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16. Canonical Transformations

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Abstract
Part sixteen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc16]

16. Canonical Transformations

Point transformations [mln88]

Point transformations $q_i \to Q_i$ are invertible coordinate transformations in configuration space.

Given a set of transformation relations between generalized coordinates:

$$q_i = q_i(Q_1, \ldots, Q_n; t), \quad i = 1, \ldots, n.$$

Generalized velocities: $\dot{q}_i(Q_1, \ldots, Q_n; \dot{Q}_1, \ldots, \dot{Q}_n; t) = \sum_j \frac{\partial q_j}{\partial Q_j}\dot{Q}_j + \frac{\partial q_j}{\partial t}.$

Transformed Lagrangian via substitution of transformation relations:

$$\tilde{L}(Q_1, \ldots, Q_n; \dot{Q}_1, \ldots, \dot{Q}_n; t) = L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t).$$

Invariance of Lagrange equations under point transformations [mex79]:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \quad \Leftrightarrow \quad \frac{\partial \tilde{L}}{\partial Q_i} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_i} = 0.$$

Canonical momenta: $p_i \doteq \frac{\partial L}{\partial \dot{q}_i}, \quad P_i \doteq \frac{\partial \tilde{L}}{\partial \dot{Q}_i}.$

Relation between canonical momenta [mex80]: $P_i = \sum_j p_j \frac{\partial q_j}{\partial Q_i}.$

Relation between Hamiltonians [mex80]:

$$\tilde{H}(Q_1, \ldots, Q_n, P_1, \ldots, P_n, t) = H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) - \sum_j p_j \frac{\partial q_j}{\partial t}.$$

Invariance of canonical equations under point transformations [mex82]:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Leftrightarrow \quad \dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}.$$

[mex80] Effect of point transformation on Hamiltonian

Consider the point transformation $q_i = q_i(Q_1, \ldots, Q_n, t), i = 1, \ldots, n$ between two sets of generalized coordinates. The original and transformed Lagrangians are $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \tilde{L}(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t)$. By comparing the differentials dL and $d\tilde{L}$ show that the following relations hold between the canonical momenta $\{P_i\}, \{p_i\}$ and between the Hamiltonians $H, \tilde{H}$ before and after the transformation:

$$
\begin{aligned}
P_i(Q_1, \ldots, Q_n, p_1, \ldots, p_n, t) &= \sum_j p_j \frac{\partial q_j}{\partial Q_i}, \\
\tilde{H}(Q_1, \ldots, Q_n, P_1, \ldots, P_n, t) &= H(q_1, \ldots, q_n, p_1, \ldots, p_n, t) - \sum_j p_j \frac{\partial q_j}{\partial t}.
\end{aligned}
$$

Solution:

[mex82] Effect of point transformation on canonical equations

Consider the point transformation $q_i = q_i(Q_1, \ldots, Q_n, t), i = 1, \ldots, n$ between two sets of generalized coordinates. Use the results stated in [mex80] to show (by substitution of coordinates) that the structure of the canonical equations is the same before and after the point transformation:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Leftrightarrow \quad \dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}.$$

Solution:

[mex193] Hamiltonian of free particle in rotating frame

Consider a particle of mass m that is free to move in the xy-plane.
(a) Find the Hamiltonian $H(r,\theta,p_r,p_\theta)$, where $x = r\cos\theta, y = r\sin\theta$.
(b) Convert the resulting canonical equations into two 2$^{\text{nd}}$-order ODEs for r and θ.
(c) Perform a point transformation $R = r, \phi = \theta + \omega t$ to a rotating frame with $\omega = \text{const}$. Find the transformed Hamiltonian $\tilde{H}(r,\phi,p_R,p_\phi)$ following the prescription derived in [mex80] and convert the resulting canonical equations into two 2$^{\text{nd}}$-order ODEs for R and ϕ.
(d) Derive the equations of motion found in (c) from those found in (b) through direct substitution of the transformation relations.

Solution:

Canonical Transformations [mln89]

Canonical transformations $(q;p) \to (Q;P)$ operate in phase space.

Notation: $(q;p) \equiv (q_1, \ldots, q_n; p_1, \ldots, p_n)$ etc.

Not every transformation $q_i = q_i(Q;P;t)$, $p_i = p_i(Q;P;t)$ preserves the structure of the canonical equations.

Canonicity of transformation $(q;p) \to (Q;P)$ hinges on relation between Hamiltonians $H(q;p;t)$ and $K(Q;P;t)$ such that

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Rightarrow \quad \dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

Canonicity enforced via modified Hamilton's principle [mln83]:

$$\delta \int_{t_1}^{t_2} dt \left[\sum_j p_j \dot{q}_j - H(q;p;t) \right] = \delta \int_{t_1}^{t_2} dt \left[\sum_j P_j \dot{Q}_j - K(Q;P;t) \right] = 0.$$

$$\Rightarrow \quad \sum_j p_j \dot{q}_j - H(q;p;t) = \sum_j P_j \dot{Q}_j - K(Q;P;t) + \frac{dF}{dt}.$$

Generating function $F(x;Y;t)$ depends on n old and n new coordinates.
For example: $(x;Y) \equiv (q;Q)$, $(q;P)$, $(p;Q)$, $(p;P)$.

Total time derivative of F has vanishing variation:

$$\delta \int_{t_1}^{t_2} dt \, \frac{dF}{dt} = \Big[\delta F \Big]_{t_1}^{t_2} = 0.$$

Different generating functions for the same canonical transformation are related to each other via Legendre transform.

The four basic types of generating functions are

$$\begin{aligned} F_1(q;Q;t) &= F_2(q;P;t) - \sum_j P_j Q_j \\ &= F_3(p;Q;t) + \sum_j p_j q_j \\ &= F_4(p;P;t) - \sum_j P_j Q_j + \sum_j p_j q_j. \end{aligned}$$

Implementation of canonical transformation specified by $F_1(q;Q;t)$:

$$\frac{d}{dt}\,F_1(q;Q;t) = \sum_j (p_j\dot{q}_j - P_j\dot{Q}_j) - [H(q;p;t) - K(Q;P;t)]\,.$$

$$\Rightarrow \quad \sum_j \left(\frac{\partial F_1}{\partial q_j}\,dq_j + \frac{\partial F_1}{\partial Q_j}\,dQ_j\right) + \frac{\partial F_1}{\partial t}\,dt \;=\; \sum_j (p_j dq_j - P_j dQ_j) \;-\; [H(q;p;t) - K(Q;P;t)]\,dt.$$

Comparison of coefficients yields

$$p_j = \frac{\partial F_1}{\partial q_j}, \quad P_j = -\frac{\partial F_1}{\partial Q_j}, \quad K - H = \frac{\partial F_1}{\partial t}.$$

Transformation relations:

- Invert relations $P_j(q;Q;t)$ into $q_j(Q;P;t)$.
- Combine relations $p_j(q;Q,t)$ with $q_j(Q;P;t)$ to get $p_j(Q;P;t)$.

Transformed Hamiltonian:

- $K(Q;P;t) = H(q;p;t) + \frac{\partial}{\partial t}\,F_1(q;Q;t)$.

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generating function	**transformation of coordinates**		**transformation of Hamiltonian**
$F_1(q,Q,t)$	$p_j = \frac{\partial F_1}{\partial q_j}$	$P_j = -\frac{\partial F_1}{\partial Q_j}$	$K = H + \frac{\partial F_1}{\partial t}$
$F_2(q,P,t)$	$p_j = \frac{\partial F_2}{\partial q_j}$	$Q_j = \frac{\partial F_2}{\partial P_j}$	$K = H + \frac{\partial F_2}{\partial t}$
$F_3(p,Q,t)$	$q_j = -\frac{\partial F_3}{\partial p_j}$	$P_j = -\frac{\partial F_3}{\partial Q_j}$	$K = H + \frac{\partial F_3}{\partial t}$
$F_4(p,P,t)$	$q_j = -\frac{\partial F_4}{\partial p_j}$	$Q_j = \frac{\partial F_4}{\partial P_j}$	$K = H + \frac{\partial F_4}{\partial t}$

Canonicity and Volume Preservation [mln90]

Illustration for one degree of freedom (2D phase space).

Consider transformation $(q,p) \to (Q,P)$.

Area preservation: Jacobian determinant $D = 1$ or, equivalently, area inside any closed path $\mathcal{C}$ is invariant.

Canonicity: There exists a generating function, e.g. $F_1(q,Q)$.

Canonicity implies area preservation:

- Given canonicity specified by $F_1(q,Q)$.
- $\Rightarrow\ p = \dfrac{\partial F_1}{\partial q}, \quad P = -\dfrac{\partial F_1}{\partial Q}.$
- $\Rightarrow\ \dfrac{\partial Q}{\partial q} = 0$ (Q and q are independent); $\dfrac{\partial P}{\partial p}$ is finite, in general.
- $\Rightarrow\ \dfrac{\partial P}{\partial q} = -\dfrac{\partial^2 F_1}{\partial Q \partial q} = -\left(\dfrac{\partial Q}{\partial p}\right)^{-1}.$
- $\Rightarrow\ D \doteq \dfrac{\partial(Q,P)}{\partial(q,p)} = \begin{vmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{vmatrix} = \dfrac{\partial Q}{\partial q}\dfrac{\partial P}{\partial p} - \dfrac{\partial Q}{\partial p}\dfrac{\partial P}{\partial q} = 1.$

Area preservation implies canonicity:

- Transformation: $p = p(q,Q) \quad P = P(q,Q)$.
- Area inside closed path $\mathcal{C}$: $\displaystyle\oint_{\mathcal{C}} p\,dq = \oint_{\mathcal{C}} P\,dQ.$
- $\Rightarrow\ \displaystyle\oint_{\mathcal{C}} [p(q,Q)dq - P(q,Q)dQ] = 0$ for arbitrary closed paths $\mathcal{C}$.
- $\Rightarrow$ Integrand must be perfect differential $dF_1(q,Q)$:

$$p\,dq - P\,dQ = dF_1 = \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial Q}dQ.$$

- $\Rightarrow\ p(q,Q) = \dfrac{\partial F_1}{\partial q}, \quad P(q,Q) = -\dfrac{\partial F_1}{\partial Q}.$

[mex87] Determine canonicity and generating functions I

Consider the following transformation from a set of canonical coordinates (q, p) to a new set of coordinates (Q, P):

$$Q = \ln\left(\frac{\sin p}{q}\right), \qquad P = q \cot p.$$

(a) Verify that this transformation is canonical by investigating its Jacobian determinant. (b) Determine the generating function $F_3(p, Q)$ by integration of the total differential dF_3. (c) Determine the generating function $F_2(q, P)$ from $F_3(p, Q)$ via Legendre transform.

Solution:

[mex90] Determine canonicity and generating functions II

Consider the following transformation from a set of canonical coordinates (q,p) to a new set of coordinates (Q,P):

$$Q = \frac{1}{2}(q^2 + p^2), \qquad P = -\arctan\frac{q}{p}.$$

(a) Verify that this transformation is canonical by investigating its Jacobian determinant. (b) Determine the generating function $F_4(p,P)$ by integration of the total differential dF_4. (c) Determine the generating function $F_1(q,Q)$ from $F_4(p,P)$ via Legendre transform.

Solution:

[mex194] Determine canonicity and generating functions III

Consider the following transformation from a set of canonical coordinates (q, p) to a new set of coordinates (Q, P):

$$Q = \ln p, \qquad P = -qp.$$

(a) Verify that this transformation is canonical by investigating its Jacobian determinant. (b) Determine the generating function $F_1(q, Q)$ by integration of the total differential dF_1. (c) Determine the generating function $F_2(q, P)$ by integration of the total differential dF_2. (d) Determine the generating function $F_2(q, P)$ from $F_1(q, Q)$ via Legendre transform.

Solution:

[mex198] Determine canonicity and generating functions IV

Consider the following transformation from a set of canonical coordinates (q, p) to a new set of coordinates (Q, P):

$$Q = q^k p^l, \qquad P = q^m p^n.$$

(a) For what values of the exponents k, l, m, n is this transformation canonical? (b) Find the generating function $F_1(q, Q)$ for those values. (c) One canonical case cannot be covered by the function $F_1(q, Q)$. Why not?

Solution:

Infinitesimal canonical transformations [mln91]

Consider a canonical transformation generated by

$$F_2(q,P;\epsilon) = qP + \epsilon W(q,P;\epsilon),$$

where ϵ is a continuous parameter. The first term represents the *identity transformation*.

Transformed canonical coordinates:

$$Q(\epsilon) = q + \epsilon\,\frac{\partial W}{\partial P}, \qquad p(\epsilon) = P + \epsilon\,\frac{\partial W}{\partial q}.$$

Generator: $G(Q,P) \doteq \lim_{\epsilon\to 0} W(q,P;\epsilon)$ (Lie generating function).

Transformed canonical coordinates [to O(ϵ)]:

$$Q(\epsilon) = q + \epsilon\,\frac{\partial G}{\partial P}, \qquad P(\epsilon) = p - \epsilon\,\frac{\partial G}{\partial Q}.$$

Dependence of coordinates Q, P on ϵ expressed via two first-order ODEs:

$$\Rightarrow \;\; \frac{dQ}{d\epsilon} = \frac{\partial G}{\partial P}, \qquad \frac{dP}{d\epsilon} = -\frac{\partial G}{\partial Q}. \tag{1}$$

Solutions $Q(\epsilon),\; P(\epsilon)$ with initial conditions $Q(0)=q,\; P(0)=p$.

For time evolution set $\epsilon = t$ and $G(Q,P) = H(Q,P)$.

The generator of the time evolution is the Hamiltonian and the differential equations (1) that determine this particular canonical transformation become the canonical equations:

$$\dot{Q} = \frac{\partial H}{\partial P}, \qquad \dot{P} = -\frac{\partial H}{\partial Q}.$$

The volume preservation of the time evolution in phase space is expressed by the Liouville theorem [tln45] [tln46].

Classical Hamiltonian system [tln45]

Consider an autonomous classical dynamical system with $3N$ degrees of freedom (e.g. N particles in a 3D box with reflecting walls). The dynamics is fully described by $6N$ independent variables, e.g. by a set of *canonical coordinates* $q_1, \ldots, q_{3N}; p_1, \ldots, p_{3N}$.

The time evolution of these coordinates is specified by a *Hamiltonian function* $H(q_1, \ldots, q_{3N}; p_1, \ldots, p_{3N})$ and determined by the *canonical equations*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad i = 1, \ldots, 3N$$

The time evolution of an arbitrary dynamical variable $f(q_1, \ldots, q_{3N}; p_1, \ldots, p_{3N})$ is determined by *Hamilton's equation of motion*:

$$\frac{df}{dt} = \sum_{i=1}^{3N} \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) = \sum_{i=1}^{3N} \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \equiv \{f, H\},$$

here expressed in terms of a Poisson bracket [msl30].

Conserved quantity: $\dfrac{df}{dt} = 0 \;\Leftrightarrow\; \{f, H\} = 0.$
Energy conservation is guaranteed: $\dfrac{dH}{dt} = 0$ because $\{H, H\} = 0$.

The microstate of the system is specified by one point in the $6N$-dimensional *phase space* (Γ-space): $\mathbf{X} \equiv (q_1, \ldots, q_{3N}; p_1, \ldots, p_{3N})$. As time evolves, this point traces a trajectory through Γ-space.

The conservation law $H(q_1, \ldots, q_{3N}; p_1, \ldots, p_{3N}) =$ const confines the motion of any phase point to a $6N-1$-dimensional hypersurface in Γ-space. Other conservation laws, provided they exist, will further reduce the dimensionality of the manifold to which phase-space trajectories are confined.

Our knowledge of the instantaneous microstate of the system is expressed by a probability density $\rho(\mathbf{X}, t)$ in Γ-space.

Normalization: $\displaystyle\int_\Gamma d^{6N}X\, \rho(\mathbf{X}, t) = 1.$

Instantaneous expectation value: $\displaystyle\langle f \rangle = \int_\Gamma d^{6N}X\, f(\mathbf{X})\rho(\mathbf{X}, t).$

Maximum knowledge about microstate realized for $\rho(\mathbf{X}, 0) = \delta(\mathbf{X} - \mathbf{X_0})$.

Classical Liouville operator [tln46]

To describe the time evolution of $\rho(\mathbf{X},t)$ we consider a volume V_0 with surface S_0 in Γ-space. The following equations relate the change of probability inside V_0 to the flow of probability through S_0 and use Gauss' theorem.

$$\frac{\partial}{\partial t}\int_{V_0} d^{6N}X\,\rho(\mathbf{X},t) = -\oint_{S_0} d\mathbf{s}\cdot\dot{\mathbf{X}}\rho(\mathbf{X},t) = -\int_{V_0} d^{6N}X\,\nabla_{\mathbf{X}}\cdot[\dot{\mathbf{X}}\rho(\mathbf{X},t)].$$

Balance equation: $\dfrac{\partial}{\partial t}\rho(\mathbf{X},t) + \nabla_{\mathbf{X}}\cdot[\dot{\mathbf{X}}\rho(\mathbf{X},t)] = 0.$

Use $\nabla_{\mathbf{X}}\cdot[\dot{\mathbf{X}}\rho] = \rho\nabla_{\mathbf{X}}\cdot\dot{\mathbf{X}} + \dot{\mathbf{X}}\cdot\nabla_{\mathbf{X}}\rho$ and $\nabla_{\mathbf{X}}\cdot\dot{\mathbf{X}} = 0.$

$$\Rightarrow\ \frac{\partial}{\partial t}\rho(\mathbf{X},t) + \dot{\mathbf{X}}\cdot\nabla_{\mathbf{X}}\rho(\mathbf{X},t) = 0.$$

Introduce convective derivative: $\dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} + \dot{\mathbf{X}}\cdot\nabla_{\mathbf{X}}.$

Liouville theorem: $\dfrac{d}{dt}\rho(\mathbf{X},t) = 0.$

Use $\dot{\mathbf{X}}\cdot\nabla_{\mathbf{X}}\rho = \displaystyle\sum_{i=1}^{3N}\left(\dot{q}_i\frac{\partial\rho}{\partial q_i} + \dot{p}_i\frac{\partial\rho}{\partial p_i}\right) = \sum_{i=1}^{3N}\left(\frac{\partial\rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \{\rho, H\}.$

Liouville operator: $L \equiv i\{H,\ \} = i\displaystyle\sum_{i=1}^{3N}\left(\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}\right).$

Liouville equation: $i\dfrac{\partial\rho}{\partial t} = i\{H,\rho\} = L\rho.$

Formal solution: $\rho(\mathbf{X},t) = e^{-iLt}\rho(\mathbf{X},0).$

L is a Hermitian operator. Hence all its eigenvalues are real. Hence $\rho(\mathbf{X},t)$ cannot relax to equilibrium in any obvious way. The Liouville equation reflects the time reversal symmetry of the underlying microscopic dynamics. Obtaining the broken time reversal symmetry of irreversible processes from the Liouville equation is a central problem in statistical mechanics (topic of ergodic theory).

Nevertheless: the thermal equilibrium is described by a stationary (time-independent) probability density:

$$\frac{\partial\rho}{\partial t} = 0\ \Rightarrow\ L\rho = 0\ \Rightarrow\ \{H,\rho\} = 0.$$

A stationary ρ is an eigenfunction of L with eigenvalue zero. If $\rho = \rho(H)$ then $\{H,\rho\} = 0$. Hence ρ is time-independent.

[mex195] Canonicity of gauge transformation

Consider the gauge transformation $L(q,\dot{q},t) \to \tilde{L}(q,\dot{q},t)$ with

$$\tilde{L}(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}f(q,t),$$

which we have shown in [mex21] to leave the Lagrange equations invariant. Show that this transformation is canonical and find its generating function $F_2(q,P,t)$. Find also the gauge-transformed Hamiltonian $\tilde{H}$.

Solution:

[mex196] Electromagnetic gauge transformation

Consider a particle of mass m and electric charge q moving (nonrelativistically) in an electromagnetic field described by the vector potential $\mathbf{A}(\mathbf{r},t)$ and the scalar potential $\phi(\mathbf{r},t)$. Show that the gauge transformation

$$\mathbf{A} \to \mathbf{A}' \doteq \mathbf{A} + \nabla f(\mathbf{r},t), \qquad \phi \to \phi' \doteq \phi - \frac{1}{c}\frac{\partial}{\partial t} f(\mathbf{r},t)$$

is a canonical transformation. Find the generating function $F_2(\mathbf{r},\mathbf{P},t)$.

Solution:

[mex84] Check the canonicity of coordinate transformations

(a) The relations $q = P\cos Q,\ p = P\sin Q$ describe a transformation between Cartesian coordiantes and polar coordinates in the phase space of a system with one degree of freedom. Determine whether this transformation is canonical or not. (b) Determine the values of the parameters α, β the linear transformation $Q = q + \alpha p,\ P = p + \beta q$ is canonical. (c) Verify that the transformation $Q = \sqrt{p - t^2},\ P = -2q\sqrt{p - t^2}$ to be used in [mex83] is canonical.

Solution:

[mex83] Time-dependence of generating functions

Consider the generating functions $F_1(q, Q, t)$ and $F_2(q, P, t) = F_1 + QP$ of a time-dependent canonical transformation $Q = Q(q, p, t), \;\; P = P(q, p, t)$. (a) Show that both generating functions have the same explicit time-dependence: $\partial F_1/\partial t = \partial F_2/\partial t$. (b) Find the generating functions $F_1(q, Q, t)$ and $F_2(q, P, t)$ for the specific transformation, which was shown in [mex84] to be canonical, $Q = \sqrt{p - t^2}, \;\; P = -2q\sqrt{p - t^2}$, and verify that $\partial F_1/\partial t = \partial F_2/\partial t$ is indeed satisfied.

Solution:

[mex85] Canonical transformation from rest frame to moving frame

Consider a particle of mass m moving along a straight line in a scalar potential. The Hamiltonian in the rest frame reads

$$H(q,p) = \frac{p^2}{2m} + V(q).$$

The function $F_2(q,P,t) = P[q - d(t)]$ generates a canonical transformation between the rest frame and a frame whose origin is displaced a distance $d(t)$ from the origin of the rest frame. (a) Determine the Hamiltonian $K(Q,P,t)$ after the transformation. (b) Compare the momenta p, P and the generalized velocities $\dot{q}, \dot{Q}$ of the particle in the two frames. (c) Determine the equations of motion in the form $m\ddot{q} = \ldots$ and $m\ddot{Q} = \ldots$ in the two frames of reference and explain the nature of all terms on the right-hand side of both equations.

Solution:

[mex86] Canonical transformation applied to harmonic oscillator

Subject the Hamiltonian $H(q,p)$ of the harmonic oscillator to the canonical transformation specified by the generating function $F_1(q,Q)$:

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2, \qquad F_1(q,Q) = \frac{1}{2}m\omega_0 q^2 \cot Q.$$

(a) Determine the transformation relations in the form $q = q(P,Q),\ p = p(P,Q)$. (b) Determine the transformed Hamiltonian $K(Q,P)$. (c) Solve the canonical equations for $K(Q,P)$ to get the coordinates $Q(t), P(t)$. (d) Substitute these solutions into the transformation relations to obtain the time evolution of the original coordinates $q(t), p(t)$ for a given value of energy E.

Solution: