2015

Asymptotic Approximations of the Stable and Unstable Manifolds of Fixed Points of a Two-dimensional Cubic Map

Jasmin Betešević

Mustafa Kulenovic
University of Rhode Island, mkulenovic@uri.edu

See next page for additional authors

Citation/Publisher Attribution
Bektešević, J., Kulenović, M. R. S., & Pilav, E. (2015). Asymptotic Approximations of the Stable and Unstable Manifolds of Fixed Points of a Two-dimensional Cubic Map. *International Journal of Difference Equations, 10*(1), 39-58.
Available at: http://campus.mst.edu/ijde/contents/v10n1p3.pdf

Authors
Jasmin Betešević, Mustafa Kulenovic, and Esmir Pilav

Asymptotic approximations of a stable and unstable manifolds of a two-dimensional quadratic map

J. Bektešević[†] and M.R.S Kulenović[‡1] and E. Pilav[§2]

[†]Division of Mathematics
Faculty of Mechanical Engineering, University of Sarajevo, Bosnia and Herzegovina

[‡]Department of Mathematics
University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

[§]Department of Mathematics
University of Sarajevo, Sarajevo, Bosnia and Herzegovina

Abstract. We find the asymptotic approximations of the stable and unstable manifolds of the saddle equilibrium solutions and the saddle period-two solutions of the following difference equation $x_{n+1} = cx_{n-1}^2 + dx_n + 1$, where the parameters c and d are positive numbers and initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. These manifolds determine completely global dynamics of this equation.

Keywords. Basin of attraction, cooperative, difference equation, local stable manifold, local unstable manifold, monotonicity, period-two solutions;

AMS 2000 Mathematics Subject Classification: Primary: 37B25, 37D10 Secondary: 37M99, 65P40, 65Q30 .

1 Introduction

In this paper we consider the difference equation

$$x_{n+1} = cx_{n-1}^2 + dx_n + 1, \tag{1}$$

where the parameters c and d are positive numbers and initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. Set

$$u_n = x_{n-1} \text{ and } v_n = x_n \text{ for } n = 0, 1, \ldots \tag{2}$$

and write Eq.(1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= cu_n^2 + dv_n + 1. \end{aligned} \tag{3}$$

Let T be the corresponding map defined by:

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ cu^2 + dv + 1 \end{pmatrix}. \tag{4}$$

It is easy to see that

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = T\left(T\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} cu^2 + dv + 1 \\ d\left(cu^2 + dv + 1\right) + cv^2 + 1 \end{pmatrix}. \tag{5}$$

The local dynamics of the map T was derived in [1] where it was shown that the following holds:

Theorem 1 *If*

$$d < 1 \textit{ and } (d-1)^2 - 4c \geq 0$$

---

[1]Corresponding author, *e-mail: mkulenovic@mail.uri.edu*

[2]Partially supported by FMON Grant No. 05-39-3632–1/14

then Eq.(1) *has the equilibrium points* $\bar{x}_1$ *and* $\bar{x}_2$ *where*

$$\bar{x}_1 = \frac{1-d-\sqrt{(d-1)^2-4c}}{2c}, \quad \bar{x}_2 = \frac{1-d+\sqrt{(d-1)^2-4c}}{2c}$$

and the following holds:

i) $\bar{x}_1$ *is locally asymptotically stable if*

$$c < \frac{(d-1)^2}{4}.$$

ii) $\bar{x}_1$ *a non-hyperbolic point if*

$$c = \frac{(d-1)^2}{4}.$$

iii) $\bar{x}_2$ *is a repeller if*

$$c < \frac{(1-3d)(d+1)}{4}.$$

iv) $\bar{x}_2$ *is a saddle point if*

$$\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}$$

v) $\bar{x}_2$ *a non-hyperbolic point if*

$$c = \frac{(1-3d)(d+1)}{4} \ \text{or}\ c = \frac{(d-1)^2}{4}.$$

Theorem 2 *If*

$$c < \frac{(1-3d)(d+1)}{4}$$

then Eq.(1) *has the minimal period-two solution*

$$P = \left\{ \frac{d+1-\sqrt{1-4c-d(3d+2)}}{2c}, \frac{d+1+\sqrt{1-4c-d(3d+2)}}{2c} \right\}$$

which is a saddle point.

The global dynamics of Eq.(1) is delicate and is described by the following theorem [1].

Theorem 3 *Consider Eq.*(1)*. Then the following holds:*

(i) If $c < \frac{(1+d)(1-3d)}{4}$ *then Eq.*(1) *has two equilibrium solutions* $0 < \bar{x}_- < \bar{x}_+$*, where* x_- *is locally asymptotically stable,* $\bar{x}_+$ *is a repeller and the minimal period-two solution* $\ldots, \Phi, \Psi, \ldots,$ $\Phi < \Psi$ *is a saddle point. All non-equilibrium solutions* $\{x_n\}$ *converge to* x_-*, or to the period-two solution or are asymptotic to* ∞*. More precisely, there exist four continuous curves* $W^s(P_1), W^s(P_2)$ *(stable manifolds of* $P_1(\Phi,\Psi)$ *and* $P_2(\Psi,\Phi)$*),* $W^u(P_1), W^u(P_2)$*, (unstable manifolds of* P_1 *and* P_2*) where* $W^s(P_1), W^s(P_2)$ *are passing through the point* $E_+(\bar{x}_+, \bar{x}_+)$*, and are graphs of decreasing functions. The curves* $W^u(P_1), W^u(P_2)$ *are the graphs of increasing functions and are starting at* $E_-(\bar{x}_-, \bar{x}_-)$*. Every solution* $\{x_n\}$ *which starts below* $W^s(P_1) \cup W^s(P_2)$ *in North-east ordering converges to* $E_-(\bar{x}_-, \bar{x}_-)$ *and every solution* $\{x_n\}$ *which starts above* $W^s(P_1) \cup W^s(P_2)$ *in North-east ordering satisfies* $\lim x_n = \infty$*.*

(ii) If $c = \frac{(1+d)(1-3d)}{4}$ *then Eq.*(1) *has two equilibrium solutions* $0 < \bar{x}_- < \bar{x}_+$*, where* x_- *is locally asymptotically stable and* $\bar{x}_+$ *is the non-hyperbolic equilibrium solution. There exist the continuous decreasing curve* $W^s(E_+)$ *passing through the point* $E_+ = (\bar{x}_+, \bar{x}_+)$*, such that every solution* $\{x_n\}$ *which starts below* $W^s(E_+)$ *in North-east ordering converges to* $E_-(\bar{x}_-, \bar{x}_-)$ *and every solution* $\{x_n\}$ *which starts above* $W^s(E_+)$ *in North-east ordering satisfies* $\lim x_n = \infty$*.*

(iii) If $\frac{(1+d)(1-3d)}{4} < c < \frac{(1-d)^2}{4}$ *then Eq.*(1) *has two equilibrium solutions* $0 < \bar{x}_- < \bar{x}_+$ *and no minimal period-two solutions. If* $\bar{x}_+$ *is a saddle equilibrium solution, then there exist two continuous curves* $W^s(E_+)$ *and* $W^u(E_+)$*, both passing through the point* $E_+ = (\bar{x}_+, \bar{x}_+)$*, such that* $W^s(E_+)$ *is a graph of decreasing function and* $W^u(E_+)$ *is a graph of an increasing*

function. The first quadrant of initial condition $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is is the union of three disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_-) \cup \mathcal{B}(E_+) \cup \mathcal{B}(E_\infty),$$

where E_- and E_∞ denote the points (x_-, x_-) and (∞, ∞) respectively, and $\mathcal{B}(E_+) = W^s(E_+)$,

$$\mathcal{B}(E_-) = \{(x,y)|(x,y) \preceq_{ne} (x_{E_+}, y_{E_+}) \quad \textit{for some} \quad (x_{E_+}, y_{E_+}) \in W^s(E_+)\},$$

$$\mathcal{B}(E_\infty) = \{(x,y)|(x_{E_+}, y_{E_+}) \preceq_{ne} (x,y) \quad \textit{for some} \quad (x_{E_+}, y_{E_+}) \in W^s(E_+)\}.$$

In addition, for every $(x_{-1}, x_0) \in Q_1 \setminus W^s(E_+)$ every solution is asymptotic to $W^u(E_+)$.

(iv) If $c = \frac{(1-d)^2}{4}$ then Eq.(1) has one non-hyperbolic equilibrium solution $\bar{x}$ and there exists an invariant continuous curve $W^s(E)$, where $E(\bar{x}, \bar{x})$, which is the graph of a decreasing function, such that every solution $\{x_n\}$ of Eq.(1) for which $(x_{-1}, x_0) \in W^s(E)$ is attracted to E as well as every solution $\{x_n\}$ of Eq.(1) for which $(x_{-1}, x_0) \preceq_{ne} W^s(E)$.

Every solution $\{x_n\}$ of Eq.(1) for which there exists $(x_W, y_W) \in W^s(E)$ such that $(x_W, y_W) \preceq_{ne} (x_{-1}, x_0), (x_{-1}, x_0) \notin W^s(E)$ satisfies $\lim x_n = \infty$.

(v) If $c > \frac{(1-d)^2}{4}$ then Eq.(1) neither has an equilibrium solution nor the minimal period-two solution and every solution $\{x_n\}$ of Eq.(1) satisfies $\lim_{n\to\infty} x_n = \infty$.

As one may see from Theorem 3 the boundaries of the basins of attraction of all attractors of Eq.(1) are the stable manifolds of either equilibrium points or of the period-two solution. In addition, by using the results from [9] one can see that the solutions which are asymptotic to the locally asymptotically stable equilibrium solutions are approaching the unstable manifolds of the neighboring saddle equilibrium points or period-two point. The monotonicity and smoothness of stable and unstable manifolds for the map T given with (4) is guaranteed by Theorems 4, 5, 6 of [9]. See [4, 7, 9, 12, 13] for related results about the stable manifolds for competitive maps. Our main goal here is to get the local asymptotic estimates for these manifolds for both equilibrium solutions and the period-two solutions. We will bring the considered map to the normal form around the equilibrium solutions and the period-two solutions and then use the method of undetermined coefficients to find the local approximations of the considered manifolds. Since the map T is cooperative, it is guaranteed that both stable and unstable manifolds are as smooth as the functions of the considered map and that are monotonic such that the stable manifold is decreasing and unstable manifold is increasing, see [2, 9]. See [4, 10, 14] for similar local approximations of stable and unstable manifolds. See [3, 5, 6, 11, 14] for basic results on stable ad unstable manifolds for general maps.

2 Preliminaries

In this section we present some basic results for the cooperative maps which describe the existence and the properties of their invariant manifolds.

A first order system of difference equations

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, 2, \ldots, \quad (x_0, y_0) \in \mathcal{S}, \tag{6}$$

where $\mathcal{S} \subset \mathbb{R}^2$ is nonempty, $(f,g) : \mathcal{S} \to \mathcal{S}$, f, g are continuous functions is *cooperative* if $f(x,y)$ and $g(x,y)$ are non-decreasing in x and y. *Strongly cooperative* systems of difference equations or strongly cooperative maps are those for which the functions f and g are coordinate-wise strictly monotone.

If $\mathbf{v} = (u,v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1,2,3,4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x,y) \in \mathbb{R}^2 : x \geq u,\ y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x,y) \in \mathbb{R}^2 : x \leq u,\ y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x,y) \preceq_{se} (s,t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x,y) \preceq_{ne} (s,t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $\mathrm{x} \in \mathbb{R}^2$, define the *distance from* x *to* $\mathcal{A}$ as $\mathrm{dist}(\mathrm{x}, \mathcal{A}) := \inf\{\|\mathrm{x} - \mathrm{y}\| : \mathrm{y} \in \mathcal{A}\}$. By $\mathrm{int}\,\mathcal{A}$ we denote the interior of a set $\mathcal{A}$.

It is easy to show that a map F is cooperative if it is non-decreasing with respect to the North-East partial order, that is if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{7}$$

The following five results were proved by Kulenović and Merino [8, 9] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or non-hyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. We give the analogue versions for cooperative maps.

A region $\mathcal{R} \subset \mathbb{R}^2$ is *rectangular* if it is the cartesian product of two intervals in $\mathbb{R}$.

Theorem 4 *Let T be a cooperative map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \mathrm{int}\,(\mathcal{Q}_2(\overline{\mathrm{x}}) \cup \mathcal{Q}_4(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NE or SW vertex of $\mathcal{R}$), and T is strongly cooperative on Δ. Suppose that the following statements are true.*

a. *The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.*
b. *The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly decreasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Corollary 1 *If T has no fixed point nor periodic points of minimal period two in Δ, then the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$.*

For maps that are strongly cooperative near the fixed point, hypothesis (b). of Theorem 4 reduces just to $|\lambda| < 1$. This follows from a change of variables [13] that allows the Perron-Frobenius Theorem to be applied to give that at any point, the Jacobian matrix of a strongly cooperative map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such a case no associated eigenvector is aligned with a coordinate axis.

Theorem 5 *Under the hypotheses of Theorem 4, suppose there exists a neighborhood $\mathcal{U}$ of $\overline{\mathrm{x}}$ in $\mathbb{R}^2$ such that T is of class C^k on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian of T at each $\mathrm{x} \in \Delta$ is invertible. Then the curve $\mathcal{C}$ in the conclusion of Theorem 4 is of class C^k.*

The following result gives a description of the global stable and unstable manifolds of a saddle point of a cooperative map. The result is the modification of Theorem 5 from [7]. See also [8].

Theorem 6 *In addition to the hypotheses of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 4 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the global stable manifold $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the global unstable manifold $\mathcal{W}^u(\overline{\mathrm{x}})$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly increasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Theorem 7 *Assume the hypotheses of Theorem 4, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 4. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \textit{ with } \mathrm{x} \preceq_{ne} \mathrm{y}\} \quad \text{and} \quad \mathcal{W}_+ := \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \textit{with } \mathrm{y} \preceq_{ne} \mathrm{x}\}, \tag{8}$$

such that the following statements are true.

(i) *$\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(\mathrm{x}), \mathcal{Q}_1(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $\mathrm{x} \in \mathcal{W}_-$.*

(ii) *$\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(\mathrm{x}), \mathcal{Q}_3(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $\mathrm{x} \in \mathcal{W}_+$.*

If, in addition, $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly cooperative in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $\mathcal{Q}_2(\overline{\mathrm{x}}) \cup \mathcal{Q}_4(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $\mathrm{x} \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_1(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $\mathrm{x} \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_3(\overline{\mathrm{x}})$ for $n \geq n_0$.

Remark 1 The map T defined with (4) is strongly cooperative in the first quadrant of initial conditions. Theorems 4, 5 and 6 show that the stable and unstable manifolds of cooperative maps, which satisfies certain conditions, are simple monotonic curves which are as smooth as the functions of the map. Thus the assumed forms of these manifolds are justified. As is well-known the stable and unstable manifolds of general maps can have complicated structure consisting of many branches or being strange attractors, see [3, 5, 10, 14] for some examples of polynomial maps such as Henon with unstable manifold being a starnge attractor. Finally, see [13] for examples of competitive and so cooperative maps in the plane with chaotic attractors.

3 Invariant manifolds and Normal Forms

Let

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + \begin{pmatrix} g_1(\xi_n, \eta_n) \\ g_2(\xi_n, \eta_n) \end{pmatrix}, \tag{9}$$

where

$$g_1(0,0) = 0, \quad g_2(0,0) = 0, \quad Dg_1(0,0) = 0 \text{ and } Dg_2(0,0) = 0.$$

Suppose that $|\mu_1| < 1$ and $|\mu_2| > 1$. Then, there are two unique invariant manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ tangents to $(1,0)$ and $(0,1)$ at (0,0), which are graphs of the maps

$$\varphi : E_1 \to E_2 \text{ and } \psi : E_1 \to E_2,$$

such that

$$\varphi(0) = \psi(0) = 0 \text{ and } \varphi'(0) = \psi'(0) = 0.$$

See [4, 5, 10, 14]. Letting $\eta_n = \varphi(\xi_n)$ yields

$$\eta_{n+1} = \varphi(\xi_{n+1}) = \varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))). \tag{10}$$

On the other hand by (9)

$$\eta_{n+1} = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \tag{11}$$

Equating equations (10) and (11) yields

$$\varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))) = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \tag{12}$$

Similarly, letting $\xi_n = \psi(\eta_n)$ yields

$$\xi_{n+1} = \psi(\eta_{n+1}) = \psi(\mu_2 \eta_n + g_2(\psi(\eta_n), \eta_n)). \tag{13}$$

By using (9) we obtain

$$\xi_{n+1} = \mu_1 \psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \tag{14}$$

Equating equations (13) and (14) yields

$$\psi(\mu_2 \eta_n + g_2(\psi(\eta_n), \eta_n)) = \mu_1 \psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \tag{15}$$

Thus the functional equations (12) and (15), define the local stable manifold

$$\mathcal{W}^s = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \varphi(\xi)\},$$

and the local unstable manifold

$$\mathcal{W}^u = \{(\xi, \eta) \in \mathbb{R}^2 : \xi = \psi(\eta)\}.$$

Without loss generality, we can assume that solutions of the functional equations (12) and (15) take the forms

$$\psi(\eta) = \alpha_2 \eta^2 + \beta_2 \eta^3 + O(|\eta|^4)$$

and

$$\varphi(\xi) = \alpha_1 \xi^2 + \beta_1 \xi^3 + O(|\xi|^4),$$

where $\alpha_i, \beta_i, i = 1, 2$ are undetermined coefficients.

3.1 Normal form of the map T at $\bar{x}_2$

Put $y_n = x_n - \bar{x}_2$. Then Eq(1) becomes

$$y_{n+1} = c\left(\bar{x}_2 + y_{n-1}\right)^2 + d\left(\bar{x}_2 + y_n\right) - \bar{x}_2 + 1. \tag{16}$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots \tag{17}$$

and write Eq(16) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= c\left(\bar{x}_2 + u_n\right)^2 + d\left(\bar{x}_2 + v_n\right) - \bar{x}_2 + 1. \end{aligned} \tag{18}$$

Let F be the function defined by:

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ c\left(\bar{x}_2 + u\right)^2 + d\left(\bar{x}_2 + v\right) - \bar{x}_2 + 1 \end{pmatrix}. \tag{19}$$

Then F has the fixed point $(0,0)$ and maps $(-\bar{x}_2, \infty)^2$ into $(-\bar{x}_2, \infty)^2$. The Jacobian matrix of F is given by

$$Jac_F(u, v) = \begin{pmatrix} 0 & 1 \\ 2c\left(u + \bar{x}_2\right) & d \end{pmatrix}.$$

At $(0,0)$, $Jac_F(u, v)$ has the form

$$J_0 = Jac_F(0,0) = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}. \tag{20}$$

The eigenvalues of (20) are $\mu_{1,2}$ where

$$\mu_1 = \frac{1}{2}\left(d - \sqrt{8c\bar{x}_2 + d^2}\right) \text{ and } \mu_2 = \frac{1}{2}\left(d + \sqrt{8c\bar{x}_2 + d^2}\right),$$

and the corresponding eigenvectors are given by

$$v_1 = \left(-\frac{d + \sqrt{8c\bar{x}_2 + d^2}}{4c\bar{x}_2}, 1\right)^T \text{ and } v_2 = \left(-\frac{d - \sqrt{8c\bar{x}_2 + d^2}}{4c\bar{x}_2}, 1\right)^T,$$

respectively.

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u, v) \\ g_1(u, v) \end{pmatrix}, \tag{21}$$

where

$$\begin{gathered} f_1(u, v) = 0 \\ g_1(u, v) = \bar{x}_2\left(c\bar{x}_2 + d - 1\right) + cu^2 + 1. \end{gathered}$$

Then, the system (16) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n, v_n) \\ g_1(u_n, v_n) \end{pmatrix}. \tag{22}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} -\frac{d + \sqrt{d^2 + 8c\bar{x}_2}}{4c\bar{x}_2} & -\frac{d - \sqrt{d^2 + 8c\bar{x}_2}}{4c\bar{x}_2} \\ 1 & 1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} -\frac{2c\bar{x}_2}{\sqrt{d^2 + 8c\bar{x}_2}} & \frac{\sqrt{d^2 + 8c\bar{x}_2} - d}{2\sqrt{d^2 + 8c\bar{x}_2}} \\ \frac{2c\bar{x}_2}{\sqrt{d^2 + 8c\bar{x}_2}} & \frac{d + \sqrt{d^2 + 8c\bar{x}_2}}{2\sqrt{d^2 + 8c\bar{x}_2}} \end{pmatrix}.$$

Then system (22) is equivalent to

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + P^{-1} \cdot H_1\left(P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}\right), \tag{23}$$

where

$$H_1 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} f_1(u,v) \\ g_1(u,v) \end{pmatrix}.$$

Let

$$G_1 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \tilde{f}_1(u,v) \\ \tilde{g}_1(u,v) \end{pmatrix} = P^{-1} \cdot H_1 \left(P \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

By straightforward calculation we obtain that

$$\tilde{f}_1(u,v) = \frac{\Upsilon_1(u,v)\left(\sqrt{8c\bar{x}_2 + d^2} - d\right)}{16c\bar{x}_2^2\sqrt{8c\bar{x}_2 + d^2}},$$

$$\tilde{g}_1(u,v) = \frac{\Upsilon_1(u,v)\left(\sqrt{8c\bar{x}_2 + d^2} + d\right)}{16c\bar{x}_2^2\sqrt{8c\bar{x}_2 + d^2}},$$

where

$$\Upsilon_1(u,v) = 8c^2\bar{x}_2^4 + d\left(u^2 - v^2\right)\sqrt{8c\bar{x}_2 + d^2} + 4c\bar{x}_2\left(2(d-1)\bar{x}_2^2 + 2\bar{x}_2 + (u-v)^2\right) + d^2\left(u^2 + v^2\right).$$

3.2 Stable and unstable manifolds corresponding to $\bar{x}_2$

Assume that $d < 1$ and $(d-1)^2 - 4c \geq 0$. Then Eq.(1) has the equilibrium point $\bar{x}_2$ where

$$\bar{x}_2 = \frac{1 - d + \sqrt{(d-1)^2 - 4c}}{2c}$$

which is a saddle point if

$$\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}.$$

Let us assume that the local stable manifold is the graph of the map φ_1 of the form

$$\varphi_1(\xi) = \alpha_1\xi^2 + \beta_1\xi^3 + O(|\xi|^4), \quad \alpha_1, \beta_1 \in \mathbb{R}.$$

Now we compute the constants α_1 and β_1. The function φ_1 must satisfy the stable manifold equation

$$\varphi_1\left(\mu_1\xi + \tilde{f}_1\left(\xi, \varphi_1(\xi)\right)\right) = \mu_2\varphi_1(\xi) + \tilde{g}_1\left(\xi, \varphi_1(\xi)\right),$$

This leads to the following polynomial equation

$$p_1\xi^2 + p_2\xi^3 + \cdots + p_{18}\xi^{18} = 0$$

where the coefficients p_1 and p_2, obtain by using *Mathematica* are in appendix A. Substituting $\bar{x}_2$ into (42) and (43) and solving system $p_1 = 0$ and $p_2 = 0$, we obtain the values

$$\alpha_1 = \frac{-8c^2}{\Upsilon_1(c,d) + \Upsilon_2(c,d)\sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

and

$$\beta_1 = \frac{4\alpha_1 c(4c + (d+1)(3d-1))}{\Upsilon_3(c,d) + \Upsilon_4(c,d)\sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

where

$$\begin{aligned}
\Upsilon_1(c,d) =& \sqrt{(d-1)^2 - 4c}\left(d^6 - 4c\left(13d^2 - 4d + 8\right)\right) \\
& - 4c\left(7d^4 - 12d^3 + 17d^2 - 12d + 8\right) + (d-1)d^6 + 64c^2, \\
\Upsilon_2(c,d) =& \sqrt{(d-1)^2 - 4c}\left(4c(5d+2) - d^5\right) + 4c\left(5d^3 - 4d^2 + 3d + 2\right) + (1-d)d^5, \\
\Upsilon_3(c,d) =& \sqrt{(d-1)^2 - 4c}\left(-4c\left(5d^2 - 12d + 16\right) - 15d^4 + 27d^3 - 13d^2 - 32d + 24\right) \\
& + 64c^2 - 4c\left(d^4 - 6d^3 + 7d^2 - 32d + 28\right) - 3d^6 + 16d^5 - 42d^4 + 40d^3 + 19d^2 - 56d + 24, \\
\Upsilon_4(c,d) =& \sqrt{(d-1)^2 - 4c}\left(4c(3d-2) + 9d^3 + d^2 - 3d + 6\right) + 3d^5 - 10d^4 + 8d^3 + 4d^2 - 9d + 6 \\
& + 4c\left(d^3 - 4d^2 + 5d - 6\right).
\end{aligned} \tag{24}$$

Since

$$\eta_n = \alpha_1 \xi_n^2 + \beta_1 \xi_n^3,$$
$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

and

$$u_n = x_{n-1} - \bar{x}_2 \text{ and } v_n = x_n - \bar{x}_2$$

we can approximate locally the local stable manifold $\mathcal{W}^s_{loc}(\bar{x}_2, \bar{x}_2)$ as the graph of the map $\tilde{\varphi}_1(u)$ such that $S(u, \tilde{\varphi}_1(u)) = 0$ where

$$\begin{aligned} S(u,v) :=& \alpha_1 \left(\frac{(v-\bar{x}_2)\left(\sqrt{8c\bar{x}_2+d^2}-d\right)}{2\sqrt{8c\bar{x}_2+d^2}} - \frac{2c\bar{x}_2\,(u-\bar{x}_2)}{\sqrt{8c\bar{x}_2+d^2}} \right)^2 \\ &+ \beta_1 \left(\frac{(v-\bar{x}_2)\left(\sqrt{8c\bar{x}_2+d^2}-d\right)}{2\sqrt{8c\bar{x}_2+d^2}} - \frac{2c\bar{x}_2\,(u-\bar{x}_2)}{\sqrt{8c\bar{x}_2+d^2}} \right)^3 \\ &- \frac{2c\bar{x}_2\,(u-\bar{x}_2)}{\sqrt{8c\bar{x}_2+d^2}} - \frac{(v-\bar{x}_2)\left(\sqrt{8c\bar{x}_2+d^2}+d\right)}{2\sqrt{8c\bar{x}_2+d^2}} \end{aligned} \tag{25}$$

and which satisfies

$$\tilde{\varphi}_1(\bar{x}_2) = \bar{x}_2 \text{ and } \tilde{\varphi}_1'(\bar{x}_2) = -\frac{4c\bar{x}_2}{\sqrt{8c\bar{x}_2+d^2}+d}.$$

Let us assume that the local unstable manifold is the graph of the map ψ that has the form

$$\psi_1(\eta) = \alpha_2\eta^2 + \beta_2\eta^3 + O(|\eta|^4), \quad \alpha_1, \beta_1 \in \mathbb{R}.$$

Now we compute the constants α_2 and β_2. The function ψ_1 must satisfy the unstable manifold equation

$$\psi_1(\mu_2\eta + \tilde{g}_1(\psi_1(\eta), \eta)) = \mu_1\psi_1(\eta) + \tilde{f}_1(\psi_1(\eta), \eta),$$

This leads to the following polynomial equation

$$q_1\eta^2 + q_2\eta^3 + \cdots + q_{18}\eta^{18} = 0$$

where the coefficients q_1 and q_2 are in appendix A.

Substituting $\bar{x}_2$ into (44) and (45) and solving system $q_1 = 0$ and $q_2 = 0$, we obtain the values

$$\alpha_2 = \frac{-8c^2}{\Gamma_1(c,d) + \Gamma_2(c,d)\sqrt{4\sqrt{(d-1)^2-4c}+d^2-4d+4}}$$

and

$$\beta_2 = \frac{\alpha_2\Gamma_5(c,d)}{\Gamma_3(c,d) + \Gamma_4(c,d)\sqrt{4\sqrt{(d-1)^2-4c}+d^2-4d+4}}$$

where

$$\begin{aligned} \Gamma_1(c,d) =& 64c^2 + \sqrt{(d-1)^2-4c}\left(d^6 - 4c\left(13d^2-4d+8\right)\right) \\ &- 4c\left(7d^4-12d^3+17d^2-12d+8\right) + (d-1)d^6, \\ \Gamma_2(c,d) =& \sqrt{(d-1)^2-4c}\left(d^5-4c(5d+2)\right) - 4c\left(5d^3-4d^2+3d+2\right) + (d-1)d^5, \\ \Gamma_3(c,d) =& 256c^2 + \sqrt{(d-1)^2-4c}\left(\left(d^2-8d+8\right)^2\left(d^2-2d+3\right) - 32c\left(3d^2-8d+10\right)\right) \\ &- 4c\left(9d^4-72d^3+208d^2-304d+176\right) - \left(d^2-8d+8\right)^2\left(d^3-3d^2+5d-3\right), \\ \Gamma_4(c,d) =& d\sqrt{(d-1)^2-4c}\left(-48c+d^4-14d^3+61d^2-88d+40\right) \\ &- d\left(4c\left(7d^2-36d+32\right) + \left(d^3-13d^2+48d-40\right)(d-1)^2\right), \\ \Gamma_5(c,d) =& \sqrt{4\sqrt{(d-1)^2-4c}+d^2-4d+4}\left(-2c(d-2)^2 - 8c\sqrt{(d-1)^2-4c}\right) \\ &- 4c\left(d^2-10d+8\right)\sqrt{(d-1)^2-4c} + 2c\left(32c+3d^3-22d^2+36d-16\right). \end{aligned} \tag{26}$$

Since

$$\xi_n = \alpha_2\eta_n^2 + \beta_2\eta_n^3,$$
$$\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix} = P^{-1}\cdot\begin{pmatrix}u_n\\ v_n\end{pmatrix},$$

and

$$u_n = x_{n-1} - \bar{x}_2 \text{ and } v_n = x_n - \bar{x}_2$$

we can approximate locally the local unstable manifold $\mathcal{W}^u_{loc}(\bar{x}_2,\bar{x}_2)$ as the graph of the map $\tilde{\psi}_1(u)$ such that $U(\tilde{\psi}_1(v), v) = 0$ where

$$\begin{aligned} U(u,v) :=& \alpha_2\left(\frac{2c\bar{x}_2\,(u-\bar{x}_2)}{\sqrt{8c\bar{x}_2+d^2}} + \frac{(v-\bar{x}_2)\left(\sqrt{8c\bar{x}_2+d^2}+d\right)}{2\sqrt{8c\bar{x}_2+d^2}}\right)^2 \\ &+ \beta_2\left(\frac{2c\bar{x}_2\,(u-\bar{x}_2)}{\sqrt{8c\bar{x}_2+d^2}} + \frac{(v-\bar{x}_2)\left(\sqrt{8c\bar{x}_2+d^2}+d\right)}{2\sqrt{8c\bar{x}_2+d^2}}\right)^3 \\ &+ \frac{2c\bar{x}_2\,(u-\bar{x}_2)}{\sqrt{8c\bar{x}_2+d^2}} - \frac{(v-\bar{x}_2)\left(\sqrt{8c\bar{x}_2+d^2}-d\right)}{2\sqrt{8c\bar{x}_2+d^2}} \end{aligned} \tag{27}$$

and which satisfies

$$\tilde{\psi}_1(\bar{x}_2) = \bar{x}_2 \text{ and } \tilde{\psi}_1'(\bar{x}_2) = \frac{4c\bar{x}_2}{\sqrt{8c\bar{x}_2+d^2}-d}.$$

Thus we proved the following result

Theorem 8 *Consider Eq.*(1) *subject to the condition* $\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}$. *Then the local stable and unstable manifolds are given with the asymptotic expansions* (25) *and* (27) *respectively.*

3.3 Some numerical examples

In this section we provide some numerical examples and we compare visually the asymptotic approximations of stable and unstable manifolds, obtained by using *Mathematica*, with the boundaries of the basins of attraction obtained by using the software package *Dynamica* 3 [6].
For $c = 0.06$ and $d = 0.3$ we have that

$$\begin{aligned} S_1(u,v) =& -0.0000205931x^3 + x^2(0.0000492004y + 0.0322121) \\ &+ x\left(-0.0000391827y^2 - 0.0513067y + 0.411741\right) \\ &+ 0.0000104016y^3 + 0.0204302y^2 + 0.672096y - 10.9718 \end{aligned}$$

and for $c = 0.075$ and $d = 0.42$

$$\begin{aligned} S_2(u,v) =& y^2(0.0227887 - 0.000731949x) + 0.000814449(x-62.2685)xy \\ &- 0.000302082(x-105.844)x(x+12.4415) \\ &+ 0.000219269y^3 + 0.642493y - 6.35325. \end{aligned}$$

Figures 1 and 2 show the graph of the functions $S_1(u,v) = 0$ and $S_2(u,v) = 0$ with the basins of attraction created with *Dynamica* 3. Figure 3 shows the graph of the functions $S_1(u,v) = 0$ and $S_2(u,v) = 0$ for different values of the parameters c and d.
For $c = 0.06$ and $d = 0.3$ we have that

$$\begin{aligned} U_1(u,v) =& (0.000113205x - 0.00441057)y^2 + 0.000108186(x-77.922)xy \\ &+ 0.0000344633(x-183.477)x(x+66.5943) \\ &+ 0.0000394854y^3 + 0.559375y + 0.00870931 \end{aligned}$$

and for $c = 0.075$ and $d = 0.42$

$$\begin{aligned} U_2(u,v) =& (0.000446197x - 0.010175)y^2 + 0.000309085(x-45.6077)xy \\ &+ 0.000071369(x-111.063)x(x+42.6512) \\ &+ 0.00021471y^3 + 0.511958y - 0.265003 \end{aligned}$$

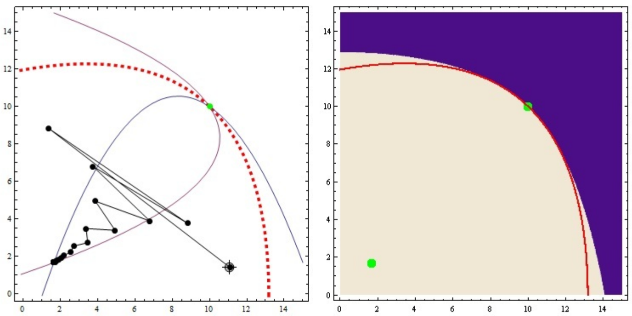

Figure 1: The graph of the function $S_1(u,v)=0$ (red curve) for $c=0.06$ and $d=0.3$ with the basins of attraction generated by *Dynamica* 3.

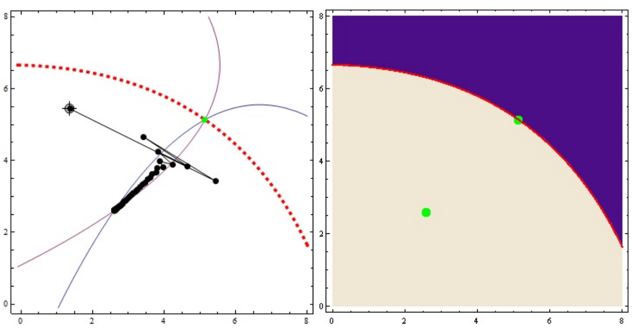

Figure 2: The graph of the function $S_2(u,v)=0$ (red curve) for $c=0.075$ and $d=0.42$ with the basins of attraction generated by *Dynamica* 3.

3.4 Normal form of the map T^2 at the period-two solution

The period-two solution of (1) is given as

$$\bar{u}_1=\frac{d+1-D}{2c} \text{ and } \bar{v}_1=\frac{d+1+D}{2c},$$

where

$$D=\sqrt{1-4c-2d-3d^2}.$$

First we transform the period two solution $(\bar{u}_1,\bar{v}_1)$ of (1) to the origin by the translation

$$\tilde{u}=u-\bar{u}_1 \text{ and } \tilde{v}=v-\bar{v}_1$$

under which the corresponding map (5) becomes

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}\to\tilde{F}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=T^2\begin{pmatrix}\tilde{u}+\bar{u}_1\\ \tilde{v}+\bar{v}_1\end{pmatrix}-\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=\begin{pmatrix}c\tilde{u}^2+\tilde{u}(d-D+1)+d\tilde{v}\\ \left(c\tilde{u}^2-D\tilde{u}+\tilde{u}+\tilde{v}\right)+\tilde{v}(c\tilde{v}+D+1)+d^2(\tilde{u}+\tilde{v})\end{pmatrix}. \tag{28}$$

Then $\tilde{F}$ has the fixed point at $(0,0)$. The Jacobian matrix of $\tilde{F}$ is given by

$$Jac_{\tilde{F}}(\tilde{u},\tilde{v})=\begin{pmatrix}d-D+2c\tilde{u}+1 & d\\ d^2+(-D+2c\tilde{u}+1)d & d^2+d+D+2c\tilde{v}+1\end{pmatrix}.$$

At $(0,0)$, $Jac_{\tilde{F}}(\tilde{u},\tilde{v})$ has the form

$$J_0=Jac_{\tilde{F}}(0,0)=\begin{pmatrix}d-D+1 & d\\ d^2+(1-D)d & d^2+d+D+1\end{pmatrix}. \tag{29}$$

The eigenvalues of (29) are

$$\nu_1=\frac{1}{2}(d(d+2)+2-C) \text{ and } \nu_2=\frac{1}{2}(d(d+2)+2+C),$$

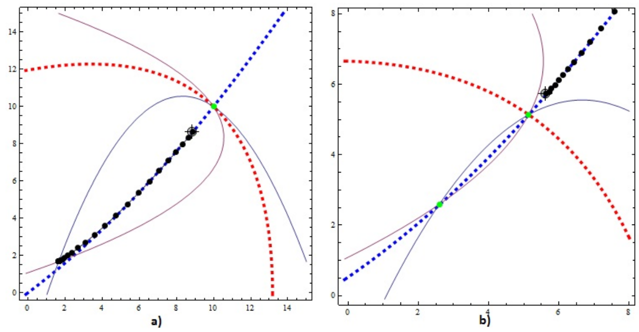

Figure 3: a) The graph of the functions $S_1(u,v)=0$ (red curve) and $U_1(u,v)=0$ (blue curve) for $c=0.06$ and $d=0.3$. b) The graph of the functions $S_2(u,v)=0$ (red curve) and $U_2(u,v)=0$ (blue curve) for $c=0.075$ and $d=0.42$.

where

$$C=\sqrt{-16c+(d-2)d(d(d+6)+4)+4}.$$

The eigenvectors corresponding to the eigenvalues $\nu_{1,2}$ are given by

$$\mathbf{v}_1=\left(\frac{2d}{-C+d^2+2D},1\right)^T \text{ and } \mathbf{v}_2=\left(\frac{2d}{C+d^2+2D},1\right)^T,$$

respectively.

Then we have that

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}\to\tilde{F}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=\begin{pmatrix} d-D+1 & d\\ d^2+(1-D)d & d^2+d+D+1\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}+\begin{pmatrix}f_2(\tilde{u},\tilde{v})\\ g_2(\tilde{u},\tilde{v})\end{pmatrix}, \tag{30}$$

where

$$f_2(\tilde{u},\tilde{v})=c\tilde{u}^2,\quad g_2(\tilde{u},\tilde{v})=c(d\tilde{u}^2+\tilde{v}^2).$$

Let

$$\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=P\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

where

$$P=\begin{pmatrix}\frac{2d}{d^2-C+2D} & \frac{2d}{d^2+C+2D}\\ 1 & 1\end{pmatrix} \text{ and } P^{-1}=\begin{pmatrix}\frac{\left(d^2+2D\right)^2-C^2}{4Cd} & -\frac{d^2-C+2D}{2C}\\ \frac{C^2-\left(d^2+2D\right)^2}{4Cd} & \frac{d^2+C+2D}{2C}\end{pmatrix}.$$

Then (30) leads to the corresponding normal form

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\begin{pmatrix}\nu_1 & 0\\ 0 & \nu_2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}+P^{-1}\cdot H_2\left(P\cdot\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right), \tag{31}$$

where

$$H_2\begin{pmatrix}u\\ v\end{pmatrix}:=\begin{pmatrix}f_2(u,v)\\ g_2(u,v)\end{pmatrix}.$$

Let

$$G_2\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\tilde{f}_2(u,v)\\ \tilde{g}_2(u,v)\end{pmatrix}=P^{-1}\cdot H_2\left(P\cdot\begin{pmatrix}u\\ v\end{pmatrix}\right).$$

By straightforward calculation we obtain that

$$\tilde{f}_2(u,v)=c\Upsilon_2(u,v)\left(\left(d^2+2D\right)^2-C^2\right),$$

$$\tilde{g}_2(u,v)=-c\Upsilon_2(u,v)\left(\left(d^2+2D\right)^2-C^2\right),$$

where

$$\Upsilon_2(u,v)=\frac{1}{2C}\left(2d\left(\frac{u}{-C+d^2+2D}+\frac{v}{C+d^2+2D}\right)^2-\frac{4d^3\left(\frac{u}{-C+d^2+2D}+\frac{v}{C+d^2+2D}\right)^2+(u+v)^2}{C+d^2+2D}\right).$$

3.5 Stable and unstable manifolds corresponding to the saddle period-two solution

If $c<\frac{(1-3d)(d+1)}{4}$ then Eq.(1) has minimal period-two solution $\{P_1(\bar{u}_1,\bar{v}_1),P_2(\bar{v}_1,\bar{u}_1)\}$ which is a saddle point.

In view of the fact that the T^2 is cooperative map if T is cooperative map, we can assume that the stable manifold W^s_{loc} at the period-two solution $(0,0)$, which corresponding to $(\bar{u}_1,\bar{v}_1)$, is the graph of the map

$$\varphi_2(\xi)=\alpha_3\xi^2+\beta_3\xi^3,\quad \alpha_3,\beta_3\in\mathbb{R}.$$

Now we compute the constants α_3 and β_3. The function φ_2 must satisfy the stable manifold equation

$$\varphi_2\left(\mu_1\xi+\tilde{g}_1\left(\xi,\varphi_2(\xi)\right)\right)=\mu_2\varphi_2(\xi)+\tilde{g}_2\left(\xi,\varphi_2(\xi)\right),$$

This leads to the following polynomial equation

$$\tilde{p}_1\xi^2+\tilde{p}_2\xi^3+\cdots+\tilde{p}_{18}\xi^{18}=0$$

where

$$\begin{aligned}\tilde{p}_1=&\frac{1}{4}\alpha_3(-C+d(d+2)+2)^2+\frac{1}{2}\alpha_3(-C-d(d+2)-2)\\&+\frac{cd\left(\left(d^2+2D\right)^2-C^2\right)}{C\left(-C+d^2+2D\right)^2}-\frac{c\left(\left(d^2+2D\right)^2-C^2\right)}{2C\left(-C+d^2+2D\right)}-\frac{2cd^3\left(\left(d^2+2D\right)^2-C^2\right)}{C\left(-C+d^2+2D\right)^3}\end{aligned}\tag{32}$$

and

$$\begin{aligned}\tilde{p}_2=&\frac{1}{8}\beta_3\left(-C^3+3C^2(d(d+2)+2)-C(3d(d+2)(d(d+2)+4)+16)\right.\\&\left.+d(d+2)(d(d+2)+2)(d(d+2)+4)\right)+\frac{4\alpha_3c(-d(d+2)-4)\left((d+2)d^3+4(d-1)dD+4D^2\right)}{2C\left(-C+d^2+2D\right)}\\&+\frac{4\alpha_3c\left(C^3-C^2(d(3d+4)+4D)+C\left(3d^4+8d^3+8d^2(D+1)+4dD+4D(D+2)\right)\right)}{2C\left(-C+d^2+2D\right)}.\end{aligned}\tag{33}$$

By solving system $\tilde{p}_1=0$ and $\tilde{p}_2=0$, we obtain the values

$$\alpha_3=\frac{2c\left(C+d^2+2D\right)\left(d^2(4D-2C)+2d(C-2D)+(C-2D)^2+d^4+2d^3\right)}{C\left(C^2-2C(d(d+2)+3)+d(d+2)(d(d+2)+2)\right)\left(-C+d^2+2D\right)^2}$$

and

$$\beta_3=\frac{\alpha_3\Phi_1(c,d)}{\Phi_2(c,d)},$$

where

$$\begin{aligned}\Phi_1(c,d)=&4c\left(-C^3+C^2(d(3d+4)+4D)-C\left(3d^4+8d^3+8d^2(D+1)+4dD+4D(D+2)\right)\right.\\&\left.+(d(d+2)+4)\left((d+2)d^3+4(d-1)dD+4D^2\right)\right),\end{aligned}\tag{34}$$

$$\begin{aligned}\Phi_2(c,d)=&4C\left(-C^3+3C^2(d(d+2)+2)-C(3d(d+2)(d(d+2)+4)+16)\right.\\&\left.+d(d+2)(d(d+2)+2)(d(d+2)+4)\right)\left(-C+d^2+2D\right).\end{aligned}\tag{35}$$

Let us assume that the unstable manifold at the period two solution $(0,0)$, which corresponding to $(\bar{u}_1,\bar{v}_1)$, is the graph of the map

$$\psi_2(\eta)=\alpha_4\eta^2+\beta_4\eta^3,\quad \alpha_3,\beta_3\in\mathbb{R}.$$

Now we compute the constants α_4 and β_4. The function ψ_2 must satisfy the stable manifold equation

$$\psi_2(\mu_2\eta + \tilde{g}_2(\psi_2(\eta), \eta)) = \mu_1\psi_2(\eta) + \tilde{f}_2(\psi_2(\eta), \eta),$$

This leads to the following polynomial equation

$$\tilde{q}_1\eta^2 + \tilde{q}_2\eta^3 + \cdots + \tilde{q}_{18}\eta^{18} = 0$$

where

$$\begin{aligned}\tilde{q}_1 = &\frac{1}{4}A(C + d(d+2) + 2)^2 + \frac{1}{2}A(C - d(d+2) - 2)\\ &- \frac{cd\left(\left(d^2+2D\right)^2 - C^2\right)}{C\left(C+d^2+2D\right)^2} + \frac{c\left(\left(d^2+2D\right)^2 - C^2\right)}{2C\left(C+d^2+2D\right)} + \frac{2cd^3\left(\left(d^2+2D\right)^2 - C^2\right)}{C\left(C+d^2+2D\right)^3}\end{aligned} \quad (36)$$

and

$$\begin{aligned}\tilde{q}_2 = &\frac{1}{8}B\left(C^3 + 3C^2(d(d+2)+2) + C(3d(d+2)(d(d+2)+4)+16)\right.\\ &\left.+d(d+2)(d(d+2)+2)(d(d+2)+4)\right) + \frac{\alpha_3 c(d(d+2)+4)\left((d+2)d^3 + 4(d-1)dD + 4D^2\right)}{2C\left(C+d^2+2D\right)}\\ &+ \frac{Ac\left(C^3 + C^2(d(3d+4)+4D) + C\left(3d^4 + 8d^3 + 8d^2(D+1) + 4dD + 4D(D+2)\right)\right)}{2C\left(C+d^2+2D\right)}.\end{aligned} \quad (37)$$

By solving system $\tilde{q}_1 = 0$ and $\tilde{q}_2 = 0$, we obtain the values

$$\alpha_4 = -\frac{2c\left(-C+d^2+2D\right)\left(2d^2(C+2D) - 2d(C+2D) + (C+2D)^2 + d^4 + 2d^3\right)}{C\left(C^2 + 2C(d(d+2)+3) + d(d+2)(d(d+2)+2)\right)\left(C+d^2+2D\right)^2}$$

and

$$\beta_4 = \frac{\alpha_4\tilde{\Phi}_1(c,d)}{\tilde{\Phi}_2(c,d)},$$

where

$$\begin{aligned}\tilde{\Phi}_1(c,d) = 4c&\left(-C^3 - C^2(d(3d+4)+4D)\right.\\ &-C\left(3d^4 + 8d^3 + 8d^2(D+1) + 4dD + 4D(D+2)\right) + (-d(d+2)-4)\\ &\left.\left((d+2)d^3 + 4(d-1)dD + 4D^2\right)\right),\end{aligned} \quad (38)$$

$$\begin{aligned}\tilde{\Phi}_2(c,d) = &C\left(C^3 + 3C^2(d(d+2)+2)\right.\\ &\left.+C(3d(d+2)(d(d+2)+4)+16) + d(d+2)(d(d+2)+2)(d(d+2)+4)\right)\left(C+d^2+2D\right).\end{aligned} \quad (39)$$

As in the case of the saddle point equilibrium, one can show that we can approximate locally local stable manifold $\mathcal{W}^s_{loc}(\bar{u}_1, \bar{v}_1)$ and local unstable manifold $\mathcal{W}^u_{loc}(\bar{u}_1, \bar{v}_1)$ as the graph of the maps $\tilde{\varphi}_2(u)$ and $\tilde{\psi}_2(u)$ such that $\tilde{S}(u, \tilde{\varphi}_2(u)) = 0$ and $\tilde{U}(\tilde{\psi}_2(v), v) = 0$ hold, where

$$\begin{aligned}\tilde{S}(u,v) := \alpha_3&\left(\frac{(u-\bar{u}_1)\left(\left(d^2+2D\right)^2 - C^2\right)}{4Cd} - \frac{(v-\bar{v}_1)\left(-C+d^2+2D\right)}{2C}\right)^2\\ &+\beta_3\left(\frac{(u-\bar{u}_1)\left(\left(d^2+2D\right)^2 - C^2\right)}{4Cd} - \frac{(v-\bar{v}_1)\left(-C+d^2+2D\right)}{2C}\right)^3\\ &-\frac{(u-\bar{u}_1)\left(C^2 - \left(d^2+2D\right)^2\right)}{4Cd} - \frac{(v-\bar{v}_1)\left(C+d^2+2D\right)}{2C},\end{aligned} \quad (40)$$

$$\tilde{U}(u,v) := \alpha_4 \left(\frac{(u-\bar{u}_1)\left(C^2-\left(d^2+2D\right)^2\right)}{4Cd} + \frac{(v-\bar{v}_1)\left(C+d^2+2D\right)}{2C} \right)^2 + \beta_4 \left(\frac{(u-\bar{u}_1)\left(C^2-\left(d^2+2D\right)^2\right)}{4Cd} + \frac{(v-\bar{v}_1)\left(C+d^2+2D\right)}{2C} \right)^3 - \frac{(u-\bar{u}_1)\left(\left(d^2+2D\right)^2-C^2\right)}{4Cd} + \frac{(v-\bar{v}_1)\left(-C+d^2+2D\right)}{2C} \quad (41)$$

Thus we proved the following result

Theorem 9 *Consider Eq.*(1) *subject to the condition* $c < \frac{(1-3d)(d+1)}{4}$. *Then the local stable and local unstable manifolds of the unique period-two solution are given with the asymptotic expansions* (40) *and* (41) *respectively.*

3.6 Some numerical examples

For $c = 0.09$ and $d = 0.23$ we have that

$$\begin{aligned}\tilde{S}_1(u,v) =& 0.147835(0.207875(v-7.64414) - 0.422497(u-6.02253))^3 \\ &- 0.418625(0.207875(v-7.64414) - 0.422497(u-6.02253))^2 \\ &- 0.422497(u-6.02253) - 0.792125(v-7.64414)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_1(u,v) =& -0.00042726(0.422497(u-6.02253) + 0.792125(v-7.64414))^3 \\ &+ 0.00729364(0.422497(u-6.02253) + 0.792125(v-7.64414))^2 \\ &+ 0.422497(u-6.02253) - 0.207875(v-7.64414).\end{aligned}$$

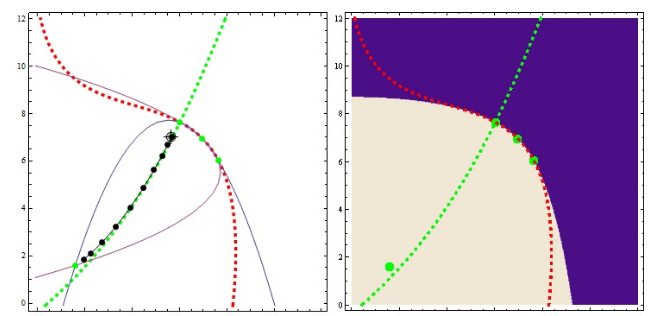

Figure 4: The graphs of the functions $\tilde{S}_1(u,v) = 0$ (red curve) and $\tilde{U}_1(u,v) = 0$ (green curve) for $c = 0.09$ and $d = 0.23$ with the basins of attraction generated by *Dynamica* 3.

For $c = 0.03$ and $d = 0.22$ we have that

$$\begin{aligned}\tilde{S}_2(u,v) =& 0.0912789(0.0236741(v-29.3826) - 0.125096(u-11.2841))^3 \\ &- 0.458495(0.0236741(v-29.3826) - 0.125096(u-11.2841))^2 \\ &- 0.125096(u-11.2841) - 0.976326(v-29.3826)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_2(u,v) =& 3.7\times 10^{-6}(0.125096(u-11.2841) + 0.976326(v-29.3826))^3 \\ &+ 0.000212577(0.125096(u-11.2841) + 0.976326(v-29.3826))^2 \\ &+ 0.125096(u-11.2841) - 0.0236741(v-29.3826)\end{aligned}$$

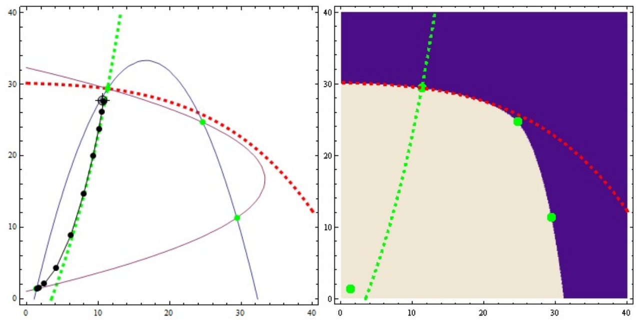

Figure 5: The graphs of the functions $\tilde{S}_2(u, v) = 0$ (red curve) and $\tilde{U}_2(u, v) = 0$ (green curve) for $c = 0.03$ and $d = 0.22$ with the basins of attraction generated by *Dynamica* 3.

Figures 4 and 5 show the graph of the functions $\tilde{S}_2(u, v) = 0$ and $\tilde{U}_2(u, v) = 0$ with the basins of attraction created with *Dynamica* 3. for different values of the parameters c and d.

References

[1] J. Bektešević, M. R. S. Kulenović and E. Pilav, Global Dynamics of Quadratic Second Order Difference Equation in the First Quadrant, *Appl. Math. Comp.*, 227(2014), 50–65.

[2] A. Brett and M. R. S. Kulenović, Basins of attraction of equilibrium points of monotone difference equations. *Sarajevo J. Math.* 5(18) (2009), 211–233.

[3] S. Elaydi, *Discrete chaos. With applications in science and engineering.* Second edition. With a foreword by Robert M. May., Chapman & Hall/CRC, Boca Raton, FL, 2008.

[4] M. Guzowska, R. Luis and S. Elaydi, Bifurcation and invariant manifolds of the logistic competition model, *Journal of Difference Equations and Applications*, 2011, 17:12, 1851–1872.

[5] J. K. Hale and H. Kocak, *Dynamics and bifurcations.* Texts in Applied Mathematics, 3. Springer-Verlag, New York, 1991.

[6] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[7] M. R. S. Kulenović and O. Merino, Competitive-Exclusion versus Competitive-Coexistence for Systems in the Plane, *Discrete Contin. Dyn. Syst. Ser. B* 6(2006), 1141-1156.

[8] M. R. S. Kulenović and O. Merino, Global Bifurcation for Competitive Systems in the Plane, *Discrete Contin. Dyn. Syst. B* 12(2009), 133-149.

[9] M. R. S. Kulenović and O. Merino, Invariant manifolds for competitive discrete systems in the plane. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 20 (2010), no. 8, 2471–2486.

[10] H. A. Lauwerier, Two-dimensional iterative maps. *Chaos, 5895, Nonlinear Sci. Theory Appl.*, Manchester Univ. Press, Manchester, 1986.

[11] C. Robinson, *Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1995.

[12] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations* 64 (1986), 165-194.

[13] H. L. Smith, Planar Competitive and Cooperative Difference Equations,*J. Differ. Equ. Appl.* 3(1998), 335-357.

[14] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos.* Second edition. Texts in Applied Mathematics, 2. Springer-Verlag, New York, 2003.

A Values of coefficients p_1, p_2, q_1 and q_2

$$
\begin{aligned}
p_1 =& -12c\beta_1\bar{x}_2^3d^6 - 12c^2\beta_1\bar{x}_2^4d^5 + 12c\beta_1\bar{x}_2^3d^5 - 4c\alpha_1\bar{x}_2^2d^5 - 12c\beta_1\bar{x}_2^2d^5 - d^5 - 168c^2\beta_1\bar{x}_2^4d^4 \\
& - 4c\alpha_1\bar{x}_2^2d^4 - 168c^3\beta_1\bar{x}_2^5d^3 + 156c^2\beta_1\bar{x}_2^4d^3 - 64c^2\alpha_1\bar{x}_2^3d^3 - 168c^2\beta_1\bar{x}_2^3d^3 - 14c\bar{x}_2d^3 \\
& - 600c^3\beta_1\bar{x}_2^5d^2 + 24c^2\beta_1\bar{x}_2^4d^2 - 64c^2\alpha_1\bar{x}_2^3d^2 - 24c^2\beta_1\bar{x}_2^3d^2 - 588c^4\beta_1\bar{x}_2^6d + 600c^3\beta_1\bar{x}_2^5d \\
& - 256c^3\alpha_1\bar{x}_2^4d - 600c^3\beta_1\bar{x}_2^4d - 12c^2\beta_1\bar{x}_2^4d + 24c^2\beta_1\bar{x}_2^3d - 48c^2\bar{x}_2^2d - 12c^2\beta_1\bar{x}_2^2d - 256c^3\alpha_1\bar{x}_2^4 \\
& + \sqrt{d^2+8c\bar{x}_2}\,\big(204c^4\beta_1\bar{x}_2^6 - 216c^3\beta_1\bar{x}_2^5 + 120c^3d^2\beta_1\bar{x}_2^5 + 216c^3d\beta_1\bar{x}_2^5 + 144c^3\alpha_1\bar{x}_2^4 + 12c^2d^4\beta_1\bar{x}_2^4 \\
& + 216c^3\beta_1\bar{x}_2^4 + 120c^2d^3\beta_1\bar{x}_2^4 + 12c^2\beta_1\bar{x}_2^4 - 108c^2d^2\beta_1\bar{x}_2^4 - 24c^2d\beta_1\bar{x}_2^4 - 16c^2\alpha_1\bar{x}_2^3 \\
& + 48c^2d^2\alpha_1\bar{x}_2^3 - 16c^2d\alpha_1\bar{x}_2^3 + 12cd^5\beta_1\bar{x}_2^3 - 12cd^4\beta_1\bar{x}_2^3 - 24c^2\beta_1\bar{x}_2^3 + 120c^2d^2\beta_1\bar{x}_2^3 + 24c^2d\beta_1\bar{x}_2^3 \\
& -16c^2\bar{x}_2^2 + 4cd^4\alpha_1\bar{x}_2^2 - 4cd^3\alpha_1\bar{x}_2^2 + 16c^2\alpha_1\bar{x}_2^2 + 12cd^4\beta_1\bar{x}_2^2 + 12c^2\beta_1\bar{x}_2^2 - 10cd^2\bar{x}_2 - d^4\big),
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
p_2 =& -\beta_1\bar{x}_2d^6 + 3\alpha_1\beta_1\bar{x}_2^2d^5 + 6c\alpha_1\beta_1\bar{x}_2^3d^4 - 18c\beta_1\bar{x}_2^2d^4 - 6\alpha_1\beta_1\bar{x}_2^2d^4 + 2\alpha_1^2\bar{x}_2d^4 + 6\alpha_1\beta_1\bar{x}_2d^4 \\
& - \beta_1\bar{x}_2d^4 + 3c^2\alpha_1\beta_1\bar{x}_2^4d^3 + 12c\alpha_1\beta_1\bar{x}_2^3d^3 + 2\alpha_1^2d^3 + 2c\alpha_1^2\bar{x}_2^2d^3 - 6c\beta_1\bar{x}_2^2d^3 + 6c\alpha_1\beta_1\bar{x}_2^2d^3 \\
& + 3\alpha_1\beta_1\bar{x}_2^2d^3 + \alpha_1d^3 + 3\alpha_1\beta_1d^3 - 2\alpha_1^2\bar{x}_2d^3 - 6\alpha_1\beta_1\bar{x}_2d^3 + 36c^2\alpha_1\beta_1\bar{x}_2^4d^2 - 102c^2\beta_1\bar{x}_2^3d^2 \\
& - 36c\alpha_1\beta_1\bar{x}_2^3d^2 + 16c\alpha_1^2\bar{x}_2^2d^2 - 10c\beta_1\bar{x}_2^2d^2 + 36c\alpha_1\beta_1\bar{x}_2^2d^2 - 4c\alpha_1\bar{x}_2d^2 - 6c\beta_1\bar{x}_2d^2 \\
& + 18c^3\alpha_1\beta_1\bar{x}_2^5d - 36c^2\alpha_1\beta_1\bar{x}_2^4d + 16c^2\alpha_1^2\bar{x}_2^3d - 48c^2\beta_1\bar{x}_2^3d + 36c^2\alpha_1\beta_1\bar{x}_2^3d + 18c\alpha_1\beta_1\bar{x}_2^3d \\
& - 16c\alpha_1^2\bar{x}_2^2d - 36c\alpha_1\beta_1\bar{x}_2^2d + 16c\alpha_1^2\bar{x}_2d + 8c\alpha_1\bar{x}_2d + 18c\alpha_1\beta_1\bar{x}_2d - 176c^3\beta_1\bar{x}_2^4 - 16c^2\beta_1\bar{x}_2^3 \\
& - 32c^2\alpha_1\bar{x}_2^2 - 48c^2\beta_1\bar{x}_2^2 + \sqrt{d^2+8c\bar{x}_2}\,\big(\beta_1\bar{x}_2d^5 - 3\alpha_1\beta_1\bar{x}_2^2d^4 - 6c\alpha_1\beta_1\bar{x}_2^3d^3 + 14c\beta_1\bar{x}_2^2d^3 \\
& + 6\alpha_1\beta_1\bar{x}_2^2d^3 - 2\alpha_1^2\bar{x}_2d^3 - 6\alpha_1\beta_1\bar{x}_2d^3 - \beta_1\bar{x}_2d^3 - 3c^2\alpha_1\beta_1\bar{x}_2^4d^2 - 2\alpha_1^2d^2 - 2c\alpha_1^2\bar{x}_2^2d^2 \\
& + 6c\beta_1\bar{x}_2^2d^2 - 6c\alpha_1\beta_1\bar{x}_2^2d^2 - 3\alpha_1\beta_1\bar{x}_2^2d^2 + \alpha_1d^2 - 3\alpha_1\beta_1d^2 + 2\alpha_1^2\bar{x}_2d^2 + 6\alpha_1\beta_1\bar{x}_2d^2 \\
& - 12c^2\alpha_1\beta_1\bar{x}_2^4d + 54c^2\beta_1\bar{x}_2^3d + 12c\alpha_1\beta_1\bar{x}_2^3d - 8c\alpha_1^2\bar{x}_2^2d - 14c\beta_1\bar{x}_2^2d - 12c\alpha_1\beta_1\bar{x}_2^2d \\
& + 6c\beta_1\bar{x}_2d - 6c^3\alpha_1\beta_1\bar{x}_2^5 + 12c^2\alpha_1\beta_1\bar{x}_2^4 - 8c^2\alpha_1^2\bar{x}_2^3 - 12c^2\alpha_1\beta_1\bar{x}_2^3 - 6c\alpha_1\beta_1\bar{x}_2^3 + 8c\alpha_1^2\bar{x}_2^2 \\
& +12c\alpha_1\beta_1\bar{x}_2^2 - 8c\alpha_1^2\bar{x}_2 + 8c\alpha_1\bar{x}_2 - 6c\alpha_1\beta_1\bar{x}_2\big)
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
q_1 =& 12\beta_2c\bar{x}_2^3d^7 + 12\beta_2c^2\bar{x}_2^4d^6 - 12\beta_2c\bar{x}_2^3d^6 + 4\alpha_2c\bar{x}_2^2d^6 + 12\beta_2c\bar{x}_2^2d^6 - d^6 + 216\beta_2c^2\bar{x}_2^4d^5 - 4\alpha_2c\bar{x}_2^2d^5 \\
& + 216\beta_2c^3\bar{x}_2^5d^4 - 204\beta_2c^2\bar{x}_2^4d^4 + 80\alpha_2c^2\bar{x}_2^3d^4 + 216\beta_2c^2\bar{x}_2^3d^4 - 18c\bar{x}_2d^4 + 1176\beta_2c^3\bar{x}_2^5d^3 \\
& - 24\beta_2c^2\bar{x}_2^4d^3 - 48\alpha_2c^2\bar{x}_2^3d^3 + 24\beta_2c^2\bar{x}_2^3d^3 + 1164\beta_2c^4\bar{x}_2^6d^2 - 1080\beta_2c^3\bar{x}_2^5d^2 + 528\alpha_2c^3\bar{x}_2^4d^2 \\
& + 1176\beta_2c^3\bar{x}_2^4d^2 + 12\beta_2c^2\bar{x}_2^4d^2 - 16\alpha_2c^2\bar{x}_2^3d^2 - 24\beta_2c^2\bar{x}_2^3d^2 + 16\alpha_2c^2\bar{x}_2^2d^2 + 12\beta_2c^2\bar{x}_2^2d^2 \\
& - 96c^2\bar{x}_2^2d^2 + 1728\beta_2c^4\bar{x}_2^6d - 192\beta_2c^3\bar{x}_2^5d - 128\alpha_2c^3\bar{x}_2^4d + 192\beta_2c^3\bar{x}_2^4d + 1632\beta_2c^5\bar{x}_2^7 \\
& - 1728\beta_2c^4\bar{x}_2^6 + 1152\alpha_2c^4\bar{x}_2^5 + 1728\beta_2c^4\bar{x}_2^5 + 96\beta_2c^3\bar{x}_2^5 - 128\alpha_2c^3\bar{x}_2^4 - 192\beta_2c^3\bar{x}_2^4 + 128\alpha_2c^3\bar{x}_2^3 \\
& + 96\beta_2c^3\bar{x}_2^3 - 128c^3\bar{x}_2^3 + \sqrt{d^2+8c\bar{x}_2}\,\big(12\beta_2c\bar{x}_2^3d^6 + 12\beta_2c^2\bar{x}_2^4d^5 - 12\beta_2c\bar{x}_2^3d^5 + 4\alpha_2c\bar{x}_2^2d^5 \\
& + 12\beta_2c\bar{x}_2^2d^5 + d^5 + 168\beta_2c^2\bar{x}_2^4d^4 + 4\alpha_2c\bar{x}_2^2d^4 + 168\beta_2c^3\bar{x}_2^5d^3 - 156\beta_2c^2\bar{x}_2^4d^3 + 64\alpha_2c^2\bar{x}_2^3d^3 \\
& + 168\beta_2c^2\bar{x}_2^3d^3 + 14c\bar{x}_2d^3 + 600\beta_2c^3\bar{x}_2^5d^2 - 24\beta_2c^2\bar{x}_2^4d^2 + 64\alpha_2c^2\bar{x}_2^3d^2 + 24\beta_2c^2\bar{x}_2^3d^2 \\
& + 588\beta_2c^4\bar{x}_2^6d - 600\beta_2c^3\bar{x}_2^5d + 256\alpha_2c^3\bar{x}_2^4d + 600\beta_2c^3\bar{x}_2^4d + 12\beta_2c^2\bar{x}_2^4d - 24\beta_2c^2\bar{x}_2^3d \\
& +12\beta_2c^2\bar{x}_2^2d + 48c^2\bar{x}_2^2d + 256\alpha_2c^3\bar{x}_2^4\big),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
q_2 =&\beta_2\bar{x}_2d^6 - 3\alpha_2\beta_2\bar{x}_2^2d^5 - 6c\alpha_2\beta_2\bar{x}_2^3d^4 + 18c\beta_2\bar{x}_2^2d^4 + 6\alpha_2\beta_2\bar{x}_2^2d^4 - 2\alpha_2^2\bar{x}_2d^4 - 6\alpha_2\beta_2\bar{x}_2d^4\\
&+ \beta_2\bar{x}_2d^4 - 3c^2\alpha_2\beta_2\bar{x}_2^4d^3 - 12c\alpha_2\beta_2\bar{x}_2^3d^3 - 2\alpha_2^2d^3 - 2c\alpha_2^2\bar{x}_2^2d^3 + 6c\beta_2\bar{x}_2^2d^3 - 6c\alpha_2\beta_2\bar{x}_2^2d^3\\
&- 3\alpha_2\beta_2\bar{x}_2^2d^3 - \alpha_2d^3 - 3\alpha_2\beta_2d^3 + 2\alpha_2^2\bar{x}_2d^3 + 6\alpha_2\beta_2\bar{x}_2d^3 - 36c^2\alpha_2\beta_2\bar{x}_2^4d^2 + 102c^2\beta_2\bar{x}_2^3d^2\\
&+ 36c\alpha_2\beta_2\bar{x}_2^3d^2 - 16c\alpha_2^2\bar{x}_2^2d^2 + 10c\beta_2\bar{x}_2^2d^2 - 36c\alpha_2\beta_2\bar{x}_2^2d^2 + 4c\alpha_2\bar{x}_2d^2 + 6c\beta_2\bar{x}_2d^2\\
&- 18c^3\alpha_2\beta_2\bar{x}_2^5d + 36c^2\alpha_2\beta_2\bar{x}_2^4d - 16c^2\alpha_2^2\bar{x}_2^3d + 48c^2\beta_2\bar{x}_2^3d - 36c^2\alpha_2\beta_2\bar{x}_2^3d - 18c\alpha_2\beta_2\bar{x}_2^3d\\
&+ 16c\alpha_2^2\bar{x}_2^2d + 36c\alpha_2\beta_2\bar{x}_2^2d - 16c\alpha_2^2\bar{x}_2d - 8c\alpha_2\bar{x}_2d - 18c\alpha_2\beta_2\bar{x}_2d + 176c^3\beta_2\bar{x}_2^4 + 16c^2\beta_2\bar{x}_2^3\\
&+ 32c^2\alpha_2\bar{x}_2^2 + 48c^2\beta_2\bar{x}_2^2 + \sqrt{d^2 + 8c\bar{x}_2}\,\big(\beta_2\bar{x}_2d^5 - 3\alpha_2\beta_2\bar{x}_2^2d^4 - 6c\alpha_2\beta_2\bar{x}_2^3d^3 + 14c\beta_2\bar{x}_2^2d^3\\
&+ 6\alpha_2\beta_2\bar{x}_2^2d^3 - 2\alpha_2^2\bar{x}_2d^3 - 6\alpha_2\beta_2\bar{x}_2d^3 - \beta_2\bar{x}_2d^3 - 3c^2\alpha_2\beta_2\bar{x}_2^4d^2 - 2\alpha_2^2d^2 - 2c\alpha_2^2\bar{x}_2^2d^2\\
&+ 6c\beta_2\bar{x}_2^2d^2 - 6c\alpha_2\beta_2\bar{x}_2^2d^2 - 3\alpha_2\beta_2\bar{x}_2^2d^2 + \alpha_2d^2 - 3\alpha_2\beta_2d^2 + 2\alpha_2^2\bar{x}_2d^2 + 6\alpha_2\beta_2\bar{x}_2d^2\\
&- 12c^2\alpha_2\beta_2\bar{x}_2^4d + 54c^2\beta_2\bar{x}_2^3d + 12c\alpha_2\beta_2\bar{x}_2^3d - 8c\alpha_2^2\bar{x}_2^2d - 14c\beta_2\bar{x}_2^2d - 12c\alpha_2\beta_2\bar{x}_2^2d\\
&+ 6c\beta_2\bar{x}_2d - 6c^3\alpha_2\beta_2\bar{x}_2^5 + 12c^2\alpha_2\beta_2\bar{x}_2^4 - 8c^2\alpha_2^2\bar{x}_2^3 - 12c^2\alpha_2\beta_2\bar{x}_2^3 - 6c\alpha_2\beta_2\bar{x}_2^3 + 8c\alpha_2^2\bar{x}_2^2\\
&+12c\alpha_2\beta_2\bar{x}_2^2 - 8c\alpha_2^2\bar{x}_2 + 8c\alpha_2\bar{x}_2 - 6c\alpha_2\beta_2\bar{x}_2\big)
\end{aligned}
\tag{45}
$$