11-19-2015

15. Gauss's Law for Magnetic Field and Ampere's Law

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Abstract
Lecture slides 15 for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island.
Some of the slides contain figures from the textbook, Paul A. Tipler and Gene Mosca. *Physics for Scientists and Engineers*, $5^{th}/6^{th}$ editions. The copyright to these figures is owned by W.H. Freeman. We acknowledge permission from W.H. Freeman to use them on this course web page. The textbook figures are not to be used or copied for any purpose outside this class without direct permission from W.H. Freeman.

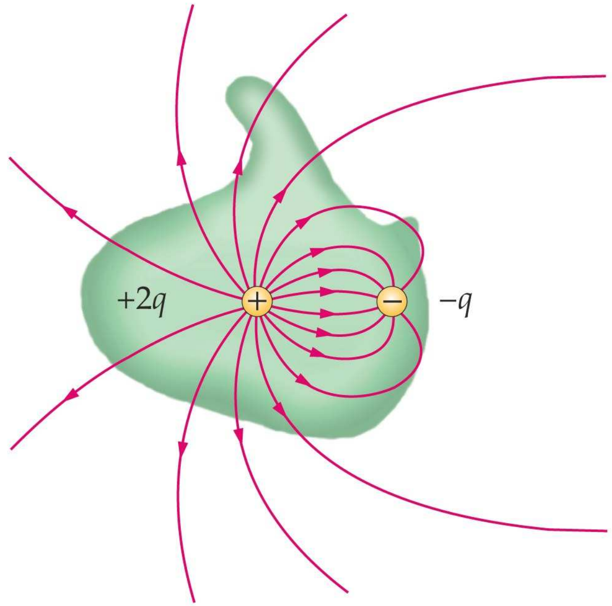
Gauss's Law for Electric Field

The net electric flux Φ_E through any closed surface is equal to the net charge Q_{in} inside divided by the permittivity constant ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{in} = \frac{Q_{in}}{\epsilon_0} \qquad \text{i.e.} \qquad \Phi_E = \frac{Q_{in}}{\epsilon_0} \qquad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12} \mathrm{C}^2 \mathrm{N}^{-1} \mathrm{m}^{-2}$$

The closed surface can be real or fictitious. It is called "Gaussian surface".
The symbol $\oint$ denotes an integral over a closed surface in this context.

- Gauss's law is a general relation between electric charge and electric field.
- In electrostatics: Gauss's law is equivalent to Coulomb's law.
- Gauss's law is one of four Maxwell's equations that govern cause and effect in electricity and magnetism.



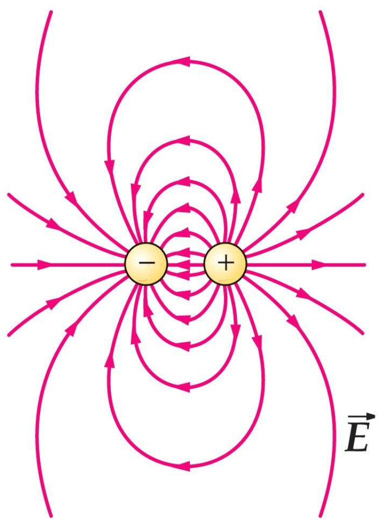
Gauss's Law for Magnetic Field

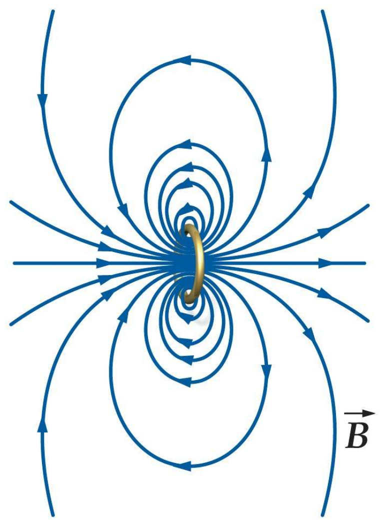
The net magnetic flux Φ_B through any closed surface is equal to zero:

$$\oint \vec{B} \cdot d\vec{A} = 0.$$

There are no magnetic charges. Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes.

The figures below illustrate Gauss's laws for the electric and magnetic fields in the context of an electric dipole (left) and a magnetic dipole (right).





Ampère's Law (Restricted Version)

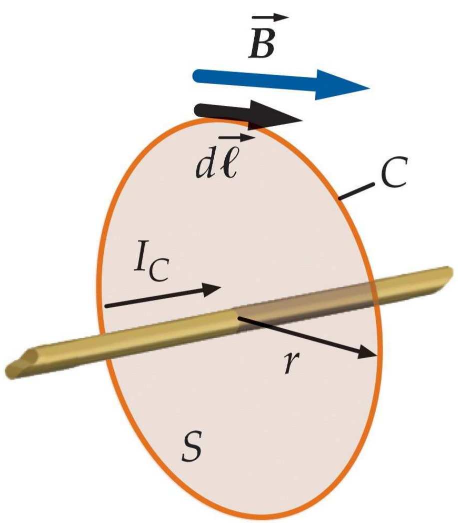
The circulation integral of the magnetic field $\vec{B}$ around any closed curve (loop) C is equal to the net electric current I_C flowing through the loop:

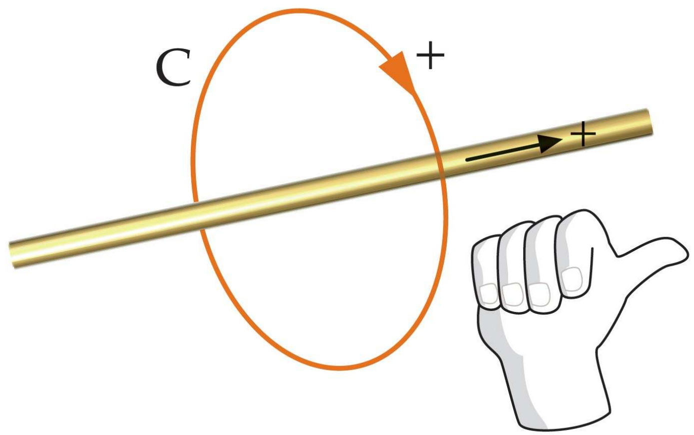
$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C, \qquad \text{with} \quad \mu_0 = 4\pi \times 10^{-7}\mathrm{Tm/A}$$

The symbol $\oint$ denotes an integral over a closed curve in this context.

Note: Only the component of $\vec{B}$ tangential to the loop contributes to the integral.

The positive current direction through the loop is determined by the right-hand rule.

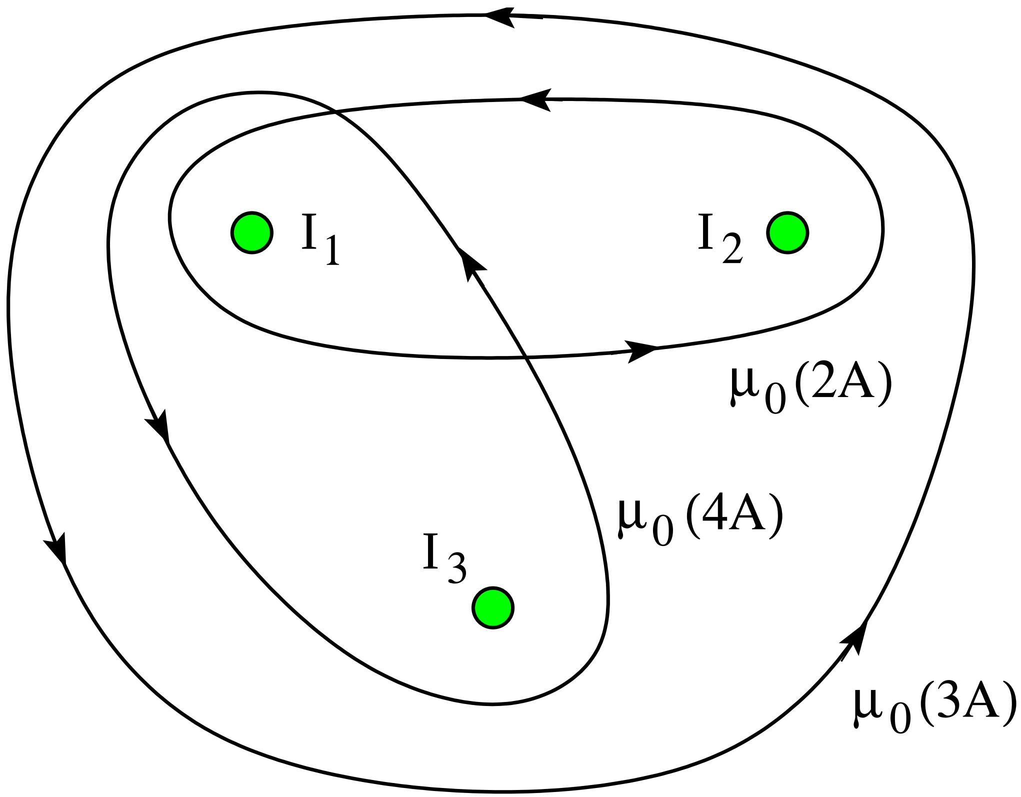




Ampère's Law: Application (1)

The line integrals $\oint \vec{B} \cdot d\vec{s}$ along the three Amperian loops are as indicated.

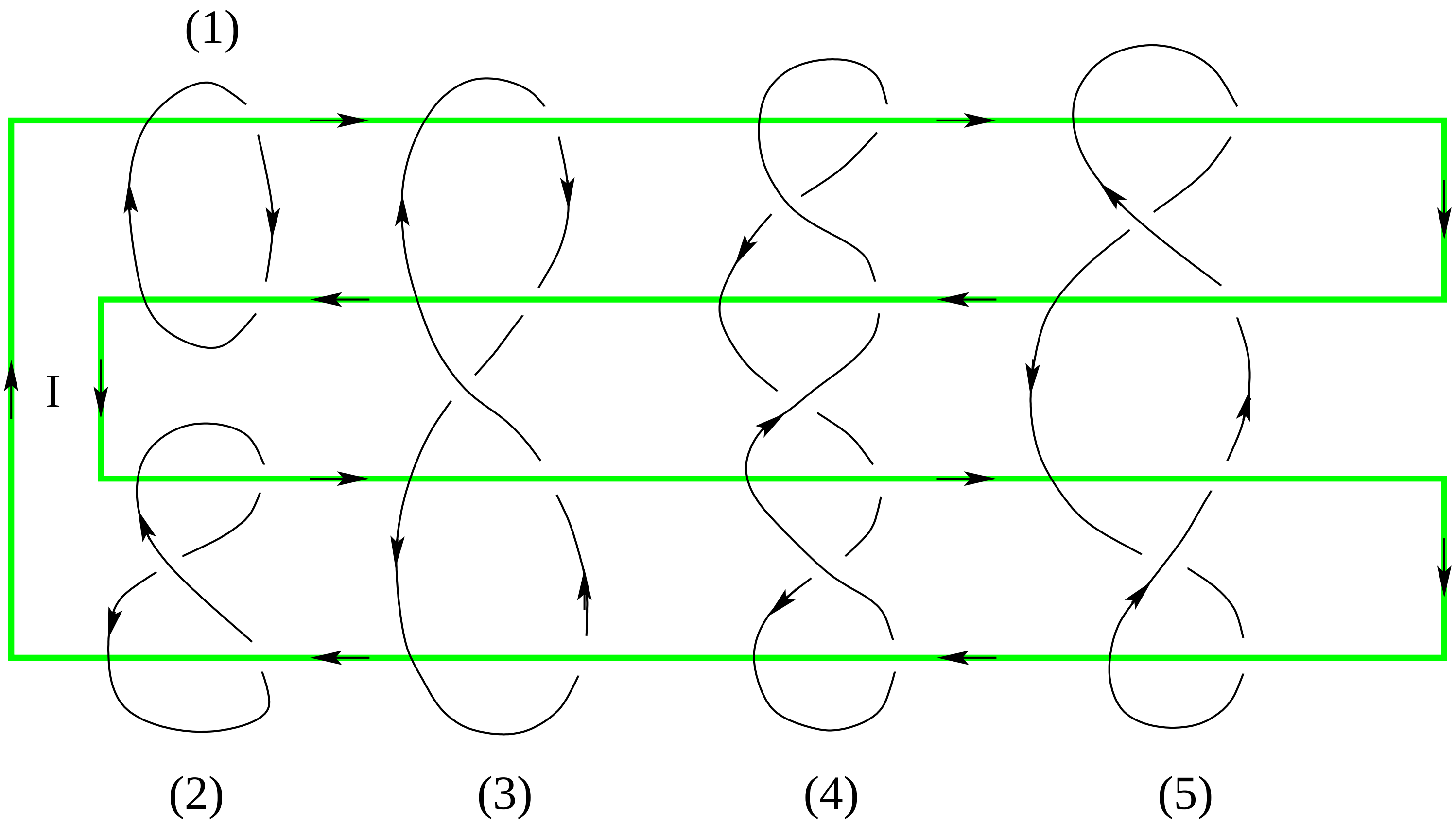
- Find the direction ($\odot, \otimes$) and the magnitude of the currents I_1, I_2, I_3.



Ampère's Law: Application (2)

An electric current I flows through the wire in the direction indicated.

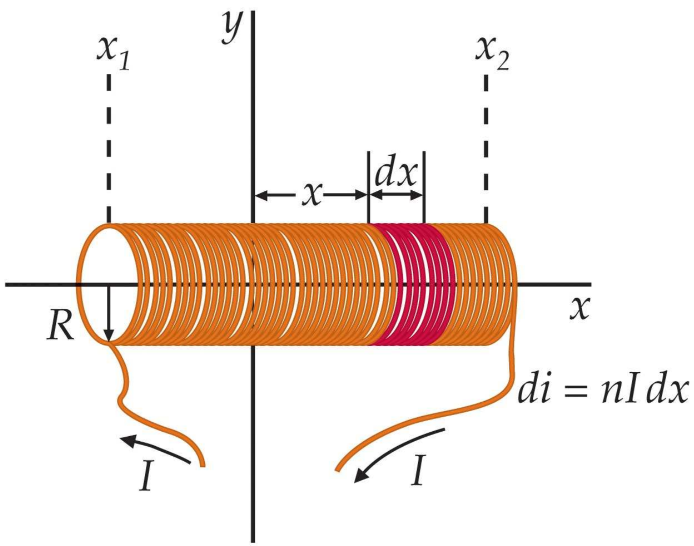
- Determine for each of the five Amperian loops whether the line integral $\oint \vec{B} \cdot d\vec{s}$ is positive, negative, or zero.

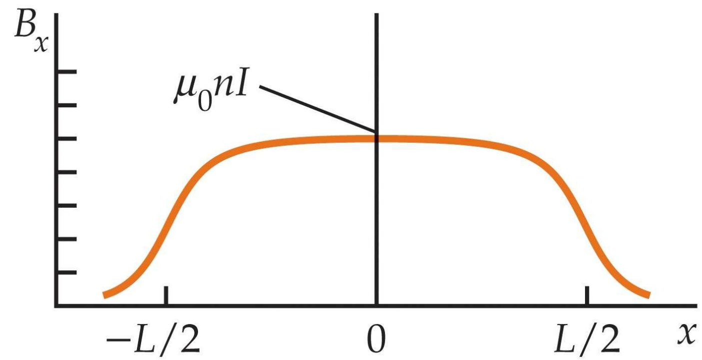


Magnetic Field on the Axis of a Solenoid

- Number of turns per unit length: $n = N/L$
- Current circulating in ring of width dx': $nI dx'$
- Magnetic field on axis of ring: $dB_x = \frac{\mu_0 (nI dx')}{2} \frac{R^2}{[(x-x')^2 + R^2]^{3/2}}$
- Magnetic field on axis of solenoid:

$$B_x = \frac{\mu_0 n I}{2} R^2 \int_{x_1}^{x_2} \frac{dx'}{[(x-x')^2 + R^2]^{3/2}} = \frac{\mu_0 n I}{2} \left(\frac{x - x_1}{\sqrt{(x-x_1)^2 + R^2}} - \frac{x - x_2}{\sqrt{(x-x_2)^2 + R^2}} \right)$$

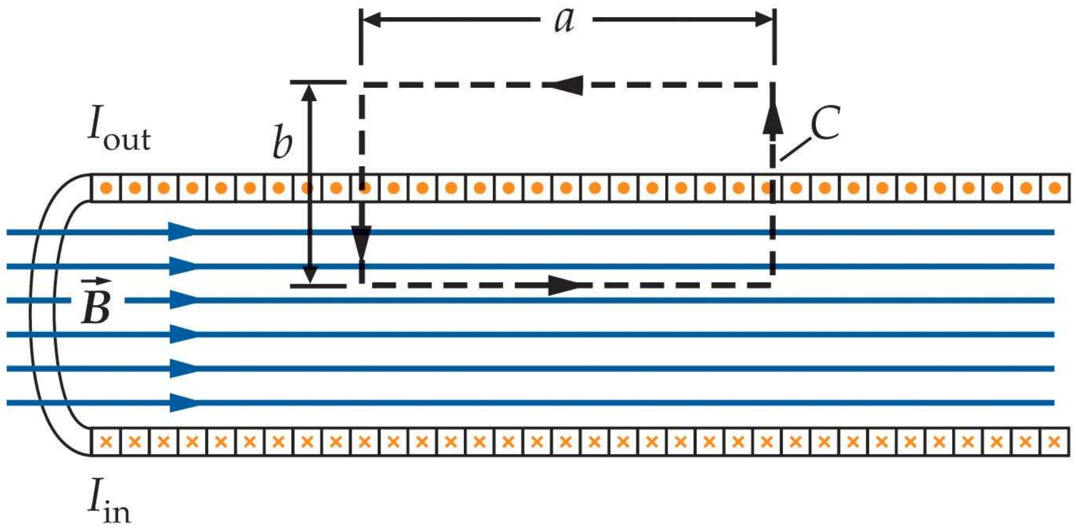




Ampère's Law: Magnetic Field Inside a Long Solenoid

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the rectangular Amperian loop shown.

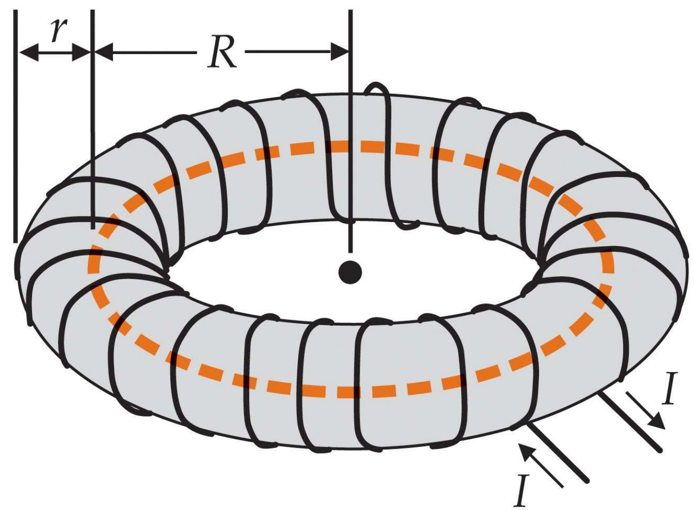
- Magnetic field inside: strong, uniform, directed along axis.
- Magnetic field outside: negligibly weak.
- Number of turns per unit length: n.
- Total current through Amperian loop: $I_C = nIa$ (I is the current in the wire).
- Ampère's law applied to rectangular loop: $Ba = \mu_0 nIa$.
- Magnetic field inside: $B = \mu_0 nI$.



Ampère's Law: Magnetic Field Inside a Toroid

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular Amperian loop shown.

- Magnetic field inside: directed tangentially with magnitude depending on R only.
- Magnetic field outside: negligibly weak.
- Number of turns: N.
- Total current through Amperian loop: $I_C = NI$ (I is the current in the wire).
- Ampère's law applied to circular loop: $B(2\pi R) = \mu_0 NI$.
- Magnetic field inside: $B = \dfrac{\mu_0 NI}{2\pi R}$.

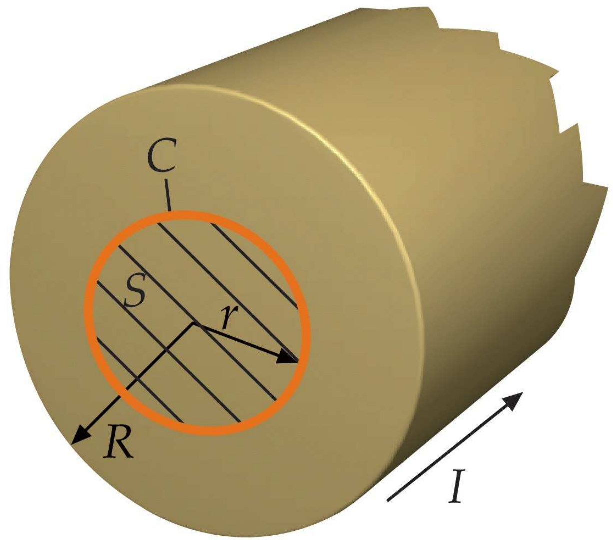


Ampère's Law: Magnetic Field Inside a Wire

Consider a long, straight wire of radius R.
The current I is distributed uniformly over the cross section.

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular loop of radius $r < R$.

- The symmetry dictates that the magnetic field $\vec{B}$ is directed tangentially with magnitude B depending on R only.
- Line integral: $\oint \vec{B} \cdot d\vec{\ell} = B(2\pi r)$.
- Fraction of current inside loop: $\dfrac{I_C}{I} = \dfrac{\pi r^2}{\pi R^2}$.
- Magnetic field at radius $r < R$: $B = \dfrac{\mu_0 I_C}{2\pi r} = \dfrac{\mu_0 I r}{2\pi R^2}$.
- B increases linearly with r from zero at the center.
- Magnetic field at the perimeter: $B = \dfrac{\mu_0 I}{2\pi R}$.

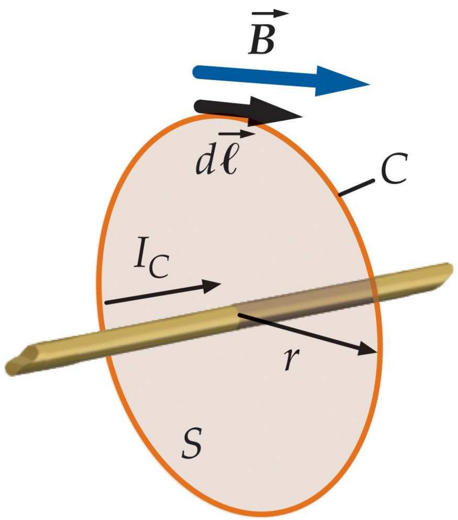


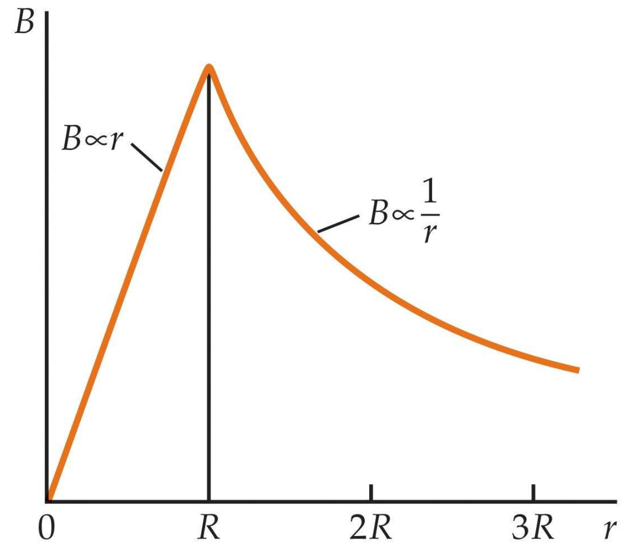
Ampère's Law: Magnetic Field Outside a Wire

Consider a long, straight wire of radius R with current I.

Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C$, to the circular loop of radius $r > R$.

- The symmetry dictates that the magnetic field $\vec{B}$ is directed tangentially with magnitude B depending on R only.
- Current inside loop: $I_C = I$.
- Ampère's law applied: $B(2\pi r) = \mu_0 I$.
- Magnetic field at radius $r > R$: $B = \dfrac{\mu_0 I}{2\pi r}$.



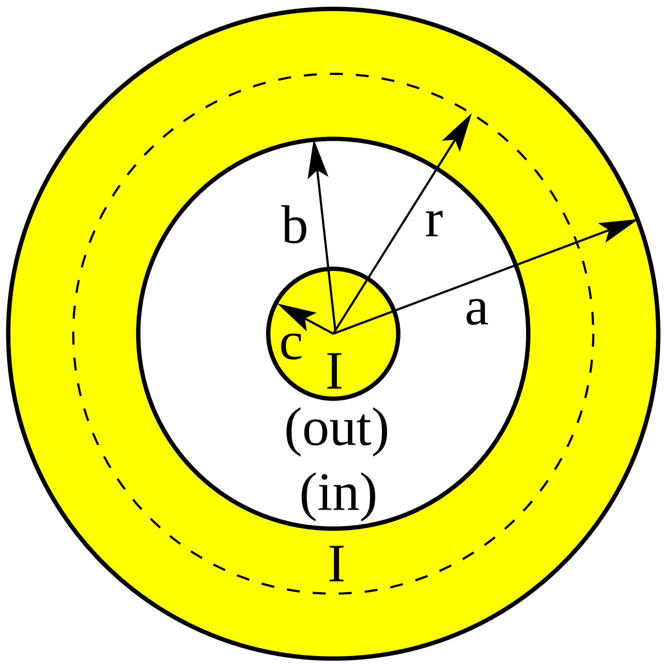


Ampère's Law: Coaxial Cable

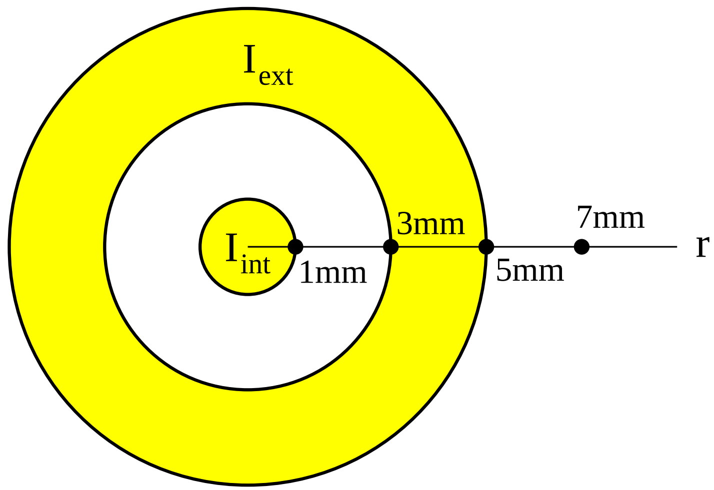
Consider a long coaxial cable, consisting of two cylindrical conductors separated by an insulator as shown in a cross-sectional view.

There is a current I flowing out of the plane in the inner conductor and a current of equal magnitude I flowing into the plane in the outer conductor.

Calculate the magnetic field B as a function of the radial coordinate r.

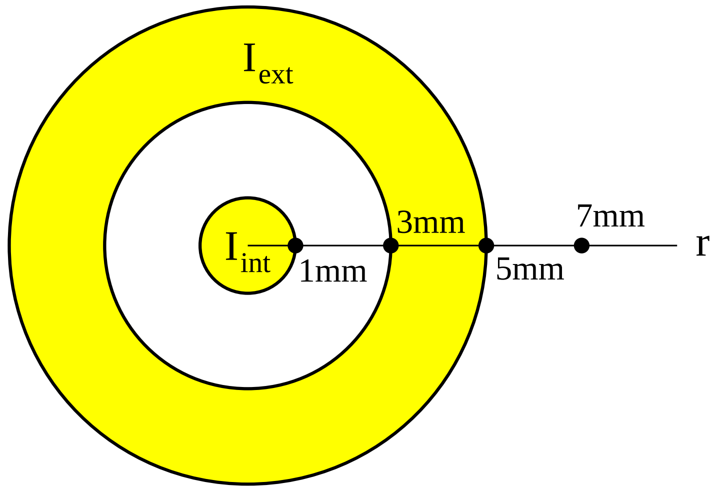


The coaxial cable shown in cross section has surfaces at radii 1mm, 3mm, and 5mm. Equal currents flow through both conductors: $I_{int} = I_{ext} = 0.03\text{A}$ $\odot$ (out). Find direction $(\uparrow, \downarrow)$ and magnitude (B_1, B_3, B_5, B_7) of the magnetic field at the four radii indicated ($\bullet$).



Unit Exam III: Problem #3 (Spring ’12)

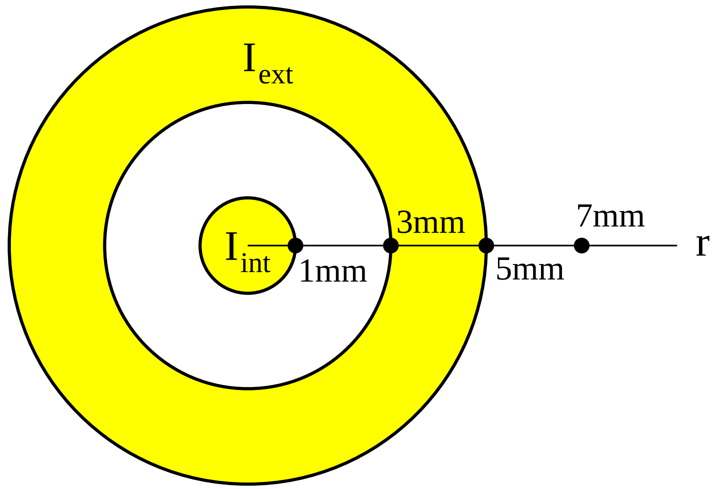
The coaxial cable shown in cross section has surfaces at radii 1mm, 3mm, and 5mm. Equal currents flow through both conductors: $I_{int} = I_{ext} = 0.03\text{A}$ $\odot$ (out). Find direction $(\uparrow, \downarrow)$ and magnitude (B_1, B_3, B_5, B_7) of the magnetic field at the four radii indicated ($\bullet$).



Solution:

$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 6\mu\text{T} \quad \uparrow$$

The coaxial cable shown in cross section has surfaces at radii 1mm, 3mm, and 5mm. Equal currents flow through both conductors: $I_{int} = I_{ext} = 0.03\text{A}$ $\odot$ (out). Find direction $(\uparrow, \downarrow)$ and magnitude (B_1, B_3, B_5, B_7) of the magnetic field at the four radii indicated $(\bullet)$.

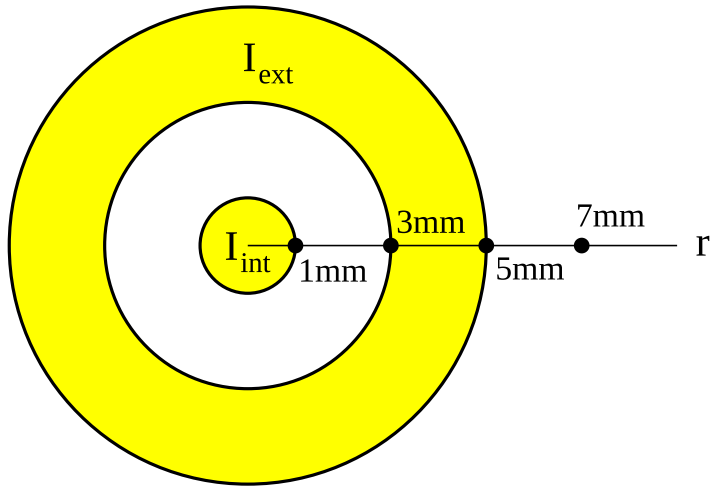


Solution:

$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 6\mu\text{T} \quad \uparrow$$

$$2\pi(3\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 2\mu\text{T} \quad \uparrow$$

The coaxial cable shown in cross section has surfaces at radii 1mm, 3mm, and 5mm. Equal currents flow through both conductors: $I_{int} = I_{ext} = 0.03\text{A}$ $\odot$ (out). Find direction $(\uparrow, \downarrow)$ and magnitude (B_1, B_3, B_5, B_7) of the magnetic field at the four radii indicated ($\bullet$).



Solution:

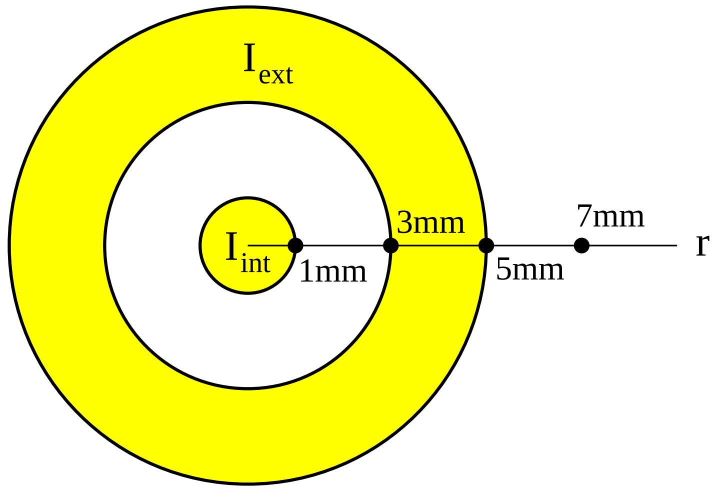
$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 6\mu\text{T} \quad \uparrow$$

$$2\pi(3\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 2\mu\text{T} \quad \uparrow$$

$$2\pi(5\text{mm})B_1 = \mu_0(0.06\text{A}) \quad \Rightarrow \quad B_1 = 2.4\mu\text{T} \quad \uparrow$$

Unit Exam III: Problem #3 (Spring ’12)

The coaxial cable shown in cross section has surfaces at radii 1mm, 3mm, and 5mm. Equal currents flow through both conductors: $I_{int} = I_{ext} = 0.03\text{A}$ $\odot$ (out). Find direction $(\uparrow, \downarrow)$ and magnitude (B_1, B_3, B_5, B_7) of the magnetic field at the four radii indicated ($\bullet$).



Solution:

$$2\pi(1\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 6\mu\text{T} \quad \uparrow$$

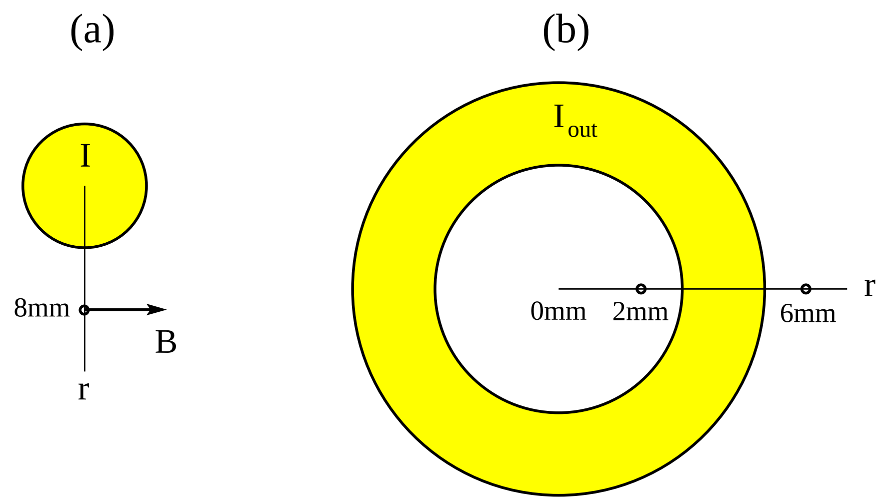
$$2\pi(3\text{mm})B_1 = \mu_0(0.03\text{A}) \quad \Rightarrow \quad B_1 = 2\mu\text{T} \quad \uparrow$$

$$2\pi(5\text{mm})B_1 = \mu_0(0.06\text{A}) \quad \Rightarrow \quad B_1 = 2.4\mu\text{T} \quad \uparrow$$

$$2\pi(7\text{mm})B_1 = \mu_0(0.06\text{A}) \quad \Rightarrow \quad B_1 = 1.71\mu\text{T} \quad \uparrow$$

Unit Exam III: Problem #2 (Spring '08)

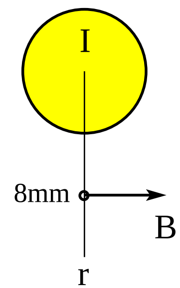
(a) Consider a solid wire of radius $R = 3\text{mm}$.
Find magnitude I and direction (in/out) that produces a magnetic field $B = 7\mu\text{T}$ at radius $r = 8\text{mm}$.
(b) Consider a hollow cable with inner radius $R_{int} = 3\text{mm}$ and outer radius $R_{ext} = 5\text{mm}$.
A current $I_{out} = 0.9\text{A}$ is directed out of the plane.
Find direction (up/down) and magnitude B_2, B_6 of the magnetic field at radius $r_2 = 2\text{mm}$ and $r_6 = 6\text{mm}$, respectively.



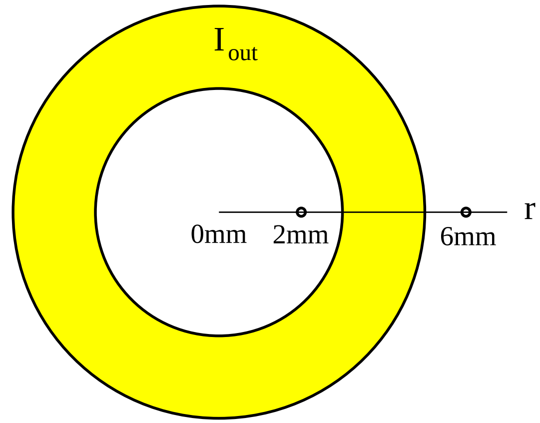
Unit Exam III: Problem #2 (Spring ’08)

(a) Consider a solid wire of radius $R = 3\text{mm}$.
Find magnitude I and direction (in/out) that produces a magnetic field $B = 7\mu\text{T}$ at radius $r = 8\text{mm}$.
(b) Consider a hollow cable with inner radius $R_{int} = 3\text{mm}$ and outer radius $R_{ext} = 5\text{mm}$.
A current $I_{out} = 0.9\text{A}$ is directed out of the plane.
Find direction (up/down) and magnitude B_2, B_6 of the magnetic field at radius $r_2 = 2\text{mm}$ and $r_6 = 6\text{mm}$, respectively.

(a)



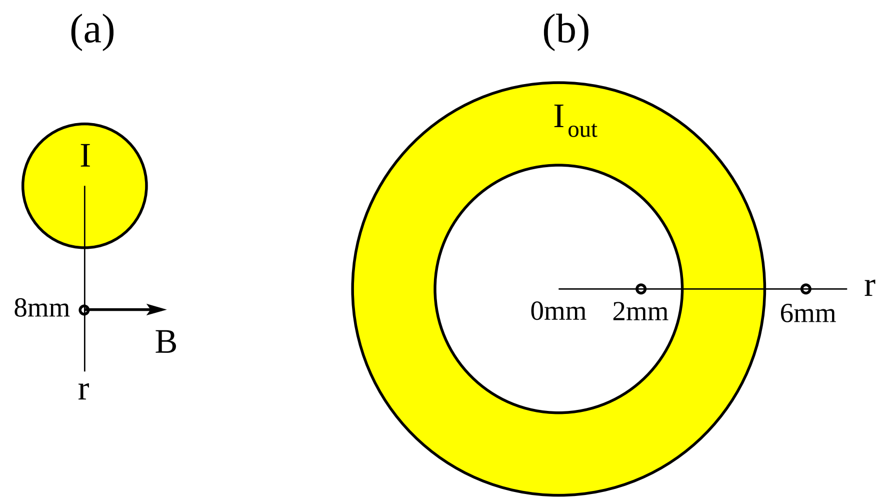
(b)



Solution:

(a) $7\mu\text{T} = \dfrac{\mu_0 I}{2\pi(8\text{mm})} \quad \Rightarrow I = 0.28\text{A}$ (out).

(a) Consider a solid wire of radius $R = 3$mm.
Find magnitude I and direction (in/out) that produces a magnetic field $B = 7\mu\mathrm{T}$ at radius $r = 8$mm.
(b) Consider a hollow cable with inner radius $R_{int} = 3$mm and outer radius $R_{ext} = 5$mm.
A current $I_{out} = 0.9$A is directed out of the plane.
Find direction (up/down) and magnitude B_2, B_6 of the magnetic field at radius $r_2 = 2$mm and $r_6 = 6$mm, respectively.



Solution:

(a) $7\mu\mathrm{T} = \dfrac{\mu_0 I}{2\pi(8\mathrm{mm})} \quad \Rightarrow \; I = 0.28\mathrm{A}$ (out).

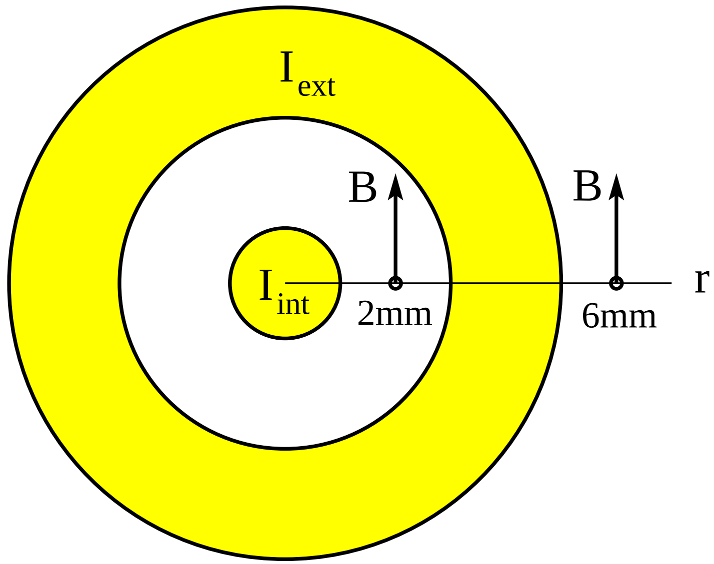
(b) $B_2 = 0,\qquad B_6 = \dfrac{\mu_0(0.9\mathrm{A})}{2\pi(6\mathrm{mm})} = 30\mu\mathrm{T}$ (up).

Unit Exam III: Problem #2 (Spring ’11)

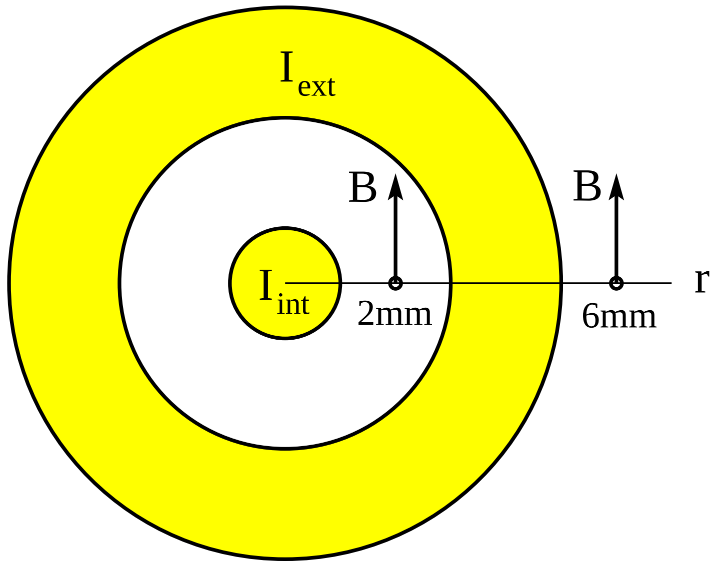
The coaxial cable shown has surfaces at radii 1mm, 3mm, and 5mm. The magnetic field is the same at radii 2mm and 6mm, namely $B = 7\mu$T in the direction shown.

(a) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm int}$ flowing through the inner conductor.

(b) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm ext}$ flowing through the outer conductor.



The coaxial cable shown has surfaces at radii 1mm, 3mm, and 5mm. The magnetic field is the same at radii 2mm and 6mm, namely $B = 7\mu$T in the direction shown.
(a) Find magnitude (in SI units) and direction (in/out) of the current I_{int} flowing through the inner conductor.
(b) Find magnitude (in SI units) and direction (in/out) of the current I_{ext} flowing through the outer conductor.

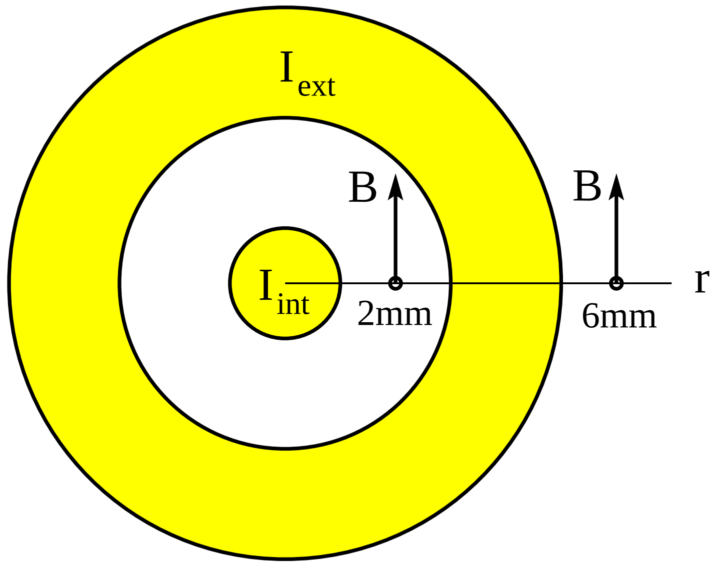


Solution:

(a) $(7\mu\text{T})(2\pi)(0.002\text{m}) = \mu_0 I_{\text{int}} \quad \Rightarrow \quad I_{\text{int}} = 0.07\text{A}$ (out)

Unit Exam III: Problem #2 (Spring '11)

The coaxial cable shown has surfaces at radii 1mm, 3mm, and 5mm. The magnetic field is the same at radii 2mm and 6mm, namely $B = 7\mu$T in the direction shown.
(a) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm int}$ flowing through the inner conductor.
(b) Find magnitude (in SI units) and direction (in/out) of the current $I_{\rm ext}$ flowing through the outer conductor.



Solution:

(a) $(7\mu\mathrm{T})(2\pi)(0.002\mathrm{m}) = \mu_0 I_{\rm int} \quad \Rightarrow \; I_{\rm int} = 0.07\mathrm{A}$ (out)

(b) $(7\mu\mathrm{T})(2\pi)(0.006\mathrm{m}) = \mu_0 (I_{\rm int} + I_{\rm ext}) \quad \Rightarrow \; I_{\rm int} + I_{\rm ext} = 0.21\mathrm{A}$ (out)
$\Rightarrow \; I_{\rm ext} = 0.14\mathrm{A}$ (out)