell+1$ points at all of the even levels since we would have two points adjacent to the point obtained from the miniature down-one move from the previous stage. Indeed, the points $(t-k)\vec{e_1}+2\ell\vec{e_2}+\vec{e_p}+(k-2\ell-1)\vec{e_q}$ and $(t-k)\vec{e_1}+2\ell\vec{e_2}+(k-2\ell)\vec{e_q}$ for $i,j\in\{3,4\}$ with p and q distinct, are both adjacent to $(t-k)\vec{e_1}+(2\ell+1)\vec{e_2}+(k-2\ell-1)\vec{e_q}$ in $\mathcal{H}$.

Note also that the selected points constitute an induced path in $\mathcal{H}$. The verification of this fact is made in the same manner as it was in the case of the up-all move and will be omitted.

Now, calculating the total number of points taken, by summing the number of points taken at each stage, we obtain the desired result.

Since there is exactly one point taken at each odd stage other than $k-1$, the odd stages contribute $\frac{k}{2}-1$ points. The even stages account for a total of

$$\begin{aligned}3+\sum_{i=2}^{\frac{k}{2}}2i &= 3+2\left(\frac{\left(\frac{k}{2}\right)\left(\frac{k}{2}+1\right)}{2}-1\right)\\ &= \left(\frac{k}{2}\right)^2+\frac{k}{2}-2+3\\ &= \left(\frac{k}{2}\right)^2+\frac{k}{2}+1\end{aligned}$$

points.

Adding the number of points from even stages and odd stages gives $\left(\frac{k^2}{4}+\frac{k}{2}+1\right)+\left(\frac{k}{2}-1\right)=\frac{k^2}{4}+k$ total points, as claimed.

Furthermore, note that a down-all move will only occur at a level $k \equiv 0 \mod 4$. This means that $k-2 \equiv 2 \mod 4$. From this observation we can see that if a down-all move were to begin from $(t-k)\vec{e_1}+(k-2)\vec{e_2}+2\vec{e_3}$, then such a move would end at the point $(t-k)\vec{e_1}+k\vec{e_3}$. Note that this must be, since if $j \equiv 2 \mod 4$, we move from a 'heavy' $\vec{e_3}$ to a 'heavy' $\vec{e_4}$ through stage j, whereas if $j \equiv 0 \mod 4$, we move from a 'heavy' $\vec{e_4}$ back to a 'heavy' $\vec{e_3}$. Since $0 \equiv 0 \mod 4$, we must have that we end at $(t-k)\vec{e_1}+k\vec{e_3}$. That is, a down-all move maintains the 'heavy' coordinate. ▼

5.5.3 Connecting the Paths Through An Entire Tube

We have identified a sequence of paths of varying lengths at each level of the isolation tube. Here we show that the collection of all of the corresponding points induces a single path through all of the levels of the isolation tube.

Recall that we began from level one by selecting the single point $(t-1)\vec{e_1}+\vec{e_3}$. Recall that, on the second level we chose the points $(t-2)\vec{e_1}+\vec{e_3}+\vec{e_4}$ and $(t-2)\vec{e_1}+2\vec{e_4}$; at level three, we perform an up-one move; at level four, a down-all move; at level five another up-one move; and at level six an up-all move. We repeat this pattern, performing an up-one move at level k if k is odd, a down-all move if $k \equiv 0 \mod 4$, and an up-all move if $k \equiv 2 \mod 4$.

We can count the number of points selected by considering cases based on the value of $\left(\lfloor \frac{t}{2} \rfloor - 1\right) \mod 4$.

Case 1: $(\lfloor \frac{t}{2} \rfloor - 1 \equiv 2 \mod 4)$ and $t \geq 14$

Since $\lfloor \frac{t}{2} \rfloor - 1 \equiv 2 \mod 4$, we end with an up-all move.

This gives that there are $\frac{\lfloor \frac{t}{2} \rfloor - 3}{2}+1$ total up-one moves, and therefore the same number of corresponding points. Note that $\frac{\lfloor \frac{t}{2} \rfloor - 3}{4}$ of the levels will correspond to down-all moves. This contributes

$$\sum_{j=1}^{\frac{\lfloor \frac{t}{2} \rfloor - 3}{4}} \frac{k^2}{4} + k; \qquad \text{where } k = 4j$$

$$\sum_{j=1}^{\frac{\lfloor \frac{t}{2} \rfloor - 3}{4}} \frac{(4j)^2}{4} + 4j$$

$$= \frac{1}{48} \left(\lfloor \frac{t}{2} \rfloor - 3 \right) \left(\lfloor \frac{t}{2} \rfloor + 1 \right) \left(\lfloor \frac{t}{2} \rfloor + 5 \right)$$

points.

Now, there will be the same number of levels corresponding to an up-all move as there are levels corresponding to down-all moves. Therefore, the up-all moves contribute

$$\sum_{j=1}^{\frac{\lfloor \frac{t}{2} \rfloor - 3}{4}} \left(\frac{k^2}{4} - 1 \right); \qquad \text{where } k = 4j + 2$$

$$\sum_{j=1}^{\frac{\lfloor \frac{t}{2} \rfloor - 3}{4}} \frac{(4j+2)^2}{4} - 1$$

$$= \frac{1}{48} \left(\lfloor \frac{t}{2} \rfloor - 3 \right) \left(\lfloor \frac{t}{2} \rfloor + 1 \right) \left(\lfloor \frac{t}{2} \rfloor + 5 \right)$$

points

Notice that we accounted for one fewer level than we were supposed to. This is because we did not categorize the points taken at level two as resulting from an up-all or down-all move. Remember that we selected 2 points at level two. Adding the length of the extension to the length of the original path straight down one side of the tube gives that total number of points on the path we constructed is

$$\left(\lfloor\frac{t}{2}\rfloor-1\right)+\left(\frac{\lfloor\frac{t}{2}\rfloor-3}{2}+1\right)+2\cdot\frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor-3\right)\left(\lfloor\frac{t}{2}\rfloor+1\right)\left(\lfloor\frac{t}{2}\rfloor+5\right)+2 \qquad (Q_1)$$

Note, however, that use of this formula required $t \geq 14$, since this is the smallest value for t such that $\lfloor\frac{t}{2}\rfloor - 1 \equiv 2 \mod 4$ and $\lfloor\frac{t}{2}\rfloor - 1 \geq 6$, the first level at which an up-all move occurs.

Case 2: $(\lfloor\frac{t}{2}\rfloor - 1 \equiv 3 \mod 4)$ and $t \geq 16$

Since $\lfloor\frac{t}{2}\rfloor - 1 \equiv 3 \mod 4$, we end with an up-one move.

This gives that there are $\frac{\lfloor\frac{t}{2}\rfloor-4}{2} + 2$ total up-one moves, and therefore the same number of corresponding points. Note that $\frac{\lfloor\frac{t}{2}\rfloor-4}{4} + 1$ of the levels will correspond to down-all moves. This contributes

$$\sum_{j=1}^{\frac{\lfloor\frac{t}{2}\rfloor-4}{4}} \frac{(4j)^2}{4} + 4j$$

$$= \frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor\right)\left(\lfloor\frac{t}{2}\rfloor - 4\right)\left(\lfloor\frac{t}{2}\rfloor + 4\right)$$

points.

Now, there will be the same number of levels corresponding to an up-all move as there are levels corresponding to down-all moves. Therefore, the up-all moves contribute

$$\sum_{j=1}^{\frac{\lfloor\frac{t}{2}\rfloor-4}{4}} \frac{(4j+2)^2}{4} - 1$$

$$= \frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor\right)\left(\lfloor\frac{t}{2}\rfloor - 4\right)\left(\lfloor\frac{t}{2}\rfloor + 4\right)$$

points.

Again, we have not accounted for level two, which contributes 2 points. Adding the length of the extension to the length of the original path straight down one side of the tube gives that total number of points on the path we constructed is

$$\left(\lfloor\frac{t}{2}\rfloor - 1\right) + \left(\frac{\lfloor\frac{t}{2}\rfloor - 4}{2} + 2\right) + 2 \cdot \frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor\right)\left(\lfloor\frac{t}{2}\rfloor - 4\right)\left(\lfloor\frac{t}{2}\rfloor + 4\right) + 2 \qquad (Q_2)$$

Case 3: $(\lfloor\frac{t}{2}\rfloor - 1 \equiv 0 \mod 4)$ and $t \geq 18$

Since $\lfloor\frac{t}{2}\rfloor - 1 \equiv 0 \mod 4$, then the last move made inside of the isolation tube is a down-all. Note that one more down-all than up-all move has been made.

In this case, exactly half of the moves made are up-one moves, contributing $\frac{\lfloor\frac{t}{2}\rfloor-1}{2}$ points. One fourth of these moves are down-all moves. The down-all moves must contribute

$$= \sum_{j=1}^{\frac{\lfloor\frac{t}{2}\rfloor-1}{4}} \left(\frac{(4j)^2}{4} + (4j)\right)$$

$$= \frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor - 1\right)\left(\lfloor\frac{t}{2}\rfloor + 3\right)\left(\lfloor\frac{t}{2}\rfloor + 7\right)$$

points. Finally, one less than one fourth of these moves are up-all moves. These up-all moves contribute

$$= \sum_{j=1}^{\frac{\lfloor\frac{t}{2}\rfloor-1}{4}-1} \left(\frac{(4j+2)^2}{4} - 1\right)$$

$$= \frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor - 5\right)\left(\lfloor\frac{t}{2}\rfloor - 1\right)\left(\lfloor\frac{t}{2}\rfloor + 3\right)$$

points.

Again, we have not accounted for level two, which contributes 2 points. Adding the length of the extension to the length of the original path straight down one side of the tube gives that total number of points on the path we constructed is

$$\left(\lfloor\frac{t}{2}\rfloor - 1\right) + \frac{3(\lfloor\frac{t}{2}\rfloor - 1)}{2} + 2 \cdot \frac{1}{48}\left(\lfloor\frac{t}{2}\rfloor - 1\right)\left(\lfloor\frac{t}{2}\rfloor + 3\right)\left(\lfloor\frac{t}{2}\rfloor + 1\right) + 2 \qquad (Q_3)$$

Case 4: $(\lfloor \frac{t}{2} \rfloor - 1 \equiv 1(\mod 4))$ and $t \geq 20$

Since $\lfloor \frac{t}{2} \rfloor - 1 \equiv 1(\mod 4)$, we end with an up-one move. This gives that there are $\frac{\lfloor \frac{t}{2} \rfloor - 2}{2} + 1$ total up-one moves, and therefore the same number of corresponding points. Note that $\frac{\lfloor \frac{t}{2} \rfloor - 2}{4}$ of the levels will correspond to down-all moves. This contributes

$$\sum_{j=1}^{\frac{\lfloor \frac{t}{2} \rfloor - 2}{4}} \frac{(4j)^2}{4} + 4j$$

$$= \frac{1}{48}\left(\lfloor \frac{t}{2} \rfloor - 2\right)\left(\lfloor \frac{t}{2} \rfloor + 2\right)\left(\lfloor \frac{t}{2} \rfloor + 6\right)$$

points.

Now, there will still be one fewer level corresponding to an up-all move than there were levels corresponding to down-alls. Therefore, the up-all's contribute

$$\sum_{j=1}^{\frac{\lfloor \frac{t}{2} \rfloor - 2}{4} - 1} \frac{(4j+2)^2}{4} - 1$$

$$= \frac{1}{48}\left(\lfloor \frac{t}{2} \rfloor - 6\right)\left(\lfloor \frac{t}{2} \rfloor - 2\right)\left(\lfloor \frac{t}{2} \rfloor + 2\right)$$

Again, we have not accounted for level two, which contributes 2 points. Adding the length of the extension to the length of the original path straight down one side of the tube gives that total number of points on the path we constructed is

$$\left(\lfloor \frac{t}{2} \rfloor - 1\right) + \left(\frac{\lfloor \frac{t}{2} \rfloor - 2}{2} + 1\right) + 2 \cdot \frac{1}{48}\left(\lfloor \frac{t}{2} \rfloor\right)\left(\lfloor \frac{t}{2} \rfloor - 2\right)\left(\lfloor \frac{t}{2} \rfloor + 2\right) + 2 \qquad (Q_4)$$

5.5.4 The Leaves of the Path

In order to ultimately connect the paths through the multiple isolation tubes, we will need to know what the leaves of each of the paths are. Pairing what was said previously, in Claims 5.5.8 and 5.5.9 about up-all and down-all moves, with the knowledge that after selecting points at level two we ended at a point which was heavy in $\vec{e_4}$, we can make the following observation.

Observation 5.5.10 *The extension path ends in a point with a heavy $\vec{e_4}$.*

This observation gives the following theorem:

Theorem 5.5.11 *The endpoints of the path constructed through the isolation tube corresponding to $\vec{e_1}$ are $\left(t - \left(\lfloor \frac{t}{2} \rfloor - 1\right)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2}$ and*

i. If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 0 \mod 4$ *the remaining leaf is* $\left(t - \left(\lfloor \frac{t}{2} \rfloor - 1\right)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_4}$

ii. If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 1 \mod 4$ *the remaining leaf is* $\left(t - \left(\lfloor \frac{t}{2} \rfloor - 1\right)\right) \vec{e_1} + \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 2\right) \vec{e_4}$

iii. If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 2 \mod 4$ *the remaining leaf is* $\left(t - \left(\lfloor \frac{t}{2} \rfloor - 1\right)\right) \vec{e_1} + \left(\left(\lfloor \frac{t}{2} \rfloor - 1\right) - 2\right) \vec{e_2} + 2\vec{e_4}$

iv. If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 3 \mod 4$ *the remaining leaf is* $\left(t - \left(\lfloor \frac{t}{2} \rfloor - 1\right)\right) \vec{e_1} + \left(\left(\lfloor \frac{t}{2} \rfloor - 1\right) - 2\right) \vec{e_2} + 2\vec{e_4}$

5.5.5 Paths Through Other Tubes

We have constructed a path through the isolation tube corresponding to $\vec{e_1}$. Now we wish to obtain similar paths through the other three isolation tubes. We can construct such paths by permuting the roles of the basis vectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$, and $\vec{e_4}$. For the isolation tube corresponding to $\vec{e_i}$, we transpose the roles of $\vec{e_1}$ and $\vec{e_i}$ and also transpose the roles of the remaining two basis vectors. This gives us a path through each of the isolation tubes.

5.5.6 Joining Paths

We now wish to join the constructed paths in order to complete a cycle. We proceed by cases, depending on the leaves of the paths constructed within the tubes. Note that we will have four situations to explore, each depending on the value of $\lfloor \frac{t}{2} \rfloor - 1 \mod 4$.

Recalling the lengths of the paths through the isolation tubes from section four, which we labeled by Q_i, we have the following main theorem.

Theorem 5.5.12 *The length of the longest $(\mathcal{K}_{1,4}; t)$-representable cycle is at least*

i) If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 0 \mod 4$, the length of the longest $(\mathcal{K}_{1,4}; t)$-representable cycle is at least $4 \cdot Q_3 + (6 + 2(-1)^{t+1})$.

ii) If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 1 \mod 4$, the length of the longest $(\mathcal{K}_{1,4}; t)$-representable cycle is at least $4 \cdot Q_4 + 4 + (6 + 2(-1)^{t+1})$.

iii) If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 2 \mod 4$, the length of the longest $(\mathcal{K}_{1,4}; t)$-representable cycle is at least $4 \cdot Q_1 + (3 + (-1)^{t+1}) + 2(t - 5)$.

iv) If $\lfloor \frac{t}{2} \rfloor - 1 \equiv 3 \mod 4$, the length of the longest $(\mathcal{K}_{1,4}; t)$-representable cycle is at least $4 \cdot Q_2 + 4 + (3 + (-1)^{t+1}) + 2(t - 5)$.

Proof. We proceed by selecting points which will join the paths through the isolation tubes in such a way that a single cycle is formed.

i) The leaves of the paths through the isolation tubes are as follows:

$$\text{Tube 1: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_4} \end{array}$$

$$\text{Tube 2: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_1} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_3} \end{array}$$

$$\text{Tube 3: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_4} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2} \end{array}$$

$$\text{Tube 4:}\ \begin{matrix} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_3} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_1} \end{matrix}$$

Using the nodes $\lceil \frac{t}{2} \rceil \vec{e_1} + \lfloor \frac{t}{2} \rfloor \vec{e_2}$ and $\lceil \frac{t}{2} \rceil \vec{e_2} + \lfloor \frac{t}{2} \rfloor \vec{e_1}$ connects the leaf $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2}$ of tube 1 to the leaf $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2}$ of tube 2. Similarly, using the nodes $\lceil \frac{t}{2} \rceil \vec{e_2} + \lfloor \frac{t}{2} \rfloor \vec{e_3}$ and $\lceil \frac{t}{2} \rceil \vec{e_3} + \lfloor \frac{t}{2} \rfloor \vec{e_2}$, $\lceil \frac{t}{2} \rceil \vec{e_3} + \lfloor \frac{t}{2} \rfloor \vec{e_4}$ and $\lceil \frac{t}{2} \rceil \vec{e_4} + \lfloor \frac{t}{2} \rfloor \vec{e_3}$, $\lceil \frac{t}{2} \rceil \vec{e_4} + \lfloor \frac{t}{2} \rfloor \vec{e_1}$ and $\lceil \frac{t}{2} \rceil \vec{e_1} + \lfloor \frac{t}{2} \rfloor \vec{e_4}$ connect leaves of tube 2 to tube 3, tube 3 to tube 4, and tube 4 to tube 1, respectively. Note that if t is odd, we truly took 8 connecting points, but if t was even we have double-counted and only took 4 connecting points. This gives us the cycle of the claimed length.

ii) The leaves of the paths through the isolation tubes are as follows:

$$\text{Tube 1:}\ \begin{matrix} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 2\right) \vec{e_4} \end{matrix}$$

$$\text{Tube 2:}\ \begin{matrix} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_1} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 2\right) \vec{e_3} \end{matrix}$$

$$\text{Tube 3:}\ \begin{matrix} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_4} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \vec{e_4} + \left(\lfloor \frac{t}{2} \rfloor - 2\right) \vec{e_2} \end{matrix}$$

$$\text{Tube 4:}\ \begin{matrix} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_3} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \vec{e_3} + \left(\lfloor \frac{t}{2} \rfloor - 2\right) \vec{e_1} \end{matrix}$$

Using the nodes $\lceil \frac{t}{2} \rceil \vec{e_1} + \lfloor \frac{t}{2} \rfloor \vec{e_2}$, $\lfloor \frac{t}{2} \rfloor \vec{e_1} + \lceil \frac{t}{2} \rceil \vec{e_2}$ and $\lceil \frac{t}{2} \rceil \vec{e_3} + \lfloor \frac{t}{2} \rfloor \vec{e_4}$, $\lfloor \frac{t}{2} \rfloor \vec{e_3} + \lceil \frac{t}{2} \rceil \vec{e_4}$ connect the path in tube 1 to the one in tube 2 and the path in tube 3 to the one in tube 4 respectively.

We can connect the remaining leaf in tube 2 to the remaining leaf in tube 3 using the nodes $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_3}$, $\lceil \frac{t}{2} \rceil \vec{e_2} + \lfloor \frac{t}{2} \rfloor \vec{e_3}$, $\lfloor \frac{t}{2} \rfloor \vec{e_2} + \lceil \frac{t}{2} \rceil \vec{e_3}$ and $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2}$. Similarly, the remaining leaf in tube 4 can be connected to the remaining leaf in tube 1 via the nodes $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_1}$, $\lceil \frac{t}{2} \rceil \vec{e_4} + \lfloor \frac{t}{2} \rfloor \vec{e_1}$, $\lfloor \frac{t}{2} \rfloor \vec{e_4} + \lceil \frac{t}{2} \rceil \vec{e_1}$, and $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_4}$.

Similarly to Case i), if t is even we have done some double-counting. Again, this completes the cycle of the claimed length.

iii) The leaves of the paths through the isolation tubes are as follows

$$\text{Tube 1: } \begin{matrix} \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_1}+\left(\lfloor\frac{t}{2}\rfloor-1\right)\vec{e_2} \\ \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_1}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_2}+2\vec{e_4} \end{matrix}$$

$$\text{Tube 2: } \begin{matrix} \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_2}+\left(\lfloor\frac{t}{2}\rfloor-1\right)\vec{e_1} \\ \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_2}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_1}+2\vec{e_3} \end{matrix}$$

$$\text{Tube 3: } \begin{matrix} \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_3}+\left(\lfloor\frac{t}{2}\rfloor-1\right)\vec{e_4} \\ \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_3}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_4}+2\vec{e_2} \end{matrix}$$

$$\text{Tube 4: } \begin{matrix} \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_4}+\left(\lfloor\frac{t}{2}\rfloor-1\right)\vec{e_3} \\ \left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_4}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_3}+2\vec{e_1} \end{matrix}$$

Now, we can connect tubes 1 and 2 and tubes 3 and 4 using the same methods as in i). We connect the remaining leaf of tube 4 to the remaining leaf of tube 2 by selecting all nodes of the form $\left(t-(\lfloor\frac{t}{2}\rfloor-1)-j\right)\vec{e_2}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_1}+2\vec{e_3}+j\vec{e_4}$ (for $j\in\{1,2,...,t-(\lfloor\frac{t}{2}\rfloor-1)\}$) and $\left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_4}+\left((\lfloor\frac{t}{2}\rfloor-1)-2-i\right)\vec{e_1}+(2+i)\vec{e_3}$ (for $i\in\{1,2,...,(\lfloor\frac{t}{2}\rfloor-1)-5\}$).

We connect the remaining leaf of tube 3 to that of tube 1 by selecting all nodes of the form $\left(t-(\lfloor\frac{t}{2}\rfloor-1)-j\right)\vec{e_1}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_2}+j\vec{e_3}+2\vec{e_4}$ (for $j\in\{1,2,...,t-(\lfloor\frac{t}{2}\rfloor-1)\}$) and $\left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_3}+\left((\lfloor\frac{t}{2}\rfloor-1)-2-i\right)\vec{e_2}+(2+i)\vec{e_4}$ (for $i\in\{1,2,...,(\lfloor\frac{t}{2}\rfloor-1)-5\}$).

Note that there are no unwanted edges induced here. It is clear that there are no edges between points of the form $\left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_2}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_1}+2\vec{e_3}+j\vec{e_4}$ and those of the form $\left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_3}+\left((\lfloor\frac{t}{2}\rfloor-1)-2-i\right)\vec{e_2}+(2+i)\vec{e_4}$ because of the difference in coefficients of $\vec{e_1}$. Similarly there are no edges between points of the form $\left(t-(\lfloor\frac{t}{2}\rfloor-1)-j\right)\vec{e_1}+\left((\lfloor\frac{t}{2}\rfloor-1)-2\right)\vec{e_2}+j\vec{e_3}+2\vec{e_4}$ and those of the form $\left(t-(\lfloor\frac{t}{2}\rfloor-1)\right)\vec{e_4}+\left((\lfloor\frac{t}{2}\rfloor-1)-2-i\right)\vec{e_1}+(2+i)\vec{e_3}$

because of the difference in coefficients of $\vec{e_2}$. For similar reasons, no edges between points of the form $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2 - i\right) \vec{e_1} + (2 + i)\vec{e_3}$ and of the form $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2 - i\right) \vec{e_2} + (2+i)\vec{e_4}$ can exist either. It is easily seen that if there were an edge between a point of the form $\left(t - (\lfloor \frac{t}{2} \rfloor - 1) - j\right) \vec{e_2} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_1} + 2\vec{e_3} + j\vec{e_4}$ and a point of the form $\left(t - (\lfloor \frac{t}{2} \rfloor - 1) - k\right) \vec{e_1} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_2} + k\vec{e_3} + 2\vec{e_4}$, then j and k must each be $1, 2$, or 3. Now if $j = 3$, then it is clear that the coefficients of $\vec{e_2}$ do not match, so this forces $k = 2$. However, when $k = 2$, the coefficients of $\vec{e_1}$ will be off by at least two. If $j = 1$ it is clear that the coefficients of $\vec{e_2}$ will be off by at least three. Finally, if $j = 2$, then the coefficients of $\vec{e_2}$ will be off by at least two. By the rules for adjacency in $\mathcal{H}$, we see that no unwanted edge exists between these newly selected points, and we have completed a cycle of the length claimed. Note that again we accounted for some double-counting in the case that t was even.

iv) The leaves of the paths through the isolation tubes are as follows

$$\text{Tube 1: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_2} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_1} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_2} + 2\vec{e_4} \end{array}$$

$$\text{Tube 2: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_1} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_2} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_1} + 2\vec{e_3} \end{array}$$

$$\text{Tube 3: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_4} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_4} + 2\vec{e_2} \end{array}$$

$$\text{Tube 4: } \begin{array}{l} \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left(\lfloor \frac{t}{2} \rfloor - 1\right) \vec{e_3} \\ \left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_3} + 2\vec{e_1} \end{array}$$

Now, we can connect tubes 1 and 2 and tubes 3 and 4 using the same methods as in ii). We connect the remaining leaf of tube 4 to the remaining leaf of tube 2 by selecting all nodes of the form $\left(t - (\lfloor \frac{t}{2} \rfloor - 1) - j\right) \vec{e_2} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_1} + 2\vec{e_3} + j\vec{e_4}$ (for $j \in \{1, 2, ..., t - (\lfloor \frac{t}{2} \rfloor - 1)\}$) and

$\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_4} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2 - i\right) \vec{e_1} + (2+i)\vec{e_3}$ (for $i \in \{1, 2, ..., (\lfloor \frac{t}{2} \rfloor - 1) - 5\}$).

We connect the remaining leaf of tube 3 to that of tube 1 by selecting all nodes of the form $\left(t - (\lfloor \frac{t}{2} \rfloor - 1) - j\right) \vec{e_1} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2\right) \vec{e_2} + j\vec{e_3} + 2\vec{e_4}$ (for $j \in \{1, 2, ..., t - (\lfloor \frac{t}{2} \rfloor - 1)\}$) and $\left(t - (\lfloor \frac{t}{2} \rfloor - 1)\right) \vec{e_3} + \left((\lfloor \frac{t}{2} \rfloor - 1) - 2 - i\right) \vec{e_2} + (2+i)\vec{e_4}$ (for $i \in \{1, 2, ..., (\lfloor \frac{t}{2} \rfloor - 1) - 5\}$).

Similarly to Case iii), one can verify that no unwanted edges exist. Again noting that some double-counting has occurred in the case where t is even gives us the cycle of the claimed length. □

Recall that for all of the results in this chapter, we assumed $t \geq 14$. Cycles for smaller values of t can be constructed using similar methods to those discussed here. Note, however, that we must have $t > 5$ for our isolation tubes to exist. For $t \leq 5$, it is conjectured that the length of the longest $(\mathcal{K}_{1,n}; t)$-representable cycle is linear in t. In all cases for $t \geq 14$ the length of the cycle we constructed is on the order of t^3, which can describe by saying the length of the cycle is $O(t^3)$. We conjecture that the length of the longest $(\mathcal{K}_{1,4}; t)$-representable cycle truly is $O(t^3)$ for large enough t.

5.6 Constructing a Cycle for Arbitrary n

Consider the construction in Section 5. The same technique used in this construction can be generalized to arbitrary n. One can construct a long path through the first isolation tube by generalizing the up-all and down-all moves to account for the extra dimensions. It is for this reason that we conjecture.

Conjecture 5.6.1 *The length of the longest $(\mathcal{K}_{1,n}; t)$-representable cycle is $O(t^{n-1})$.*

Our conjecture is supported by the work in this paper and the theorem of Eaton and Barbato which was cited in Chapter 3 and is restated below.

Theorem 5.6.2 *For* $t = 3, 4,$ *and* 5 *the maximum* n *such that* C_n *is* $(\mathcal{K}_{1,3}; t)$*-representable is* $3t-3$*. For* $t \geq 6$ *the largest such* n *satisfies the following inequality*

$$\frac{1}{4}t^2 + t + \frac{3}{4} \leq n \leq \frac{1}{4}t^2 + \frac{3}{2}t - \frac{3}{4}$$

5.7 Future Work

There are many questions left unanswered here. Again, we have only found a lower bound on the length of the longest $(\mathcal{K}_{1,n}; t)$-representable cycle. Improvements to this lower bound could be made by attempting to maximize the lengths of the connecting paths between the isolation tubes. Also of interest would be an upper bound on the length of the longest cycle having a $(\mathcal{K}_{1,n}; t)$-representation. The goal here would be to find the exact length of such a longest cycle. I am hopeful that the work done in chapters 4 and 5 will be useful stepping stones in the pursuit to completely classify those graphs which are $(\mathcal{K}_{1,n}; t)$-representable.

List of References

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CHAPTER 6

Introduction to Closure Systems

6.1 Introduction

In his new text [1], Jamison writes that 'the notion of closure is pervasive throughout mathematics'. Indeed, from grade-school it is taught that the operations of addition, subtraction, and multiplication by real numbers, as well as division by non-zero real numbers are closed. Students then learn of the open and closed intervals of the real line, which are then abstracted to open and closed sets in general. Students in college-level mathematics courses learn of vector spaces, algebraic groups, rings, and fields, which are all closed sets of objects. Closure is surely a common theme across all of mathematics.

A closure operator, ϕ, on a ground set X is a set operator satisfying the following criteria:

1. $\phi(X) = X$
2. If $A \subset B \subset X$, then $\phi(A) \subset \phi(B) \subset X$
3. $\phi(\phi(A)) = \phi(A)$

6.2 Examples of Closure Operators

There are many ways to define a closure operator on a ground set. In fact, it has been shown that if the finite ground set, X, contains at least 10 elements, then there are $|X|^{|X|}$ unique closure operators on X [2]. The following subsections explore some familiar and possibly unfamiliar notions of closure.

6.2.1 Closure on the Real Line

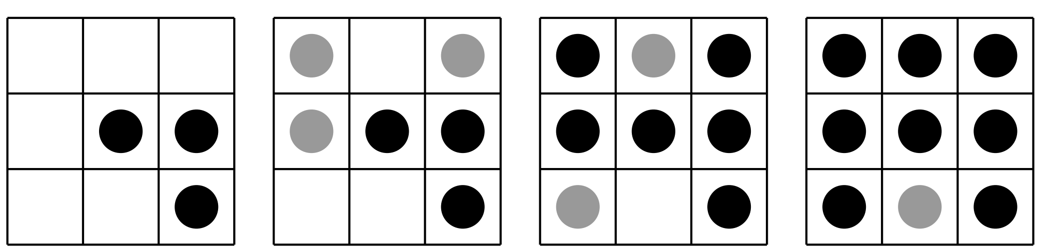
The classic examples on the real line are closed intervals. For example, a real interval is closed iff it contains both of its endpoints. A finite union of intervals on the real line is closed iff it contains all of its endpoints. We are all very familiar with this notion of closure. Let's look at closure in some more abstract spaces.

6.2.2 Poker Closure

Consider a game of poker in which the cards 5,6,7,9 are contained in a hand. A straight is completed if an 8 is obtained. This suggests that the 8 completes a hand with the four cards already possessed. In poker closure, a card, c, is said to be dependent on a hand, H, if c completes a straight with the cards already in H. Note that in poker closure, the ground set contains the 13 card denominations (without any regard to suit). Some interesting properties of poker closure are as follows: (1) any initial hand containing three or fewer cards is closed, (2) the poker closure of any initial hand containing four consecutive cards is the entire space, (3) the poker closure of any hand having a dependency extends to the whole space, (4) for any poker-closed hand, H, which is not the entire space, all subsets of H are closed.

6.2.3 Bingo Closure

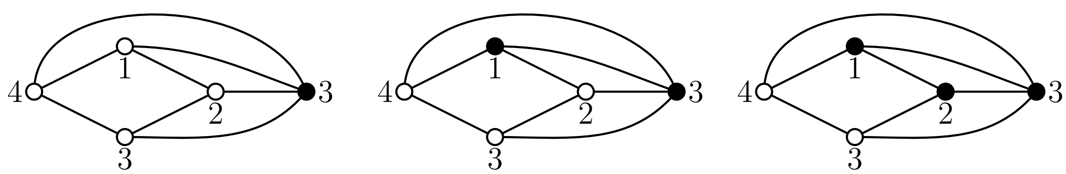
Consider an $n \times n$ grid, on which a game of Bingo can be played. Bingo is made when all of the squares in a row, column, or along one of the main diagonals are captured by a single player. One can define a square s on the Bingo board as being *dependent* on a set S of squares iff s completes a Bingo with squares that are already in S. A set S is said to be Bingo-closed if there are no squares that are dependent on S. Consider the following example on a 3×3 Bingo board.



The closure of the initial set of three squares is the entire Bingo board under this closure operator.

6.2.4 Outbreak Closure

Consider the ground set to be a social interaction network, which can be modeled by a graph with nodes representing individuals, and edges connecting individuals that interact on a daily basis. Let each vertex, v, be assigned a level of resistance $r(v)$ to a virus released into the network. A vertex, v, is said to be dependent on a subset I, of infected vertices, if $v \notin I$ and $|N(v) \cap I| \geq r(v)$. A subset of infected vertices is then closed if there are no vertices dependent on it. Consider the following small-scale example, where black vertices are infected, white vertices are uninfected, and the numbers accompanying the vertices are the corresponding resistance levels.



Note that the resistance levels of the remaining uninfected vertices keep them from becoming infected. Observe, in general, that any vertex satisfying $|N(v)| < r(v)$ is immune to the virus, unless it is one of the initial infected vertices. The idea of outbreak closure can be applied to study problems involving efficient vaccination strategies, as well as those problems centered around biological warfare.

6.3 Closure Spaces

A closure system consists of a ground set, X, a closure operator, ϕ, and the collection of all of the ϕ-closed subsets of X. If the family of subsets is closed under intersections, then the collection of all ϕ-closed subsets of X is called a closure space.

Chapter 7 will explore the closure spaces resulting from what is defined as gap-closure on the integers. One of the classical ways of categorizing topological spaces is to use the 'separation properties' (which will be defined in Chapter 7). The separation properties of a topological space ask whether or not it is possible to 'separate' two given objects within that space via open sets. Since one need only look at the complement of a closed set to find an open set, closure spaces are a natural place to consider the separation properties.

List of References

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CHAPTER 7

Separation in Gap Closures

Separation in Gap Closures

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7.1 Introduction

Gap closures are natural extensions of poker closure. In gap closure, the ground set is the integers. An ordered pair (g, h) of non-negative integers will be called a *gap type*. For $S \subset \mathbb{Z}$, a point $p \in \mathbb{Z}$ is a (g, h)-*gap point* of S provided the g integers to the left of p are in S and the h integers to the right of p also belong to S. Thus for $S := \{1, 2, 3, 5, 6, 8, 10, 11, 12, 16, 17\}$, 4 is a $(3, 2)$-gap point, 7 is a $(2, 1)$-gap point, and 9 is a $(1, 3)$-gap point. Moreover, 13 is a $(3, 0)$-gap point and 15 is a $(0, 2)$-gap point. Note that 14 is not a gap point for any non-trivial gap type. Note that even though $2 \in S$, we still say that 2 is a $(1, 1)$-gap point of S. If S contains all of its (g, h)-gap points, then we say that S is (g, h)-*closed*. If $\mathcal{G}$ is a set of gap types, then S is $\mathcal{G}$-*closed* if and only if S is (g, h)-*closed* for all $(g, h) \in \mathcal{G}$.

If $g \le j$ and $h \le k$, then every (j, k)-gap point is also a (g, h)-gap point. In particular, every (g, h)-*closed* set is also (j, k)-*closed*. This suggests ordering the gap types by $(g, h) \le (j, k)$ iff $g \le j$ and $h \le k$, making the gap types into a partially ordered set.

7.2 Preliminary Notation and a Result

In order to study the separation of disjoint $\mathcal{G}$-closed sets, some tools and notation will be useful. A *coloring* of $\mathbb{Z}$ is a map $f : \mathbb{Z} \to \{1, 2, o\}$. The set of integers colored 1 will be denoted by A and those colored 2 will be denoted by B. The o's represent uncolored integers.

Definition 7.2.1 *A* ***hole*** *in a coloring is a maximal string of contiguous integers colored by o. The width of a hole is the length of the string. A coloring is* ***complete*** *provided it has no holes; that is,* $A \cup B = \mathbb{Z}$*.*

A coloring is $\mathcal{G}$-*convex* if the sets A and B are both $\mathcal{G}$-closed. A coloring g extends a coloring f provided g is obtained from f by changing some of the o's in f from

o to either 1 or 2.

Lemma 7.2.2 *If $\mathcal{G}$ is a set of gap-types, then every pair of disjoint $\mathcal{G}$-closed sets can be extended to a maximal disjoint pair of closed sets.*

Proof. The result follows from Zorn's Lemma since pairs of disjoint closed sets can be partially ordered. Indeed, consider two pairs of disjoint closed sets (A, B) and (C, D), then $(A, B) \leq (C, D)$ iff $A \subset C$ and $B \subset D$. □

Note that this means that any partial $\mathcal{G}$-convex coloring, f, can be extended to a maximal $\mathcal{G}$-convex coloring. These partial $\mathcal{G}$-convex colorings will be our objects of study. In particular, we will be interested in partial convex colorings which extend to all of $\mathbb{Z}$. In general, though, it is not always possible to extend a partial convex coloring of $\mathbb{Z}$ to a complete convex coloring of $\mathbb{Z}$.

$$\text{(A)} \quad \begin{array}{c} \text{Let } g = h = 2 : 1121o2122 \\ \text{Let } g = 1 \text{ and } h = 3 : 1211o1222 \end{array}$$

The two color patterns in (A), above, are obstructions to completing any coloring in which they occur. More generally, given positive integers g, h, j, and k, we call the standard $[g, h, j, k]$-obstruction a string colored in the following manner: $1^g21^{h-1}o2^{j-1}12^k$, where 1^p denotes a string of p consecutive integers colored by 1, and 2^q denotes a string of q consecutive integers colored by 2. The space in this pattern cannot be filled without violating either (g, h)-convexity (if $o \to 1$) or (j, k)-convexity (if $o \to 2$).

7.3 Separation in Gap Closures

We are now ready to discuss the separation properties in the closure spaces resulting from gap closures.

7.3.1 Separation Properties

The separation properties are used as a way to categorize topological spaces. Although a closure space is not necessarily a topological space, the separation properties can still be used to classify closure spaces. We define the separation properties, and then discuss under what conditions the resulting closure spaces satisfy the different levels of separability.

Definition 7.3.1 *The separation properties are as follows:*

1. *A set X is S_0 if for any two distinct points x_1, x_2 of X, at least one has an open neighborhood not containing the other.*
2. *A set X is S_1 if for any two distinct points x_1, x_2 of X each has an open neighborhood not containing the other.*
3. *A set X is S_2 if for any two distinct points x_1, x_2 of X there exist disjoint open neighborhoods, A and B, around x_1 and x_2, respectively.*
4. *A set X is S_3 if given any point $x \in X$ and a closed set $F \subset X$, with $x \notin F$, then x and F can be separated by disjoint open sets.*
5. *A set X is S_4 if any two disjoint closed subsets of X can be separated by disjoint open sets.*

7.3.2 Exploring Separation in Gap Closures

Gap Closures and S_4

The goal of S_4 separation is to extend a partial convex coloring of $\mathbb{Z}$ to a complete convex coloring of all of $\mathbb{Z}$. As we saw earlier, this is not always possible.

Theorem 7.3.2 *All gap closures fail to satisfy S_4.*

Proof. Let $\mathcal{G}$ be a set of gap types. We begin by discussing the cases where $\mathcal{G}$ contains one or more of the following gap types: $(0,0)$, $(1,0)$, or $(0,1)$, which we call degenerate. In each case it is impossible to build two, non-empty, disjoint closed subsets of $\mathbb{Z}$. It is clear that there is only one $(0,0)$-closed subset of $\mathbb{Z}$, namely

$\mathbb{Z}$ itself. In the case of $(1,0)$-closure, if p and q are distinct integers contained in $(1,0)$-closed subsets of $\mathbb{Z}$, then both subsets must contain $\{x|x \geq \max\{p,q\}\}$. That is, there is no way to have two non-empty, disjoint, $(1,0)$-closed subsets of $\mathbb{Z}$. Similarly, it is impossible to have two, non-empty, disjoint, $(0,1)$-closed subsets of $\mathbb{Z}$.

Let $\mathcal{G}$ be a non-degenerate set of gap types, and let (g,h) be a minimal gap type in $\mathcal{G}$. Suppose first that $g=0$ and $h>1$. Consider the partial coloring $12^{h-1}o1^{h-1}$. All of the gaps in this coloring have types smaller than $(0,h)$, so since $(0,h)$ is minimal we have that this coloring is $\mathcal{G}$-convex. However, if the hole is filled with a 1, we have $12^{h-1}1^{h}$, making the rightmost original 2 into a $(0,h)$ gap point, destroying $\mathcal{G}$-convexity. Similarly if the hole is filled with a 2, we have $12^{h}1^{h-1}$, making the original first 1 into a $(0,h)$ gap point, which also destroys $\mathcal{G}$-convexity. The case that $h=0$ and $g>1$ is symmetric.

It remains to consider that $g>0$ and $h>0$. In this case, the standard $[g,h,g,h]$-obstruction establishes the result. That is, the partial coloring $1^{g}21^{h-1}o2^{g-1}12^{h}$ cannot be completed without destroying $\mathcal{G}$-convexity. □

This is not the end of the story for S_4, however. Separation by hemispaces is not possible, but how close can we get? This question will be explored further in a dissertation by A. Mia Heissan.

The Other Separation Axioms

In this section we will provide examples of gap closures which do and do not satisfy the remaining separation properties.

Theorem 7.3.3 *$(0,0)$-closure is not S_0.*

Proof. Every point is trivially a gap point in $(0,0)$-closure. Thus, the only $(0,0)$-closed set is $\mathbb{Z}$. The only set which is open, then, is the empty set, which is not a neighborhood of any points of $\mathbb{Z}$. Therefore, separation, even in the S_0 sense, is impossible. □

Theorem 7.3.4 $(0,1)$-*closure is* S_0*, but is not* S_1*.*

Proof. Consider two points, a and b, with $a < b$. Let A be the downset at a. It is clear that A is $\mathcal{G}$-closed, so A' is $\mathcal{G}$-open and contains b but not a. Note that there is no $\mathcal{G}$-closed set containing b which does not contain a, since a is in the downset at b. Thus, $(0,1)$-closure is S_0, but is not S_1. □

Note that in order to prove that a closure space is S_2, for any two integers we need two disjoint open sets, each containing one of the integers. Since we are finding open sets by looking at the compliments of closed sets, we must find two closed sets whose union is all of $\mathbb{Z}$, such that their compliments will be disjoint. That is, if $\mathcal{G}$ the set of gap types inducing our closure space, we will be looking for a complete $\mathcal{G}$-convex coloring of $\mathbb{Z}$.

Lemma 7.3.5 $\mathcal{G}$-*closure, with* $\mathcal{G} = \{(0,2),(1,1)\}$ *is* S_1*, but is not* S_2*.*

Proof. Consider two points, a and b, with $a < b$. The single element set $\{a\}$ is $\mathcal{G}$-closed, and so $\{a\}' = \mathbb{Z} \setminus \{a\}$ is a $\mathcal{G}$-open set containing b but not a. Similarly $\{b\}$ is a $\mathcal{G}$-closed set whose complement is open and contains a but not b. Thus, $\mathcal{G}$-closure is S_1.

Suppose now that we have a complete convex coloring, f of the integers, that separates a and b. Let $a < b$ and a be colored by 1, while b is colored by 2. We know there are no consecutive integers colored by 2 above a by $(0,2)$-convexity, and likewise, no consecutive integers colored by 1 above b. Hence, f must alternate 1's and 2's above b, but this is impossible since f is $\mathcal{G}$-convex. That is, no such complete $\mathcal{G}$-convex coloring exists. □

Theorem 7.3.6 $(1,2)$-*closure is* S_2*, but is not* S_3*.*

Proof. Consider two integers, a and b, colored by 1 and 2, respectively. Let $a < b$, color all of the integers to the left of b by 1, and all of the integers to the right of b by 2. Note that the resulting coloring is a complete $(2,2)$-convex coloring of $\mathbb{Z}$, which separates a and b. Thus $(1,2)$-closure is S_2.

Consider the partial coloring 121oo1, placing a 1 in the first open position gives 1211o1, making the 2 a $(1,2)$-gap point of the 1's. Thus, the first position must be filled with a 2, giving 1212o1. Note that no matter how the remaining hole is filled, we have a resulting $(1,2)$-gap point. □

Theorem 7.3.7 *$(2,2)$-closure is S_3.*

Proof. Consider trying to separate a $(2,2)$-closed set of 1's from a single given 2. We denote the original 2 by 2^*.

If the two positions to the right of 2^* are 1's, fill that entire side by 1's. Otherwise, fill all positions below 2^* by 1. Note that this is admissible since otherwise the original coloring would be of the form ...112^*11..., for which the set of 1's is not $(2,2)$-closed.

Using symmetry, assume that all positions below 2^* are 1. Note that we must have either ...112^*1o... or ...112^*o...1..., so, in either case, fill the first open position by 2, and all remaining positions by 1. The resulting coloring is a complete $(2,2)$-convex coloring of $\mathbb{Z}$. □

Some General Results

Definition 7.3.8 *The **mesh** of a set $\mathcal{G}$ of gap sizes is the minimum over all g and h such that $(g,h) \in \mathcal{G}$. The **weight** of $\mathcal{G}$ is the minimum over all $g+h$ such that $(g,h) \in \mathcal{G}$.*

Theorem 7.3.9 *Given a set of gap types $\mathcal{G}$, the weight of $\mathcal{G}$ is the largest n such* *that every set with fewer than n elements is closed.*

Proof. Any set having a (g, h)-gap point must certainly have $g+h$ elements. Therefore, every set having fewer than $g+h$ elements is certainly (g, h)-closed. The set of elements corresponding to the coloring $...oo1^g o1^h oo...$ is not (g, h)-closed. Thus, the width of a set $\mathcal{G}$ of gap types is the largest n such that any set consisting of fewer than n elements is $\mathcal{G}$-closed. □

Theorem 7.3.10 *$(0, h)$-convexity satisfies S_3 for all $h \geq 3$. $(0, 2)$-convexity does not satisfy S_3, but does satisfy S_2.*

Proof. To show S_3 we must separate a single integer colored by 1 from a $(0, h)$-closed set of integers color by 2. We must fill in any holes with 1's and 2's in such a way that convexity of the coloring is preserved. Let 1^* denote the initial 1, and fill all of the positions below 1^* with 2's. Consider a hole H that remains. This hole must be above 1^*.

a) If H has even width and 1^* bounds H, then we fill H by alternating 2's and 1's respectively.

b) If H has even width, but 1^* does not bound H, then fill H with 1's in the first two positions, followed by alternating 2's and 1's.

c) If H has odd width, fill it with alternating 1's and 2's respectively.

These rules may generate pairs of consecutive 1's, but since h is at least 3, we have created no dependencies. The rules generate no strings of 2's. Therefore, convexity is preserved.

The pattern 1^*2oo2 shows that the 1^* cannot be separated via $(0, 2)$-convex coloring from the two 2's. To prove that S_2 is satisfied, consider a pattern $1^*oo...oo2^*$. Again, color all of the integers below the 1^* by 1. Now, if the width of the hole between 1^* and 2^* is even, then fill the hole by alternating 2's and 1's. If the hole has odd width, then fill it by alternating 1's and 2's. In either case, fill all positions above 2^* by alternating 1's and 2's. $\square$

Theorem 7.3.11 *Let $\mathcal{G}$ be a set of gap types,*

a) if $\mathcal{G}$ has mesh at least 1, then the resulting closure space is S_2.

b) If $\mathcal{G}$ has mesh at least 3, then the resulting closure space is S_3.

Proof.

a): Let $\mathcal{G}$ be a set of gap types with mesh at least 1. Consider p and q, with $p < q$, to be two points that we wish two separate. Take p and its downset to be H. Since $\mathcal{G}$ has mesh at least 1, then both H and H' are $\mathcal{G}$-closed. That is, H is a separating hemispace.

b): Let $\mathcal{G}$ be a set of gap types with mesh at least 3, and let p be a point disjoint from a $\mathcal{G}$-closed set F. Let C denote the complement of F, and consider the component $K \subset C$ which contains p. Color K with 2's and F with 1's. Now, if $|K| > 1$, fill all remaining holes with 2's. Otherwise $|K| = 1$, and we proceed more carefully. If all of the remaining holes to the right of p are of width one, then color all of these holes by 1. Otherwise there are holes of width at least two to the right of p. Locate the hole of width at least two which is closest to p, and color it by 1's. Color all of the holes of width 1 between p and this newly colored sting of 1's by 1 as well. Color all of the holes above this string of 1's by 2's. Analogously fill the holes to the left of p. Note that we have extended the partial coloring to a complete coloring of $\mathbb{Z}$ while preserving $\mathcal{G}$-convexity of the coloring.

□

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