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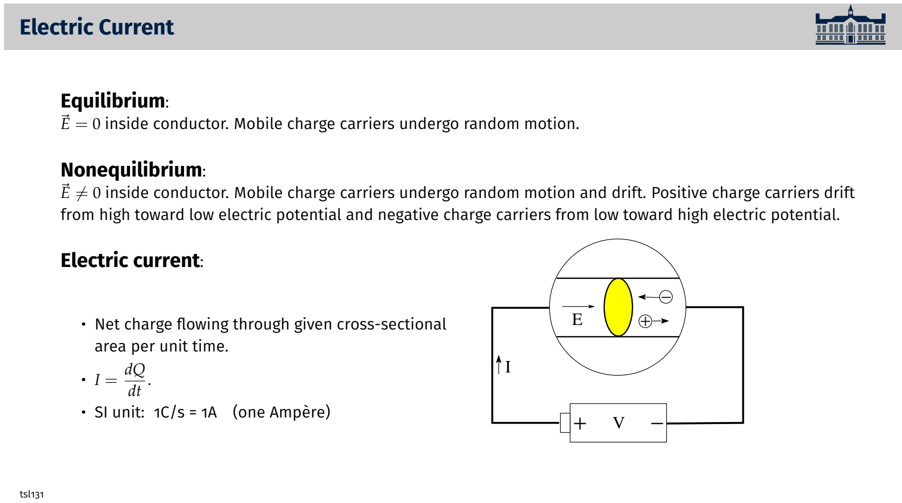
15. Electric current and current density. Resistivity, resistance, and resistor

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

PHY204 Lecture 15 [rln15]



We have reasoned previously (in lecture 6) that there can be no electric field inside a conductor at equilibrium.

When we connect a conducting wire or pipe filled with electrolyte to a power source as shown on the slide, an electric field will establish itself inside it. It will drag mobile charge carriers one way if their charge is positive and the opposite way if their charge is negative.

We say that the battery drives an electric current I through the conductor. We define electric current by picking a cross-sectional area such as highlighted on the slide and count mobile charges that cross it. The electric current is the time rate at which net charge moves across.

The slide associates an arrow with the electric current I even though it is not a vector. The arrow states that we declare the current to be clockwise. If the analysis produces a negative I, then we say we have a negative clockwise current.

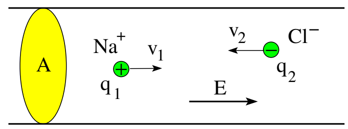
Current directions are always a matter of choice. When we declare the current I to flow clockwise, we get positive contributions from positive charges moving clockwise (cw) and from negative charges moving counterclockwise (ccw). We get negative contributions from positive charges moving ccw and negative charges moving cw.

A steady current is time-independent. Such a current is driven by an electrostatic electric field inside the conductor.

Current and Current Density

Consider drift of Na^+ and Cl^- ions in a plastic pipe filled with salt water.

- $v_1 > 0$, $v_2 < 0$: drift velocities [m/s]
- $q_1 > 0$, $q_2 < 0$: charge on ions [C]
- n_1, n_2: number of charge carriers per unit volume $[\mathrm{m}^{-3}]$



- Net charge flowing through area A in time dt: $dQ = n_1 q_1 v_1 A dt + n_2 q_2 v_2 A dt$ [C]
- Electric current through area A: $I \equiv \frac{dQ}{dt} = A(n_1 q_1 v_1 + n_2 q_2 v_2)$ [A]
- Current density: $\vec{J} = n_1 q_1 \vec{v}_1 + n_2 q_2 \vec{v}_2$ $[\mathrm{A/m^2}]$
- Current equals flux of current density: $I = \int \vec{J} \cdot d\vec{A}$ [A]

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Here we use an electrolyte for a more quantitative discussion of electric current. Unlike in a metallic conductor, where only negative charge carriers (electrons) are mobile, here we have positively charged sodium ions and negatively charged chlorine ions that are both mobile.

The current density $\vec{J}$ is a vector quantity constructed from the velocity vectors of the charge carriers as shown. Each type of charge carrier contributes one term. Both terms are vectors pointing to the right even though the Na^+ and Cl^- ions drift in opposite direction. Multiplying the vector $\vec{v}_2$, which points to the left, with the negative charge q_2, produces a vector $q_2\vec{v}_2$ pointing to the right.

The current I is the flux of the current density $\vec{J}$ through the cross-sectional surface area A. Associated with the surface is an area vector $\vec{A}$, perpendicular to the plane of the surface. We have chosen the area vector to point to the right. This choice declares that I flows from left to right.

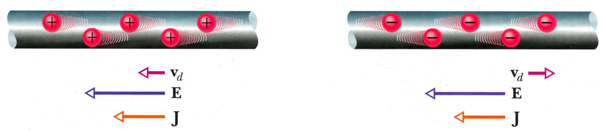
We can determine the current directly from the net charge dQ flowing from left to right across the surface in time dt as shown on the slide. Av_1dt ($|Av_2dt|$) is the volume that contains all positive (negative) charge carriers that make it through the surface from left to right (right to left) in time dt.

In a metallic conductor, e.g. a copper wire, the expressions for current I and current density $\vec{J}$ only have their second terms, representing electrons drifting from right to left. The vector $\vec{J}$ would point to the right nevertheless and, given our choice of area vector $\vec{A}$, we will have a positive current I flowing from left to right.

Current Direction

positive charge carriers

negative charge carriers



positive current I flowing toward left in both cases

negative current I flowing toward right in both cases

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Here we wish to drive home two key insights gained on the previous page. The three vectors represent drift velocity $\vec{v}_d$ of mobile charge carriers, electric field $\vec{E}$, and current density $\vec{J}$.

(i) Charge carriers do not necessarily move in the direction of the current density. In the case of positive charge carriers (shown on the left), the vectors $\vec{v}_{\rm d}$ and $\vec{J}$ are parallel. However, the two vectors are antiparallel in the case of negative charge carriers (shown on the right).

Positive and negative charge carriers are in contact with the same electric field $\vec{E}$. However, they experience an electric force $\vec{F} = q\vec{E}$ in opposite directions.

(ii) The direction of current I (not a vector) is a matter of choice. Recall that current is the flux of current density through a cross-sectional surface, which is an open surface:

$$I = \int \vec{J} \cdot d\vec{A}.$$

We have learned earlier (in lecture 4) that for opens surfaces the direction of area vectors is a matter of choice.

Choosing an area vector $\vec{A}$ pointing to the left (right) for the situations shown on the slide, means declaring left (right) to be the direction of I. When we adopt the first (second) choice, the current I comes out to positive (negative).

Resistance and Resistivity

- **Resistor**: device (material object with two terminals)
- **Resistance**: attribute of device
- **Resistivity**: attribute of conducting material

A voltage V provided by some source is applied to the terminals of a resistor and a current I is observed flowing through the resistor.

- Resistance: $R = \frac{V}{I}$ [1Ω=1V/A] (1 Ohm)

The current density $\vec{J}$ in a resistor depends on the local electric field $\vec{E}$ and on the resistivity ρ of the resistor material.

- Resistivity: $\rho = \frac{E}{J}$ $\left[\frac{1\text{V/m}}{1\text{A/m}^2} = 1\Omega\text{m}\right]$
- Conductivity: $\sigma = \frac{1}{\rho}$ $[1(\Omega\text{m})^{-1}]$
- Vector relations: $\vec{E} = \rho\vec{J}$, $\vec{J} = \sigma\vec{E}$

tsl134

We must not confuse the similarly sounding terms listed at the top of the slide. In this lecture, we focus on how the terms are related, specifically how attributes of conducting materials are related to functions of devices. The next three lectures (lectures 16, 17, and 18) will then all be on how devices function in circuits.

On the material level, we work with the quantities $\vec{J}$ (current density), $\vec{E}$ (electric field), ρ (resistivity), and σ (conductivity).

On the device level, we use different quantities: V (voltage), I (current), and R (resistance).

Resistivity of Materials

TABLE 25-1

Resistivities and Temperature Coefficients

Material	Resistivity ρ at 20°C, $\Omega\cdot$m	Temperature Coefficient α at 20°C, K^{-1}
Silver	1.6×10^{-8}	3.8×10^{-3}
Copper	1.7×10^{-8}	3.9×10^{-3}
Aluminum	2.8×10^{-8}	3.9×10^{-3}
Tungsten	5.5×10^{-8}	4.5×10^{-3}
Iron	10×10^{-8}	5.0×10^{-3}
Lead	22×10^{-8}	4.3×10^{-3}
Mercury	96×10^{-8}	0.9×10^{-3}
Nichrome	100×10^{-8}	0.4×10^{-3}
Carbon	3500×10^{-8}	-0.5×10^{-3}
Germanium	0.45	-4.8×10^{-2}
Silicon	640	-7.5×10^{-2}
Wood	$10^{8} - 10^{14}$	
Glass	$10^{10} - 10^{14}$	
Hard rubber	$10^{13} - 10^{16}$	
Amber	5×10^{14}	
Sulfur	1×10^{15}	

tsl139

- $\alpha = \dfrac{(\rho - \rho_{20})/\rho_{20}}{t_C - 20°C}$
- α: temperature coefficient at 20°C in K^{-1}
- ρ: resistivity near 20°C
- ρ_{20}: resistivity at 20°C
- t_C: temperature in °C

The list of materials is naturally divided into three groups with their resistivity ρ orders of magnitude apart.

The first group of nine are conductors, the second group of two are semiconductors, and the remaining five are insulators.

In general, the resistivity of a material varies with temperature. The temperature coefficient α gives a measure for how much ρ changes per degree of deviation from room temperature.

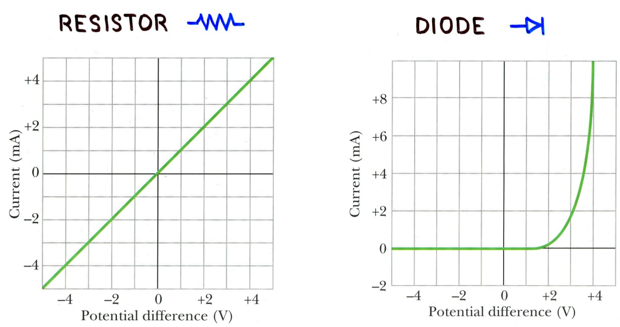
In metallic conductors, the resistivity typically increases with temperature. Here the cause of resistivity are scattering events between the mobile electrons and the localized ions. The amplitudes of lattice vibrations, which increase with temperature, enhance the scattering probabilities.

In semiconductors, by contrast, the resistivity decreases with temperature. Here, the mobility of charge carriers is of a different kind. No electrons have the mobility of conduction electrons. All electrons are bound, but some are loosely bound, such that they can hop from atom to atom when pulled by the force of an electric field. The hopping is enhanced by the lattice vibrations, as if thermal energy shakes them loose.

Ohm's Law

$V = RI$ with $R = \text{const}$

satisfied violated



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The main point of this slide is for us to appreciate the distinction between a linear and a nonlinear device, specifically, a device that obeys Ohm's law and one that does not.

The resistor (shown on the left) and the capacitor (not shown) are linear devices. The diode (shown on the right) is a nonlinear device.

The linear relation, $V = RI$, between voltage and current is shown in the diagram on the left. It represents Ohm's law. Likewise, characteristic of a capacitor is the linear relation, $Q = CV$, between charge and voltage.

The diode permits large currents in the forward direction when driven by voltages $V > 0$, but almost no current when driven in the opposite direction by voltages $V < 0$. This is illustrated in the diagram on the right. The current-voltage characteristic is nonlinear.

Diodes are realized in vacuum tubes and in semiconductor devices.

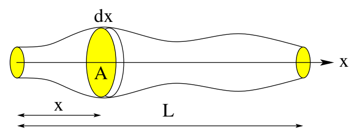
Calculating the Resistance of a Wire

Uniform cross section

- Length of wire: L
- Area of cross section: A
- Resistivity of material: ρ
- Current density: $J = \frac{E}{\rho}$ $[\mathrm{A/m^2}]$
- Current: $I = JA$ $[\mathrm{A}]$
- Voltage: $V = EL$ $[\mathrm{V}]$
- Resistance: $R \equiv \frac{V}{I} = \frac{\rho L}{A}$ $[\Omega]$

Variable cross section

- Cross-sectional profile: $A(x)$
- Resistance of slice: $dR = \frac{\rho dx}{A(x)}$
- Resistance of wire: $R = \rho \int_0^L \frac{dx}{A(x)}$



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In lecture 12 we calculated the capacitance for capacitors of different designs. Here we calculate the resistance of resistors, starting with the most common shape: a wire of some length and cross-sectional area, made of a particular conducting material.

The itemized list on the left spells out how we assemble the various ingredients using relations introduced earlier into an expression for resistance, defined as $R = V/I$.

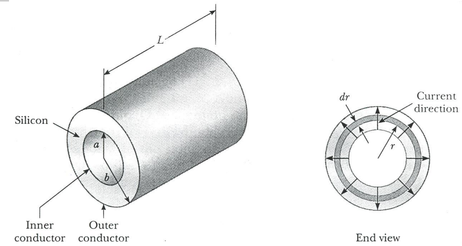
Resistance means resistance to current at given voltage. It increases if the resistor is made longer or thinner or if a material of higher resistivity is used.

When the resistor has a variable cross section, as shown on the right, we generalize this result by employing it in a creative way. We anticipate a result derived later (in the context of resistor circuits) that the resistance of resistors connected in series is the sum of individual resistances.

In the present context, we divide the wire of variable cross section into infinitesimal slices. They are connected in series. We add up the resistances of all slices, for which we use the result from the left side with L replaced by dx and with the constant cross-sectional area A replaced by a function $A(x)$. This amounts to performing n integral as shown.

The same scheme can be used if we have a wire with variable resistivity $\rho(x)$.

Radial Current in Coaxial Cable



$$dR = \frac{\rho dr}{A}, \quad A = 2\pi r L$$

$$R = \frac{\rho}{2\pi L}\int_a^b \frac{dr}{r} = \frac{\rho}{2\pi L}\ln\frac{b}{a}$$

tsl453

Consider a coaxial cable with an inner and an outer layer of materials with low resistivity ρ_{L}. They are separated by a layer of a material with much higher resistivity ρ_{H}.

A coaxial cable in use typically has the two highly conducting layers at different electric potential with currents moving in opposite directions along the cable.

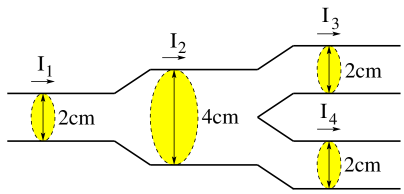
Here we are interested in the current that leaks through the more resistive layer in radial direction. For that purpose we need to calculate the resistance of a cylindrical shell of length L with terminals at radii a and b.

We adopt the method introduced on the previous page. Instead of thin flat slices we have thin cylindrical shells. Each shell has resistance dR as shown. They are connected in series again. We can add their resistances via an integral.

For a radial current in a coaxial cable, the resistance $R_\perp$ is inversely proportional to the length L length, whereas for a current along the cable the resistance $R_\|$ is proportional L. Leakage prevention requires that $R_\perp$ is high compared to $R_\|$.

Electric Current Application (1)

A steady current I is flowing through a wire from left to right. The wire first doubles its diameter and then splits into two wires of the original diameter. Both branches on the right carry the same current.



Rank the current densities $J_1, J_2, J_3 = J_4$ in the three segments.

tsl133

The point of this slide is to sharpen our understanding of current and current density.

Here we have a wire of uniform resistivity that first gets wider and then splits into to branches of equal width. How do the current I and the current density J change from position 1 on the left to position 2 in the middle and positions 3,4 on the right?

The current is steady. No charge accumulates anywhere. Therefore, the current does not change between positions 1 and 2. Between positions 2 and 3,4 it splits into two equal part:

$$I_2 = I_1, \quad I_3 = I_4 = \frac{1}{2} I_2.$$

Given that the cross-sectional area quadruples between positions 1 and 2, we conclude that the current density (current per unit area) decreases by a factor four. Between positions 2 and 3,4 the total cross-sectional area decreases to half its value for the same total current. This implies that the current density doubles:

$$J_2 = \frac{1}{2} J_1, \quad J_3 = J_4 = 2J_2.$$

Consider three wires made of the same material.

Wire 1 of length 2m and diameter 2mm has a resistance 18Ω.

(a) What resistance does wire 2 of length 4m and diameter 4mm have?
(b) How long is wire 3 of diameter 6mm with a resistance of 18Ω?

tsl147

The only resource we need for this exercise is the familiar expression for the resistance of a wire: $R = \rho L/A$.

For wire 1 we write,

$$R_1 = \frac{\rho L_1}{A_1} = 18\Omega.$$

For wire 2 with twice the length and twice the diameter we write,

$$L_2 = 2L_1, \quad A_2 = 4A_1 \quad \Rightarrow \; R_2 = \frac{\rho(2L_1)}{4A_1} = \frac{1}{2}R_1 = 9\Omega.$$

For wire 3 with three times the diameter of wire 1 and equal resistance, we write,

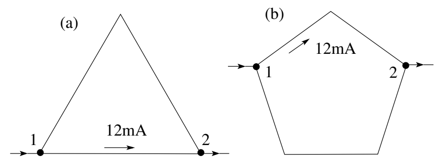
$$A_3 = 9A_1, \quad R_3 = \frac{\rho L_3}{9A_1} = \frac{\rho L_1}{A_1} = 18\Omega \quad \Rightarrow \; L_3 = 9L_1 = 18\text{m}.$$

Electric Current Application (2)

Two wires are formed into

(a) an equilateral triangle,
(b) a regular pentagon.

A voltage between points 1 and 2 produces a current of 12mA along the shorter path.



What is the current along the longer path in each case?

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This is the quiz for lecture 15.