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17. Action-Angle Coordinates

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Abstract

Part seventeen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc17]

Action-Angle Coordinates [mln92]

An elegant way of using Hamiltonian mechanics to solve a dynamical problem is to search for a canonical transformation to *action-angle* coordinates,

$$(q_1, \ldots, q_n; p_1, \ldots, p_n) \to (\underbrace{\theta_1, \ldots, \theta_n}_{\text{angles}}; \underbrace{J_1, \ldots, J_n}_{\text{actions}}),$$

such that the Hamiltonian turns into a function of the actions alone:

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n) \to K(J_1, \ldots, J_n).$$

If such a transformation exists and can be found then the solution of the canonical equations is simple:

$$\dot{J}_i = -\frac{\partial K}{\partial \theta_i} = 0 \quad \Rightarrow \; J_i = \text{const.}$$

$$\dot{\theta}_i = \frac{\partial K}{\partial J_i} \doteq \omega_i(J_1, \ldots, J_n) = \text{const.} \quad \Rightarrow \; \theta_i(t) = \omega_i t + \theta_i^{(0)}.$$

The inverse canonical transformation then yields $q_i(t)$ and $p_i(t)$.

Two or more degrees of freedom:
The existence of a transformation to action-angle coordinates is exceptional. Such systems are named *integrable*. Nonintegrable systems exhibit symptoms of *Hamiltonian chaos* (to be discussed later).

One degree of freedom:
Integrability is guaranteed. There exists a general prescription for finding the canonical transformation to action-angle coordinates.

The prescription for two modes of bounded motion is discussed in detail:

- libration (oscillation) [mln93],
- rotation [mln94].

The two modes are realized, for example, in the plane pendulum. The rotational motion can also be interpreted as unbounded motion in a periodic potential.

Actions and Angles for Librations [mln93]

Hamiltonian: $H(q,p) = \frac{p^2}{2m} + V(q) = E = \text{const.}$

Canonical momentum: $p(q,E) = \pm\sqrt{2m\,[E - V(q)]}$.

Action J and Hamiltonian $K(J)$ from area A inside trajectory:

$$A = \oint dq\, p(q,E) = 2\int_{q_1}^{q_2} dq \sqrt{2m\,[E - V(q)]} = \int_0^{2\pi} d\theta\, J = 2\pi J.$$

$$\Rightarrow\ J(E) = \frac{1}{\pi}\int_{q_1}^{q_2} dq \sqrt{2m\,[E - V(q)]} \quad \Rightarrow\ E = K(J) = H(p,q).$$

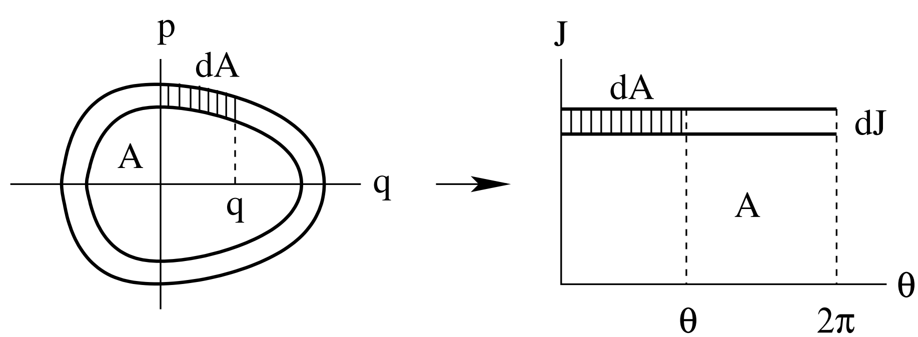
Angle variable $\theta(q,J)$ from area dA between nearby trajectories:

$$dA = \int_0^q dq\,[p(q,J+dJ) - p(q,J)] = dJ\int_0^q dq\,\frac{\partial}{\partial J}\,p(q,J) = dJ\theta(q,J).$$

$$\Rightarrow\ \theta(q,J) = \frac{\partial}{\partial J}\int_0^q dq\, p(q,J) = \frac{\partial}{\partial J}\int_0^q dq\,\sqrt{2m\,[K(J) - V(q)]}.$$

Time evolution: $J = \text{const.}, \quad \theta(t) = \omega(J)t + \theta_0, \quad \omega(J) = \frac{dK}{dJ}.$

$$\Rightarrow\ q(\theta,J) = q(t) \quad \Rightarrow\ p(q,J) = p(t).$$



Generating function of the canonical transformation $(q,p) \to (\theta,J)$:

$$F_2(q,J) = \int_0^q dq\, p(q,J).$$

Actions and Angles for Rotations [mln94]

Hamiltonian: $H(q,p) = \dfrac{p^2}{2m} + V(q) = E = \text{const.}$ with $V(q+Q_0) = V(q)$.

Canonical momentum: $p(q,E) = \sqrt{2m\,[E - V(q)]}$.

Action J and Hamiltonian $K(J)$ from area A under trajectory:

$$A = \int_0^{Q_0} dq\, p(q,E) = \int_0^{Q_0} dq \sqrt{2m\,[E - V(q)]} = \int_0^{2\pi} d\theta\, J = 2\pi J.$$

$$\Rightarrow\ J(E) = \frac{1}{2\pi}\int_0^{Q_0} dq \sqrt{2m\,[E - V(q)]} \quad \Rightarrow\ E = K(J) = H(p,q).$$

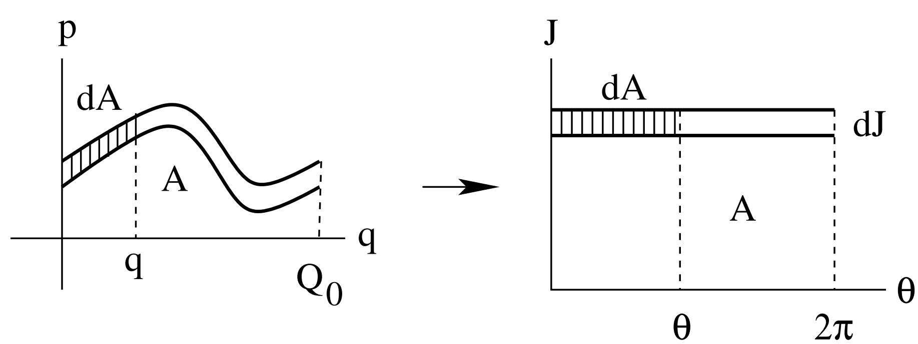
Angle variable $\theta(q,J)$ from area dA between nearby trajectories:

$$dA = \int_0^q dq\,[p(q,J+dJ) - p(q,J)] = dJ \int_0^q dq\,\frac{\partial}{\partial J}\,p(q,J) = dJ\theta(q,J).$$

$$\Rightarrow\ \theta(q,J) = \frac{\partial}{\partial J}\int_0^q dq\, p(q,J) = \frac{\partial}{\partial J}\int_0^q dq\,\sqrt{2m\,[K(J) - V(q)]}.$$

Time evolution: $J = \text{const.}$, $\quad \theta(t) = \omega(J)t + \theta_0$, $\quad \omega(J) = \dfrac{dK}{dJ}$.

$$\Rightarrow\ q(\theta,J) = q(t) \quad \Rightarrow\ p(q,J) = p(t).$$



In the case of rotations there is no natural boundary for J. Here J is only determined up to a constant.

[mex91] Action-angle coordinates of the harmonic oscillator

Determine the canonical transformation $(q,p) \to (\theta, J)$ which produces the action-angle coordinates for the harmonic oscillator:

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 \quad \to \quad K(J).$$

(a) Find the transformed Hamiltonian $K(J)$. (b) Find the transformation relations $q(\theta, J),\ p(\theta, J)$. (c) Reconstruct the generating function $F_2(q,J)$. (d) Determine from F_2 the generating function $F_1(q,\theta)$ and verify that it is equal to function $F_1(q,Q)$ used in [mex86].

Solution:

[mex92] Action-angle coordinates of an anharmonic oscillator

Determine the canonical transformation $(q,p) \to (\theta, J)$ which produces the action-angle coordinates for the anharmonic oscillator:

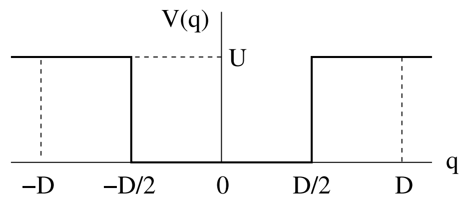
$$H(q,p) = \frac{p^2}{2m} + U\tan^2(\alpha q) \quad \to \quad K(J).$$

(a) Find the transformed Hamiltonian $K(J)$ and determine the angular frequency $\omega(J)$ which determines the linear time evolution $\theta(t) = \omega(J)t + \theta_0$ of the angle coordinates. (b) Find the transformation relations $q(\theta, J),\ p(\theta, J)$, which amount to a solution of the dynamical problem.

Solution:

[mex96] Unbounded motion in piecewise constant periodic potential

Consider a particle of mass m moving in the potential $V(q) = 0$ for $0 < |q| < D/2$ and $V(q) = U$ for $D/2 < |q| < D$ with periodicity $V(q + 2D) = V(q)$. For energies $E > U$ the motion is unbounded and can be reinterpreted as a rotational mode of bounded motion. Solve this dynamical problem via transformation $(q, p) \to (\theta, J)$ to action-angle coordinates for motion with initial conditions $q(0) = 0,\ p(0) > 0$: (a) Find the function $J(E)$, which expresses the action as a function of the energy. (b) Find the period $T \equiv 2\pi/\omega(E)$ of the rotational motion. (c) Find the function $\theta(q, E)$ for $0 < q < 2D$. (c) Plot in one diagram the functions $J = \text{const}$ and $p(t)$ for $0 < t < T$. (d) Plot in a second diagram the functions $q(t)$ and $\theta(t)$ for $0 < t < T$.



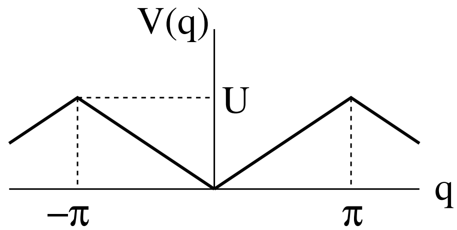
Solution:

[mex93] Unbounded motion in piecewise linear periodic potential

Consider a particle of mass m moving in a periodic potential $V(q) = (U/\pi)|q|$ for $-\pi \leq q \leq \pi$ and $V(q + 2\pi) = V(q)$. For energies $E > U > 0$, the motion is unbounded and can be reinterpreted as a rotational mode of bounded motion. Solve this dynamical problem via transformation $(q, p) \to (\theta, J)$ to action-angle coordinates by establishing the following relations:

$$p(q, E) = \sqrt{2m\,[E - (U/\pi)|q|]}, \quad J(E) = \frac{2\sqrt{2m}}{3U}\left[E^{3/2} - (E-U)^{3/2}\right],$$

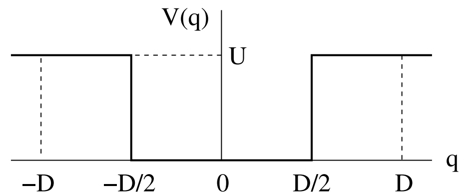
$$\omega(E) = \frac{1}{\sqrt{2m}}\left[\sqrt{E} + \sqrt{E-U}\right], \quad \theta(q, E) = \pm\pi\,\frac{\sqrt{E} - \sqrt{E - (U/\pi)|q|}}{\sqrt{E} - \sqrt{E-U}}, \quad 0 \leq \pm q \leq \pi.$$



Solution:

[mex95] Bounded motion in piecewise constant periodic potential

Consider a particle of mass m moving in the potential $V(q) = 0$ for $0 < |q| < D/2$ and $V(q) = U$ for $D/2 < |q| < D$ with periodicity $V(q+2D) = V(q)$. For energies $E < U$ the motion is bounded. Solve this dynamical problem via transformation $(q,p) \to (\theta, J)$ to action-angle coordinates for motion with initial conditions $q(0) = 0,\ p(0) > 0$. (a) Find the function $K(J)$, which expresses the Hamiltonian as a function of the action coordinate. (b) Find the period $T \equiv 2\pi/\omega(J)$ of the librational motion. (c) Find the function $q(\theta, J)$ for $0 < \theta < 2\pi$. (d) Plot in one diagram the functions $J = \text{const}$ and $p(t)$ for $0 < t < T$. (e) Plot in a second diagram the functions $q(t)$ and $\theta(t)$ for $0 < t < T$.



Solution:

Poisson Brackets [msl30]

Definition: $\{f,g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_j} \right),$

where $f(q_1,\ldots,q_n,p_1,\ldots,p_n)$, $g(q_1,\ldots,q_n,p_1,\ldots,p_n)$ are arbitrary dynamical variable expressed as functions of canonical coordinates.

Algebraic properties:

- $\{f,g\} = -\{g,f\}$,
- $\{f,c\} = 0 \quad \text{if} \;\; c = \text{const.}$,
- $\{f_1+f_2,g\} = \{f_1,g\} + \{f_2,g\}$,
- $\{f_1 f_2,g\} = f_1\{f_2,g\} + f_2\{f_1,g\}$,
- $\frac{\partial}{\partial t}\{f,g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\}$,
- $\{q_j,f\} = \frac{\partial f}{\partial p_j}, \quad \{p_j,f\} = -\frac{\partial f}{\partial q_j}$.

Fundamental Poisson brackets: $\{q_i,q_j\} = 0, \quad \{p_i,p_j\} = 0, \quad \{q_i,p_j\} = \delta_{ij}$

Invariance under canonical transformations:

$$Q_j = Q_j(q_1,\ldots,q_n,p_1,\ldots,p_n),\;\; P_j = P_j(q_1,\ldots,q_n,p_1,\ldots,p_n)$$

$$\Rightarrow\;\; \{Q_i,Q_j\}_{q,p} = 0, \quad \{P_i,P_j\}_{q,p} = 0, \quad \{Q_i,P_j\}_{q,p} = \delta_{ij}$$

Canonical equations: $\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i,H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i,H\}$.

Jacobi's identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

Poisson's theorem: $\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}$ [mex191]

Implication: If f and g are integrals of the motion, then $\{f,g\}$ is also an integral of the motion.

Specifications of Hamiltonian System [mln95]

Canonical variables:

- Canonical coordinates: $q_1, \ldots, q_n;\ p_1, \ldots, p_n$.
- Hamiltonian: $H(q_1, \ldots, q_n; p_1, \ldots, p_n)$.
- Canonical equations: $\dot{q}_i = \dfrac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\dfrac{\partial H}{\partial q_i}$.
- Dynamical variable: $f(q_1, \ldots, q_n; p_1, \ldots, p_n) = f(t)$.

Noncanonical variables:

- Elementary dynamical variables: $u_1, \ldots, u_m$.
- Energy function: $\bar{H}(u_1, \ldots, u_m)$.
- Symplectic structure: $\{u_i, u_j\} = B_{ij}(u_1, \ldots, u_m)$.
- Hamilton's equations: $\dot{u}_i = \{u_i, \bar{H}\}$.
- Dynamical variable: $f(u_1, \ldots, u_m) = f(t)$.

Link to quantum mechanics:

- Elementary operators: $u_1, \ldots, u_m$.
- Hamiltonian operator: $\bar{H}(u_1, \ldots, u_m)$.
- Commutation relations: $[u_i, u_j] = A_{ij}(u_1, \ldots, u_m)$.
- Dynamical variable: $f(u_1, \ldots, u_m) = f(t)$.
- Heisenberg equation: $\dot{f} = \dfrac{1}{i\hbar}[f, \bar{H}]$.

[mex191] Poisson's theorem

Prove Poisson's theorem for thwo dynamical variables f and g:

$$\frac{d}{dt}\{f,g\} = \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}.$$

Solution:

[mex192] Poisson brackets of angular momentum variables

Given the fundamental Poisson brackets $\{x_i, x_j\} = 0,\ \{p_i, p_j\} = 0,\ \{x_i, p_j\} = \delta_{ij}$, for the Cartesian position and momentum coordinates, determine the Poisson brackets $\{L_i, x_j\},\ \{L_i, p_j\}, \{L_i, L_j\}$ for the angular momentum variables

$$L_i \doteq \sum_{m,n=1}^{3} \epsilon_{imn} x_m p_n, \quad i = 1, 2, 3.$$

Solution:

[mex200] Action-angle coordinates of plane pendulum: librations

Determine the canonical transformation $(\phi, p) \to (\theta, J)$ which produces the action-angle coordinates for the librational motion of the plane pendulum:

$$H(\phi, p) = \frac{p^2}{2m} + G(1 - \cos\phi), \qquad M \doteq m\ell^2, \quad G \doteq mg\ell.$$

(a) Find the action $J(E)$, the angular frequency $\omega(E)$, and the angle coordinate $\theta(\phi, J)$. (b) Use this result to determine $\phi(t)$ in closed form.

Solution:

[mex94] Hamiltonian system specified by noncanonical variables

A classical dynamical system is specified by the following Hamilton's equations of motion for three noncanonical variables A, B, C:

$$\frac{d}{dt}A = -2BC, \quad \frac{d}{dt}B = -2AC, \quad \frac{d}{dt}C = 4AB.$$

The three variables satisfy the mutual Poisson brackets $\{A, B\} = C,\ \{B, C\} = A,\ \{C, A\} = B$.
(a) Determine the energy function $\bar{H}(A, B, C)$ of this system.
(b) Show that the function $I(A, B, C) = \sqrt{A^2 + B^2 + C^2}$ is an integral of the motion.
(c) Show that $q = \arctan(B/A),\ p = C$ are a pair of canonical coordinates.

Solution:

[mex197] Generating a pure Galilei transformation

Demonstrate the canonicity in phase space of a pure Galilei transformation,

$$\mathbf{R} = \mathbf{r} + \mathbf{v}t \quad \text{with } \mathbf{v} = \text{const.}$$

Find the generating function $F_2(\mathbf{r}, \mathbf{P}, t)$.

Solution:

[mex199] Exponential potential

Consider a particle of mass $m = 1/2$ moving in a straight line (x-axis) and subject to a force $F(x) = -e^x$. Find the solution $x(t),\ p(t)$ as follows:
(a) Find a generating function $F_2(x, P)$ which transforms the Hamiltonian $H(x, p) = p^2 + e^x$ into $K(Q, P) = \frac{1}{4}P^2$ and derive canonical transformation relations $Q(x, p)$ and $P(x, p)$ from $F_2(x, P)$.
(b) Solve the canonical equations for $K(Q, P)$ to get $Q(t)$ and $P(t)$ and substitute these solutions into the inverse transformation relations $x(Q, P) = x(t)$ and $p(Q, P) = p(t)$.
(c) State the solutions $x(t), p(t)$ for initial conditions $x(0) = p(0) = 0$. Verify that $x(t)$ and $p(t)$ thus found are indeed solutions of the canonical equations for $H(x, p)$.

Solution: