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GNSS Spoof Detection Using Shipboard IMU Measurements

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BIOGRAPHIES

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ABSTRACT

A variety of approaches have been proposed in the literature to detect spoofing of Global Navigation Satellite Systems (GNSS). These approaches vary widely based upon the assumed capabilities and a priori knowledge of the spoofer. This paper considers a method to detect spoofing based on comparing the relative (not absolute) platform trajectory estimated by the GNSS receiver to the relative trajectory developed from IMU measurements (specifically pitch and roll from a gyro compass).

The primary contribution of this paper is the development and analysis of a GNSS spoofing detection algorithm that exploits the unknown (to the spoofer) "high" frequency pitch/roll motion of the ship as seen by a commercial-off-the-shelf (COTS) receiver and an inertial measurement unit (IMU) that may already be in use onboard ships. We focus on generalized likelihood ratio tests using simple models of the GNSS and gyro measurements. Further, we avoid using a navigation filter, such as the extended Kalman filter, on the measurements; instead, the algorithm directly employs the instantaneous trajectories.

Experimental results are shown using a commercial GNSS receiver with data from a GNSS simulator with IMU capability. The length of time and amount of motion required to achieve low probabilities of false alarm and missed detection are analyzed.

INTRODUCTION

GNSS are well known to be accurate providers of position information across the globe; as such, they are commonly used to locate and navigate craft in various transportation modes (e.g. land vehicles, boats and ships, and aircraft). Because of high signal availabilities, capable/robust receivers, and well-populated satellite constellations, operators typically believe that the location and time information provided by their GNSS receiver is correct. More sophisticated users are concerned with the integrity of the derived information; RAIM algorithms were developed to address the possible failure of a single satellite.

Recent demonstrations have highlighted another threat to GNSS integrity, so called "spoofing" [1]. Spoofing is the intentional creation of RF signals to provide counterfeit information to the GNSS receiver.

Since spoofing might be undetected by a conventional receiver, this type of attack is considered more dangerous than a jamming attack in which the receiver is unable to provide any position, navigation, and/or time (PNT) solution.

A variety of approaches have been proposed in the literature to recognize spoofing and can vary widely based upon the assumed capabilities and a priori knowledge of the spoofer. Possible methods to detect a spoofing event at a single GNSS receiver include monitoring the power levels of the GNSS signals (absolute, relative, and across satellites), checking that the observed constellation is correct for the given time (e.g. number of and IDs of the satellites), testing the accuracy of the clock component, and checking the computed position against that derived from some non-GNSS source (e.g. an INS) [2]. Other methods include correlating the P(Y) code at the RF level [3], looking for vestigial peaks in the correlator outputs [4], comparing to trusted reference signals [5], using an antennae array to spatially locate and identify signals [6], and other multi-antenna methods [7].

This paper considers a method to detect spoofing based on comparing the relative (not absolute) platform trajectory estimated by the GNSS receiver to the relative trajectory developed from IMU measurements (specifically heading, pitch, and roll from a gyro). The motivation for this approach is three-fold: (1) the recent development of a spoof detection approach in which the GNSS antenna is intentionally wiggled so as to create carrier phase characteristics that would highlight the existence of a single point emitter spoofer [8], (2) our recognition that the combination of intended vessel motion and sea state interference (wind and current) cause the GNSS antennas on ships to move (wiggle) in unpredictable ways, and (3) the fact that existing Coast Guard vessels are equipped with a high performance IMU (a Sperry Mark 39 gyro compass).

The primary contribution of this paper is the development and analysis of a GNSS spoofing detection algorithm that exploits the unknown (to the spoofer) pitch/roll motion of the ship as seen by a commercial-off-the-shelf (COTS) receiver and an IMU that may already be in use onboard ships. While ad-hoc methods to test similarity of object movement (trajectory matching) have appeared in the computational geometry literature [9], we focus on generalized likelihood ratio tests using simple models of the GNSS and gyro measurements. Further, we avoid using a navigation filter, such as the extended Kalman filter, on the measurements; instead, the algorithm directly employs the instantaneous trajectories.

THE MEASUREMENTS

We concentrate on two dimensional (horizontal) motion of a ship and will use the variables e and n to indicate east and north position, respectively, in some coordinate frame. The pair $(e_{\cdot,k}, n_{\cdot,k})$ will be used to represent the location of some point on the ship as generated by some specific sensor; additional subscripts will be added to define the sensor and/or the location. The indexing by k recognizes that the ship is constantly moving, due to both its desired course as well as motion caused by the action of the waves and wind, and that the sensors are sampled over time.

In this work, the movement of the ship is recorded via two sets of measurements: actual location as determined by a GNSS receiver and relative movement (pitch and roll) as described by an inertial measurement unit (IMU) unit on the ship. The GNSS measurements are

$$e_{G,k} = e_{s,k} + e_{a,k} \quad \text{and} \quad n_{G,k} = n_{s,k} + n_{a,k}$$

in which the subscripts s correspond to the position of the center of mass of the ship and the subscripts a correspond to the GNSS antenna's position relative to that center. Similarly, the IMU measurements are

$$e_{I,k} = e_{a,k} \quad \text{and} \quad n_{I,k} = n_{a,k}$$

Specifically, we imagine that the IMU sees the identical antenna motion, but <u>not</u> the basic ship motion. Strictly one might think of the IMU measurements as pitch and roll; we assume that heading is also available and that the proper functional transformation to east/north as seen at the GNSS antenna's location is already accomplished. Further, we assume that the origin of the coordinate frame for the IMU measurements is at the center of mass of the ship.

THE HYPOTHESES

Our goal here is to test for spoofing which we define as the existence of GNSS signals that would result in an erroneous position solution at the GNSS receiver. We consider two situations, the null hypothesis, H_0, in which no spoofer is present, and the alternative hypothesis, H_1, in which a spoofer is present:

H_0: With no spoofer present each individual measurement is an accurate estimate of the action of the ship and its antenna:

$$\widehat{e_{G,k}} = e_{s,k} + e_{a,k} + w_{Ge,k}$$

$$\widehat{n_{G,k}} = n_{s,k} + n_{a,k} + w_{Gn,k}$$

and

$$\widehat{e_{I,k}} = e_{a,k} + w_{Ie,k} \qquad \widehat{n_{I,k}} = n_{a,k} + w_{In,k}$$

in which we use hats to indicate noisy measurements and the $w_{\cdot,k}$ are measurement errors (the first subscript identifies the sensor and direction) for $k = 0, 1, 2, \dots N-1$. Vectors of N such measurements will be denoted as $\widehat{\mathbf{e}_G}$, etc.

H_1: With a spoofer present we assume that the GNSS measurements are corrupted, but that the IMU measurements are not:

$$\widehat{e_{G,k}} = u_k + w_{Ge,k} \qquad \widehat{n_{G,k}} = v_k + w_{Gn,k}$$

and

$$\widehat{e_{I,k}} = e_{a,k} + w_{Ie,k} \qquad \widehat{n_{I,k}} = n_{a,k} + w_{In,k}$$

in which u_k and v_k describe the (unknown) spoofed location.

We will assume that the noise terms are independent, zero mean Gaussian random variables. The variances for the GNSS and IMU measurements will be σ_G^2 and σ_I^2, respectively.

HYPOTHESIS TESTING

Our goal is detection, to estimate which hypothesis is true based upon a set of measurements. We use a Neyman-Pearson formulation for this problem and wish to develop a binary hypothesis test with fixed probability of false alarm (the probability of deciding H_1 when H_0 is true) and maximum probability of detection (the probability of deciding H_1 when H_1 is true). Hypothesis testing is usually implemented by computing a scalar function of the vector observation data, $L\left(\widehat{\mathbf{e}_G}, \widehat{\mathbf{n}_G}, \widehat{\mathbf{e}_I}, \widehat{\mathbf{n}_I}\right)$, called the test statistic, and comparing this value to a constant called the threshold. If the test statistic exceeds the threshold, we decide H_1; if not, we decide H_0. Symbolically, we write this as

$$L\left(\widehat{\mathbf{e}_G}, \widehat{\mathbf{n}_G}, \widehat{\mathbf{e}_I}, \widehat{\mathbf{n}_I}\right) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda$$

The optimum test statistic for the Neyman-Pearson formulation is well known to be the likelihood ratio [10]:

$$L\left(\widehat{\mathbf{e}_G}, \widehat{\mathbf{n}_G}, \widehat{\mathbf{e}_I}, \widehat{\mathbf{n}_I}\right) = \frac{f\left(\widehat{\mathbf{e}_G}, \widehat{\mathbf{n}_G}, \widehat{\mathbf{e}_I}, \widehat{\mathbf{n}_I} | \mathrm{H}_1\right)}{f\left(\widehat{\mathbf{e}_G}, \widehat{\mathbf{n}_G}, \widehat{\mathbf{e}_I}, \widehat{\mathbf{n}_I} | \mathrm{H}_0\right)}$$

which is the ratio of the conditional probability density functions (pdfs) of the measurements under the two hypotheses. Substituting our models under the two hypotheses the test statistic is

$$L = \prod_k \frac{\frac{1}{2\pi\sigma_G\sigma_I} e^{-\frac{1}{2\sigma_G^2}\left(\widehat{e_{G,k}} - u_k\right)^2} e^{-\frac{1}{2\sigma_I^2}\left(\widehat{e_{I,k}} - e_{a,k}\right)^2}}{\frac{1}{2\pi\sigma_G\sigma_I} e^{-\frac{1}{2\sigma_G^2}\left(\widehat{e_{G,k}} - e_{s,k} - e_{a,k}\right)^2} e^{-\frac{1}{2\sigma_I^2}\left(\widehat{e_{I,k}} - e_{a,k}\right)^2}}$$

$$\times \prod_k \frac{\frac{1}{2\pi\sigma_G\sigma_I} e^{-\frac{1}{2\sigma_G^2}\left(\widehat{n_{G,k}} - v_k\right)^2} e^{-\frac{1}{2\sigma_I^2}\left(\widehat{n_{I,k}} - n_{a,k}\right)^2}}{\frac{1}{2\pi\sigma_G\sigma_I} e^{-\frac{1}{2\sigma_G^2}\left(\widehat{n_{G,k}} - n_{s,k} - n_{a,k}\right)^2} e^{-\frac{1}{2\sigma_I^2}\left(\widehat{n_{I,k}} - n_{a,k}\right)^2}}$$

(we have dropped the arguments of L for brevity). Canceling common terms (the constants and the pdfs of the IMU measurements), this is

$$L = \prod_k \frac{e^{-\frac{1}{2\sigma_G^2}\left(\widehat{e_{G,k}} - u_k\right)^2} e^{-\frac{1}{2\sigma_G^2}\left(\widehat{n_{G,k}} - v_k\right)^2}}{e^{-\frac{1}{2\sigma_G^2}\left(\widehat{e_{G,k}} - e_{s,k} - e_{a,k}\right)^2} e^{-\frac{1}{2\sigma_G^2}\left(\widehat{n_{G,k}} - n_{s,k} - n_{a,k}\right)^2}}$$

Taking a logarithm and removing more constants, the equivalent test (l instead of L) is

$$\begin{aligned} l \quad = \quad & \sum_k \widehat{e_{G,k}} \left(u_k - e_{s,k} - e_{a,k}\right) \\ & + \sum_k \widehat{n_{G,k}} \left(v_k - n_{s,k} - n_{a,k}\right) \end{aligned}$$

Not surprisingly, a spatial correlator. Of great use below is the following observation – that the correlators for east and north are additive!

Unfortunately, some of the parameters of this test are unknown; specifically u_k, v_k, $e_{s,k}$, $n_{s,k}$, $e_{a,k}$, and $n_{a,k}$. A common approach, the generalized likelihood ratio test (GLRT), replaces each of these with its maximum likelihood estimate (MLE). Unfortunately, as defined thus far, there are too many unknowns here for the GLRT to provide a useful result (this is demonstrated in the next section). To move forward, we constrain the underlying ship motion to be constant velocity. As the algebra is lengthy, we start by recasting the problem as one dimensional and employ vector-matrix notation.

THE ONE-DIMENSIONAL VERSION

Imagine that we have two sets of n observations, y_k and z_k ($k = 0, 1, \dots N-1$), and three sets of constants x_k, a_k, and c_k, all indexed by k. (Think of x as the underlying ship movement, a the antenna movement, c the spoofed movement, y the GNSS measurement, and z the IMU measurement, all in one dimension.) Let $\mathbf{y}$ and $\mathbf{z}$ be the vectors of measurements with components $\mathbf{x}$, $\mathbf{a}$, and $\mathbf{c}$. Under the two hypotheses these are related by:

H_0:

$$\mathbf{y} = \mathbf{x} + \mathbf{a} + \mathbf{w}_y \qquad \mathbf{z} = \mathbf{a} + \mathbf{w}_z$$

H_1:

$$\mathbf{y} = \mathbf{c} + \mathbf{w}_y \qquad \mathbf{z} = \mathbf{a} + \mathbf{w}_z$$

in which $\mathbf{w}_y$ and $\mathbf{w}_z$ are the measurement noise vectors. The unusual characteristic of this problem formulation is that $\mathbf{y}$ varies with the hypothesis while $\mathbf{z}$

does not; however, $\mathbf{z}$ carries information about $\mathbf{y}$ under hypothesis H_0. We assume that the noise terms $\mathbf{w}_y$ and $\mathbf{w}_z$ are composed of independent, zero mean Gaussian random variables with variances σ_y^2 and σ_z^2, respectively.

For this simpler problem the likelihood ratio is

$$\begin{aligned} L(\mathbf{y},\mathbf{z}) &= \frac{\frac{1}{(2\pi)^{N/2}\sigma_y^N} e^{-\frac{1}{2\sigma_y^2}(\mathbf{y}-\mathbf{c})^T(\mathbf{y}-\mathbf{c})}}{\frac{1}{(2\pi)^{N/2}\sigma_y^N} e^{-\frac{1}{2\sigma_y^2}(\mathbf{y}-\mathbf{x}-\mathbf{a})^T(\mathbf{y}-\mathbf{x}-\mathbf{a})}} \\ &\quad \times \frac{\frac{1}{(2\pi)^{N/2}\sigma_z^N} e^{-\frac{1}{2\sigma_z^2}(\mathbf{z}-\mathbf{a})^T(\mathbf{z}-\mathbf{a})}}{\frac{1}{(2\pi)^{N/2}\sigma_z^N} e^{-\frac{1}{2\sigma_z^2}(\mathbf{z}-\mathbf{a})^T(\mathbf{z}-\mathbf{a})}} \\ &= e^{\frac{1}{\sigma_y^2}(\mathbf{c}-\mathbf{x}-\mathbf{a})^T\mathbf{y}} e^{-\frac{1}{2\sigma_y^2}\mathbf{c}^T\mathbf{c}} e^{\frac{1}{2\sigma_y^2}(\mathbf{x}+\mathbf{a})^T(\mathbf{x}+\mathbf{a})} \end{aligned}$$

Taking a logarithm and ignoring the constants, the log-likelihood test is

$$l(\mathbf{y},\mathbf{z}) = (\mathbf{c}-\mathbf{x}-\mathbf{a})^T\mathbf{y}$$

The result in one dimension is still a correlator that ignores $\mathbf{z}$, and depends upon unknown movement vectors ($\mathbf{c}$, $\mathbf{x}$, and $\mathbf{a}$).

The GLRT for the General Case

Let's imagine that $\mathbf{c}$, $\mathbf{x}$, and $\mathbf{a}$ are all unknown and employ the GLRT; specifically, replace these unknowns with their MLEs. Since the problem is linear and Gaussian, the MLEs are characterized as follows:

H_0: The likelihood function is

$$\frac{1}{(2\pi)^N\sigma_y^N\sigma_z^N} e^{-\frac{1}{2\sigma_y^2}(\mathbf{y}-\mathbf{x}-\mathbf{a})^T(\mathbf{y}-\mathbf{x}-\mathbf{a})} e^{-\frac{1}{2\sigma_z^2}(\mathbf{z}-\mathbf{a})^T(\mathbf{z}-\mathbf{a})}$$

Maximizing over the choice of $\mathbf{a}$ and $\mathbf{x}$ yields

$$\widehat{\mathbf{a}} = \mathbf{z} \qquad \widehat{\mathbf{x}} = \mathbf{y}-\mathbf{a} = \mathbf{y}-\mathbf{z}$$

H_1: The likelihood function is

$$\frac{1}{(2\pi)^N\sigma_y^N\sigma_z^N} e^{-\frac{1}{2\sigma_y^2}(\mathbf{y}-\mathbf{c})^T(\mathbf{y}-\mathbf{c})} e^{-\frac{1}{2\sigma_z^2}(\mathbf{z}-\mathbf{a})^T(\mathbf{z}-\mathbf{a})}$$

Maximizing over the choice of $\mathbf{a}$ and $\mathbf{c}$ yields

$$\widehat{\mathbf{a}} = \mathbf{z} \qquad \widehat{\mathbf{c}} = \mathbf{y}$$

Substituting these estimates in, the GLRT is

$$l_{\mathrm{GLRT}}(\mathbf{y},\mathbf{z}) = (\mathbf{y}-(\mathbf{y}-\mathbf{z})-\mathbf{z})\,\mathbf{y} = 0\,!$$

What happened? The GLRT is a constant (zero); in order to get a fixed probability of false alarm the detection result is always a decision for H_0 and the GLRT approach fails to yield a useful result. While the noisy observation $\mathbf{z}$ seems to tell us something about $\mathbf{y}$ under H_0, the fact that $\mathbf{x}$ is any vector eliminates the information contained in $\mathbf{z}$.

A Linear Model for Gross Motion

Let's imagine that the movement observed in the y_k, but not in the z_k corresponds to a linear model (i.e. constant velocity ship movement), different under the two hypotheses

$$x_k = m_0 k + b_0 \qquad c_k = m_1 k + b_1$$

In this formulation b_0 and b_1 are the intercepts (offsets) and m_0 and m_1 are the slopes (velocities per unit time) under the two hypotheses, respectively. In vector form, these are

$$\mathbf{x} = m_0\mathbf{v} + b_0\mathbf{1} \qquad \mathbf{c} = m_1\mathbf{v} + b_1\mathbf{1}$$

in which we define $\mathbf{v} = [0,1,2,\ldots N-1]^T$ as a counting vector and $\mathbf{1} = [1,1,\ldots 1]^T$ as the vector of all ones. Under H_0 the MLEs for m_0, b_0, and the a_k jointly maximize

$$\frac{1}{(2\pi)^N\sigma_y^N\sigma_z^N} \quad e^{-\frac{1}{2\sigma_y^2}(\mathbf{y}-m_0\mathbf{v}-b_0\mathbf{1}-\mathbf{a})^T(\mathbf{y}-m_0\mathbf{v}-b_0\mathbf{1}-\mathbf{a})} \times e^{-\frac{1}{2\sigma_z^2}(\mathbf{z}-\mathbf{a})^T(\mathbf{z}-\mathbf{a})}$$

Clearly the MLE for $\mathbf{a}$ is still $\widehat{\mathbf{a}} = \mathbf{z}$, so we want to maximize

$$e^{-\frac{1}{2\sigma_y^2}(\mathbf{y}-m_0\mathbf{v}-b_0\mathbf{1}-\mathbf{z})^T(\mathbf{y}-m_0\mathbf{v}-b_0\mathbf{1}-\mathbf{z})}$$

Taking logarithms and ignoring irrelevant terms the MLEs for m_0 and b_0 are defined by

$$\begin{aligned} \left\{\widehat{m_0},\widehat{b_0}\right\} &= \arg\min_{m_0,b_0} (\mathbf{y}-\mathbf{z}-m_0\mathbf{v}-b_0\mathbf{1})^T \\ &\quad (\mathbf{y}-\mathbf{z}-m_0\mathbf{v}-b_0\mathbf{1}) \\ &= \arg\min_{m_0,b_0} \sum_k (y_k - z_k - m_0 k - b_0)^2 \end{aligned}$$

Similarly, under H_1 the MLEs for m_1 and b_1 are defined by

$$\begin{aligned} \left\{\widehat{m_1},\widehat{b_1}\right\} &= \arg\min_{m_0,b_0} (\mathbf{y}-m_1\mathbf{v}-b_1\mathbf{1})^T \\ &\quad (\mathbf{y}-m_1\mathbf{v}-b_1\mathbf{1}) \\ &= \arg\min_{m_1,b_1} \sum_k (y_k - m_1 k - b_1)^2 \end{aligned}$$

The second line of each of these expressions shows that m and b are the coefficients of a least squares linear fit of the respective data, $y_k - z_k$ and y_k.

The GLRT for the Linear Case

Normally one would find the MLEs above by taking derivatives and setting them to zero, etc.; the precise development of expressions for the four parameters appears in Appendix 1. The resulting GLRT is

$$\begin{aligned} l(\mathbf{y},\mathbf{z}) &= (\hat{\mathbf{c}}-\hat{\mathbf{x}}-\hat{\mathbf{a}})^T\mathbf{y} \\ &= \sum_k (\widehat{c_k}-\widehat{x_k}-\widehat{a_k})y_k \\ &= \sum_k \left(\widehat{m_1}k+\widehat{b_1}-\widehat{m_0}k-\widehat{b_0}-z_k\right)y_k \\ &= (\widehat{m_1}-\widehat{m_0})\sum_k ky_k+\left(\widehat{b_1}-\widehat{b_0}\right)\sum_k y_k \\ &\quad -\sum_k z_k y_k \end{aligned}$$

Define the N-by-N symmetric matrix $\mathbf{A}$ with elements A_{jk}

$$\begin{aligned} A_{jk} &= \frac{12}{N(N-1)(N+1)}jk+\frac{2(2N-1)}{N(N+1)} \\ &\quad -\frac{6}{N(N+1)}j-\frac{6}{N(N+1)}k-\delta_{jk} \end{aligned}$$

where δ_{jk} is a Kronecker delta. Using the results from Appendix 1 we can write the test statistic in matrix form as

$$l(\mathbf{y},\mathbf{z})=\mathbf{z}^T\mathbf{A}\mathbf{y}$$

Since it is scalar, we can also write it as $l(\mathbf{y},\mathbf{z}) = \mathbf{y}^T\mathbf{A}^T\mathbf{z}=\mathbf{y}^T\mathbf{A}\mathbf{z}$.

Analysis of the Result

Under the two hypotheses the measurement vectors can be written as

$$\mathbf{y}=m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y \quad \text{and} \quad \mathbf{z}=\mathbf{a}+\mathbf{w}_z$$

in which $m=m_0$, $b=b_0$, and $\mathbf{g}=\mathbf{a}$ under H_0 and $m=m_1$, $b=b_1$, and $\mathbf{g}=\mathbf{0}$ under H_1 ($\mathbf{0}$ being the all zero vector). Note that $\mathbf{y}$ and $\mathbf{z}$ are both Gaussian random vectors; hence, the test statistic is non-Gaussian due to it containing sums of products over these vectors. To proceed, two approaches come to mind:

- Invoke the central limit theorem and characterize the test by its mean and variance.
- Imagine that the IMU measurements (the z_k) are essentially *perfect*, i.e. let $\mathbf{w}_z=\mathbf{0}$, so that the test statistic reduces to a Gaussian variable.

We initially pursue the first option.

To employ the Central Limit Theorem it is necessary to compute the mean and variance of the test statistic. We begin with the mean

$$\mu\equiv E\{l\}=E\left\{\mathbf{z}^T\mathbf{A}\mathbf{y}\right\}$$

Using the fact that $\mathbf{y}$ and $\mathbf{z}$ are independent random vectors, we have

$$\mu=E\left\{\mathbf{z}^T\right\}\mathbf{A}E\{\mathbf{y}\}$$

Each of these expectations can be computed separately

$$E\left\{\mathbf{z}^T\right\}=E\left\{\mathbf{a}^T+\mathbf{w}_z^T\right\}=\mathbf{a}^T$$

and

$$E\{\mathbf{y}\}=E\{m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y\}=m\mathbf{v}+b\mathbf{1}+\mathbf{g}$$

in which the zero means of both $\mathbf{w}_y$ and $\mathbf{w}_z$ have been employed. The result is

$$\begin{aligned} \mu &= \mathbf{a}^T\mathbf{A}(m\mathbf{v}+b\mathbf{1}+\mathbf{g}) \\ &= m\mathbf{a}^T\mathbf{A}\mathbf{v}+b\mathbf{a}^T\mathbf{A}\mathbf{1}+\mathbf{a}^T\mathbf{A}\mathbf{g} \end{aligned}$$

In Appendix 2 it is shown that $\mathbf{A}\mathbf{v}=\mathbf{0}$ and $\mathbf{A}\mathbf{1}=\mathbf{0}$, so

$$\mu=\mathbf{a}^T\mathbf{A}\mathbf{g}$$

Under H_0 we have $\mathbf{g}=\mathbf{a}$ and

$$\mu_0\equiv E\{l|\mathrm{H}_0\}=\mathbf{a}^T\mathbf{A}\mathbf{a}$$

a quadratic form. In Appendix 2 is is shown that $\mathbf{A}$ is negative semi-definite; hence, $\mu_0\leq 0$ for all antenna motion vectors $\mathbf{a}$. Under H_1 we have $\mathbf{g}=\mathbf{0}$ and

$$\mu_1\equiv E\{l|\mathrm{H}_1\}=0$$

Next, let's address the variance. After involved algebra (detailed in Appendix 3) the variance of the GLRT can be shown to be

$$\mathrm{Var}\{l\}=(N-2)\sigma_y^2\sigma_z^2-\sigma_y^2\mathbf{a}^T\mathbf{A}\mathbf{a}-\sigma_z^2\mathbf{g}^T\mathbf{A}\mathbf{g}$$

Under H_0 we have $\mathbf{g}=\mathbf{a}$ so

$$\mathrm{Var}\{l\}=(N-2)\sigma_y^2\sigma_z^2-\left(\sigma_y^2+\sigma_z^2\right)\mathbf{a}^T\mathbf{A}\mathbf{a}$$

while under H_1 we have $\mathbf{g}=\mathbf{0}$ so

$$\mathrm{Var}\{l\}=(N-2)\sigma_y^2\sigma_z^2-\sigma_y^2\mathbf{a}^T\mathbf{A}\mathbf{a}$$

With these parameters the false alarm probability is

$$\begin{aligned} \mathrm{P_{fa}} &= \mathrm{Prob}\{l>\lambda|\mathrm{H}_0\} \\ &= Q\left(\frac{\lambda-\mu_0}{\sigma_0}\right) \\ &= Q\left(\frac{\lambda-\mathbf{a}^T\mathbf{A}\mathbf{a}}{\sqrt{(N-2)\sigma_y^2\sigma_z^2-\left(\sigma_y^2+\sigma_z^2\right)\mathbf{a}^T\mathbf{A}\mathbf{a}}}\right) \end{aligned}$$

($Q(\cdot)$ being the Gaussian tail probability). We could, of course, solve this expression for the threshold

$$\lambda=\mathbf{a}^T\mathbf{A}\mathbf{a}+\sqrt{(N-2)\sigma_y^2\sigma_z^2-\left(\sigma_y^2+\sigma_z^2\right)\mathbf{a}^T\mathbf{A}\mathbf{a}}\,Q^{-1}(\mathrm{P_{fa}})$$

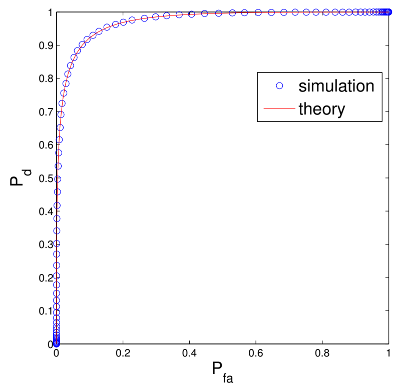

Figure 1: Typical ROC to verify the CLT assumption.

The detection probability is

$$\begin{aligned} \mathrm{P_d} &= \mathrm{Prob}\left\{ l > \lambda | \mathrm{H}_1 \right\} \\ &= Q\left(\frac{\lambda - \mu_1}{\sigma_1} \right) \\ &= Q\left(\frac{\lambda}{\sqrt{(N-2)\sigma_y^2 \sigma_z^2 - \sigma_y^2 \mathbf{a}^T \mathbf{A} \mathbf{a}}} \right) \end{aligned}$$

As expected, this is an increasing function of $-\mathbf{a}^T\mathbf{A}\mathbf{a}$, the *power* in the antenna motion. As an example, Figure 1 shows a typical receiver operating characteric (ROC). The figure compares simulation results (the blue circles) with the theoretical results to validate the Central Limit Theorem assumption.

Figure 2 shows the detection probability versus the false alarm probability for two different test durations and two different antenna distances from the ship's centroid. A mild sea state was assumed: sinusoidal roll with an amplitude of 5.14 degrees at 0.35 Hertz and sinusoidal pitch of 2.29 degrees at 0.7 Hertz. The standard deviation for the GNSS measurements was assumed to equal 0.4 meters; the IMU measurements' standard deviation was assumed to be 1.7 arc minutes. As expected, the detection performance improves as the antenna height increases or as the number of samples used in the test increases.

A Perfect IMU

Let's reconsider these results if the IMU is perfect so that $\sigma_z = 0$. The means are unchanged

$$\mu_0 \equiv E\left\{ l | \mathrm{H}_0 \right\} = \mathbf{a}^T\mathbf{A}\mathbf{a}$$

and

$$\mu_1 \equiv E\left\{ l | \mathrm{H}_1 \right\} = 0$$

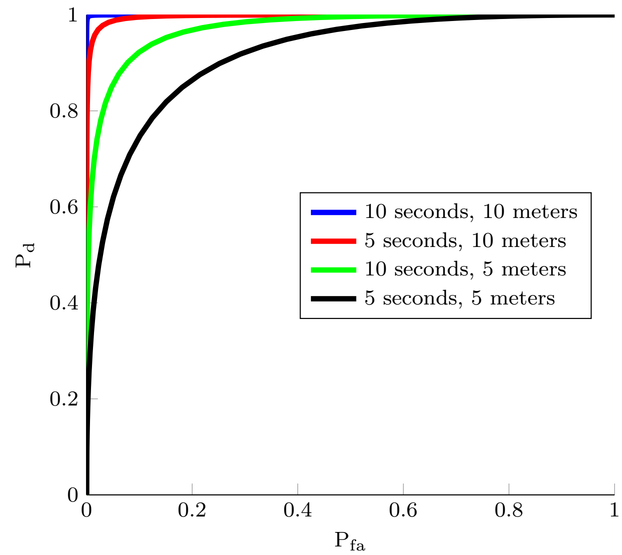

Figure 2: Typical ROC – antenna height of 5 or 10 meters, mild sea state, and a 5 or 10 second observation period (GNSS and gyro data at 1 Hz).

and the variances are equal under both hypotheses

$$\mathrm{Var}\left\{ l | \text{perfect} \quad \text{IMU} \right\} = -\sigma_y^2 \mathbf{a}^T\mathbf{A}\mathbf{a}$$

(the minus sign is fine since $\mathbf{A}$ is negative semi-definite). Further, the test statistic is Gaussian since it is linear in $\mathbf{y}$. The performance ($\mathrm{P_d}$ as a function of $\mathrm{P_{fa}}$) is

$$\mathrm{P_d} = Q\left(\frac{\sqrt{-\mathbf{a}^T\mathbf{A}\mathbf{a}}}{\sigma_y} + Q^{-1}\left(\mathrm{P_{fa}}\right) \right)$$

EXTENDING TO 2 (or 3) DIMENSIONS

Under the assumption that the noise in the East and North components are independent and identically distributed, then we saw at the beginning of this paper that the log-likelihood ratio test reduced to the sum of the tests in each individual direction. Modifying the notation from the one dimensional analysis above, the resulting test would be of the form

$$l = \widehat{\mathbf{e}_I}^T \mathbf{A} \widehat{\mathbf{e}_G} + \widehat{\mathbf{n}_I}^T \mathbf{A} \widehat{\mathbf{n}_G}$$

where $\widehat{\mathbf{e}}$ and $\widehat{\mathbf{n}}$, with appropriate subscripts, are vectorized East and North measurements. A similar extension could be made to three dimensions, although the GNSS noise variance in the Up direction is commonly assumed to be greater than either East or North.

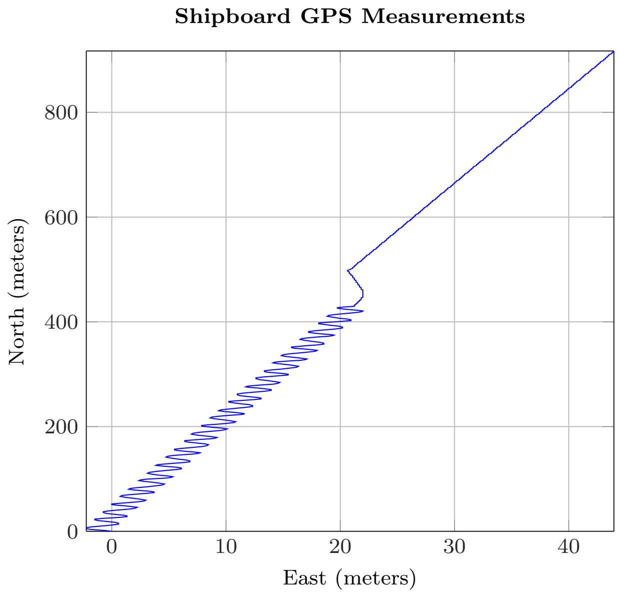

Figure 3: GNSS measurements for ship heading almost due north showing pitch and roll motion until spoofing event starts halfway through scenario.

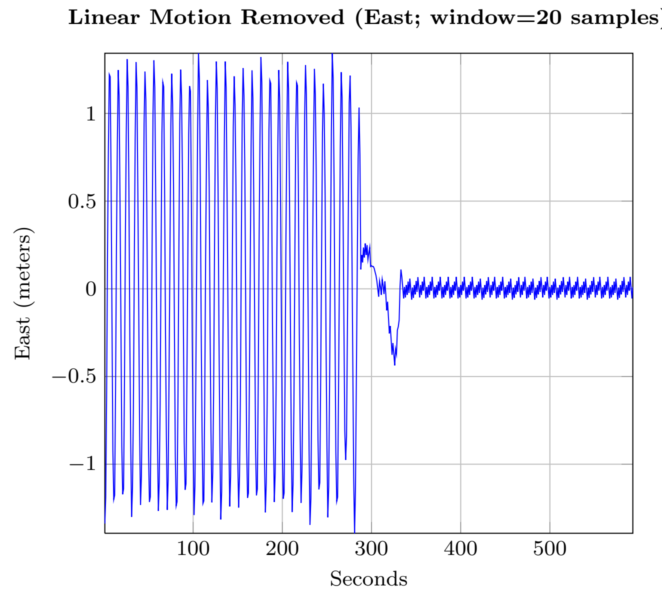

Figure 4: East motion of ship using GNSS measurements after linear motion is removed using the A matrix.

EXPERIMENTATION

The original intent of this project was to test potential algorithms with actual data collected from a Coast Guard cutter, recording both GNSS data and wind/sea related measurements from a MK-39 gyro. Unfortunately, we were unable to do so during the project's timeline, so instead we collected data from a GNSS simulator. The Spirent GSS-8000 allows for the creation of scenarios that include pitch and roll to vessel motion (mostly via setting the sea state). Typical gyro roll and pitch accuracy is 0.08 degrees RMS and the typical accuracy for the Novatel GPS receiver used in the tests is 0.4 meters RMS.

Figure 3 shows the GNSS measurements for the scenario considered here. The vessel is heading on a course of 003 degrees true with a mild sea state of 4.7 degrees of roll at 0.1 Hz and 3 degrees of pitch at 0.06 Hz. The GNSS antenna location was offset (2,5,15) meters from the ship's centroid. The spoofing event begins about halfway into the scenario in which the spoofer knew the nominal ship location and movement but did not have knowledge of the pitch and roll of the GNSS antenna.

The test statistic used for the detection of spoofing was defined earlier as

$$l(\mathbf{y}, \mathbf{z}) = \mathbf{z}^T \mathbf{A} \mathbf{y}.$$

The removal of linear motion is accomplished in the $\mathbf{Ay}$ portion of the test statistic and shown in Figure 4 for the East motion of the ship. The value of the test statistic over time is shown in Figure 5 using 10 seconds of GNSS and IMU data. We note that the detector was able to quickly spot the spoofing, the value of the test statistic jumping in value soon after the spoofing begins. Figure 6 shows the improvement in the test statistic's response (the size of the spacing between no spoofing and spoofing) when the test time is doubled to 20 seconds.

A MORE REALISTIC MODEL

Let's assume that due to averaging, the GNSS receiver low pass filters the observed position:

H_0:

$$y_k = \text{LPF}\left\{x_k + a_k\right\} + w_{y,k} \qquad z_k = a_k + w_{z,k}$$

H_1:

$$y_k = \text{LPF}\left\{c_k\right\} + w_{y,k} \qquad z_k = a_k + w_{z,k}$$

Since filtering has no effect on the linear model of gross ship motion, this is

H_0:

$$y_k = x_k + \text{LPF}\left\{a_k\right\} + w_{y,k} \qquad z_k = a_k + w_{z,k}$$

H_1:

$$y_k = c_k + w_{y,k} \qquad z_k = a_k + w_{z,k}$$

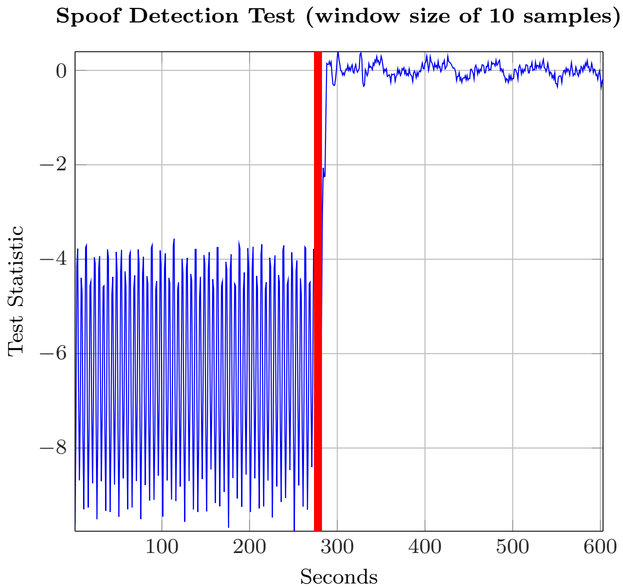

Figure 5: Value of spoof detection test statistic using 10 seconds of GNSS and IMU data at 1 Hz with spoofing event starting about halfway through the scenario at the red line.

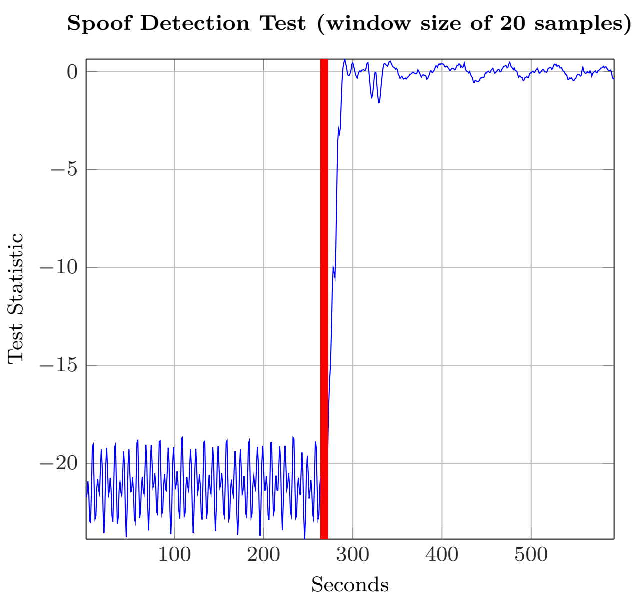

Figure 6: Value of spoof detection test statistic using 20 seconds of GNSS and IMU data at 1 Hz with spoofing event starting about halfway through the scenario at the red line.

and the resulting log-likelihood test would be

$$l(\mathbf{y}, \mathbf{z}) = \mathrm{LPF}\{\mathbf{z}\}^T \mathbf{A}\mathbf{y}$$

CONCLUSION

Using pitch and roll data from an IMU is an effective method to detect GNSS spoofing aboard ships. Motivated by recent work related to spoof detection using moving antennas, this paper developed and analyzed a GNSS spoofing detection algorithm that exploits the relative "high frequency" pitch/roll motion of the ship that is assumed to be unknown to the spoofer. Instead of using a navigation filter such as the Extended Kalman Filter, the algorithm presented here directly employs the instantaneous trajectories.

The analysis provides a method to identify the test duration for effective detection performance and showed that it provides for a short time to alarm. Mild seas provide enough ship motion for successful detection. Data from a GNSS simulator was used to provide experimental results.

This work can be extended to employ filtered data, in which the GNSS and IMU sensors have different time constants on their internal data processing. Other future work includes using real ship data and considering the impact of IMU errors (e.g. bias and/or drift) on both the test and its performance.

APPENDIX 1 – INTEGER LINEAR REGRESSION

Imagine a data sequence t_k for $k = 0, 1, 2, \ldots N-1$. The integer linear regression problem is to find constants m and b of the best straight line fit of the data under a mean-squared error criterion

$$E = \sum_{k=0}^{N-1} (t_k - mk - b)^2$$

in which m and b are the slope and intercept solutions, respectively. Taking derivatives

$$\frac{\partial E}{\partial m} = -2 \sum_{k=0}^{N-1} (t_k - mk - b)\, k$$

and

$$\frac{\partial E}{\partial b} = -2 \sum_{k=0}^{N-1} (t_k - mk - b)$$

Setting both to zero and manipulating yields a pair of simultaneous equations

$$\begin{bmatrix} \sum_{k=0}^{N-1} k^2 & \sum_{k=0}^{N-1} k \\ \sum_{k=0}^{N-1} k & N \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{N-1} kt_k \\ \sum_{k=0}^{N-1} t_k \end{bmatrix}$$

with solutions

$$m = \frac{N \sum_{k=0}^{N-1} kt_k - \sum_{k=0}^{N-1} k \sum_{k=0}^{N-1} t_k}{N \sum_{k=0}^{N-1} k^2 - \left(\sum_{k=0}^{N-1} k \right)^2}$$

and

$$b = \frac{\sum_{k=0}^{N-1} k^2 \sum_{k=0}^{N-1} t_k - \sum_{k=0}^{N-1} k \sum_{k=0}^{N-1} kt_k}{N \sum_{k=0}^{N-1} k^2 - \left(\sum_{k=0}^{N-1} k \right)^2}$$

Using the facts

$$\sum_{k=0}^{N-1} k = \frac{N(N-1)}{2}$$

and

$$\sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6}$$

these are

$$m = \frac{12 \sum_{k=0}^{N-1} kt_k - 6(N-1) \sum_{k=0}^{N-1} t_k}{N(N-1)(N+1)}$$

and

$$b = \frac{2(2N-1) \sum_{k=0}^{N-1} t_k - 6 \sum_{k=0}^{N-1} kt_k}{N(N+1)}$$

or

$$m = \frac{12}{N(N-1)(n+1)} \sum_{k=0}^{N-1} kt_k - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} t_k$$

and

$$b = \frac{2(2N-1)}{N(N+1)} \sum_{k=0}^{N-1} t_k - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} kt_k$$

For our two problems of interest in this paper we have for the H_0 regression $t_k = y_k - z_k$ while for H_1 $t_k = y_k$.

So, the MLEs are

$$\begin{aligned} \widehat{m_0} &= \frac{12}{N(N-1)(N+1)} \sum_{k=0}^{N-1} k\,(y_k - z_k) \\ &\quad - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} (y_k - z_k) \end{aligned}$$

$$\begin{aligned} \widehat{b_0} &= \frac{2(2N-1)}{N(N+1)} \sum_{k=0}^{N-1} (y_k - z_k) \\ &\quad - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} k\,(y_k - z_k) \end{aligned}$$

$$\widehat{m_1} = \frac{12}{N(N-1)(N+1)} \sum_{k=0}^{N-1} ky_k - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} y_k$$

and

$$\widehat{b_1} = \frac{2(2N-1)}{N(N+1)} \sum_{k=0}^{N-1} y_k - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} ky_k$$

Finally, we note that

$$\begin{aligned} \widehat{m_1} - \widehat{m_0} &= \frac{12}{N(N-1)(n+1)} \sum_{k=0}^{N-1} kz_k \\ &\quad - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} z_k \end{aligned}$$

and

$$\widehat{b_1} - \widehat{b_0} = \frac{2(2N-1)}{N(N+1)} \sum_{k=0}^{N-1} z_k - \frac{6}{N(N+1)} \sum_{k=0}^{N-1} kz_k$$

APPENDIX 2 – FACTS ABOUT THE MATRIX A

The matrix $\mathbf{A}$ is defined to have elements

$$\begin{aligned} A_{jk} &= \frac{12}{N(N-1)(N+1)} jk + \frac{2(2N-1)}{N(N+1)} \\ &\quad - \frac{6}{N(N+1)} j - \frac{6}{N(N+1)} k - \delta_{jk} \end{aligned}$$

where δ_{jk} is a Kronecker delta. Note that $\mathbf{A}$ is symmetric. This appendix develops some algebraic facts about this matrix. Several of these facts are just tedious algebra and rely upon the relations

$$\sum_{k=0}^{N-1} k = \frac{N(N-1)}{2}$$

and

$$\sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6}$$

- **Fact 1**: The vector of ones, $\mathbf{1}$, is in the null space of $\mathbf{A}$

$$\mathbf{A1} = \mathbf{0}$$

Equivalently, column sums of $\mathbf{A}$ equal zero. And since $\mathbf{A}$ is symmetric, both of these statements also hold for $\mathbf{A}^T$.

Proof: We explicitly show this fact by demonstrating that the j^{th} element of $\mathbf{A1}$ is zero.

$$\begin{aligned}
&[\mathbf{A1}]_j \\
&= \sum_{k=0}^{N-1} A_{jk} \times 1 \\
&= \sum_{k=0}^{N-1} \left[\begin{array}{l} \frac{12}{N(N-1)(N+1)} jk - \frac{6}{N(N+1)} j \\ + \frac{2(2N-1)}{N(N+1)} - \frac{6}{N(N+1)} k - \delta_{jk} \end{array} \right] \\
&= \left[\frac{12j}{N(N-1)(N+1)} - \frac{6}{N(N+1)} \right] \sum_{j=0}^{N-1} k \\
&\quad + \left[\frac{2(2N-1)}{N(N+1)} - \frac{6j}{N(N+1)} \right] N-1 \\
&= \left[\frac{12k}{N(N-1)(N+1)} - \frac{6}{N(N+1)} \right] \frac{N(N-1)}{2} \\
&\quad + \left[\frac{2(2N-1)}{N(N+1)} - \frac{6k}{N(N+1)} \right] N-1 \\
&= 0
\end{aligned}$$

- **Fact 2**: The vector, $\mathbf{v} = [0, 1, 2, \ldots N-1]^T$, is also in the null space of $\mathbf{A}$

$$\mathbf{Av} = \mathbf{0}$$

And since $\mathbf{A}$ is symmetric, it also hold for $\mathbf{A}^T$.

Proof: Again we explicitly show that the j^{th} element of this product is zero.

$$\begin{aligned}
&[\mathbf{Av}]_j \\
&= \sum_{k=0}^{N-1} k\, A_{jk} \\
&= \sum_{k=0}^{N-1} k \left[\begin{array}{l} \frac{12}{N(N-1)(N+1)} jk - \frac{6}{N(N+1)} j \\ + \frac{2(2N-1)}{N(N+1)} - \frac{6}{N(N+1)} k - \delta_{jk} \end{array} \right] \\
&= \left[\frac{12j}{N(N-1)(N+1)} - \frac{6}{N(N+1)} \right] \sum_{k=0}^{N-1} k^2 \\
&\quad + \left[\frac{2(2N-1)}{N(N+1)} - \frac{6k}{N(N+1)} \right] \sum_{k=0}^{N-1} k - \sum_{k=0}^{N-1} k\delta_{jk} \\
&= \left[\frac{12j}{N(N-1)(N+1)} - \frac{6}{N(N+1)} \right] \frac{N(N-1)(2N-1)}{6} \\
&\quad \left[\frac{2(2N-1)}{N(N+1)} - \frac{6j}{N(N+1)} \right] \frac{N(N-1)}{2} - j \\
&= 0
\end{aligned}$$

The next several facts concern the negative of the matrix; defined as $\mathbf{B} = -\mathbf{A}$ with elements

$$\begin{aligned}
B_{jk} &= -\frac{12}{N(N-1)(N+1)} jk - \frac{2(2N-1)}{N(N+1)} \\
&\quad + \frac{6}{N(N+1)} j + \frac{6}{N(N+1)} k + \delta_{jk}
\end{aligned}$$

- **Fact 3**: The matrix $\mathbf{B}$ satisfies $\mathbf{BB} = \mathbf{B}$; in other words, $\mathbf{B}$ is a projection matrix.

Proof: We accomplish this by showing that every element of $\mathbf{BB}$ is the corresponding element of $\mathbf{B}$. Consider the j, k element of the product

$$\begin{aligned}
&[\mathbf{BB}]_{j,k} \\
&= \sum_m B_{j,m} B_{m,k} \\
&= \sum_m \left[-\frac{12jm}{N(N-1)(N+1)} + \frac{6j}{N(N+1)} - \frac{2(2N-1)}{N(N+1)} + \frac{6m}{N(N+1)} + \delta_{jm} \right] \\
&\quad \left[-\frac{12mk}{N(N-1)(N+1)} + \frac{6m}{N(N+1)} - \frac{2(2N-1)}{N(N+1)} + \frac{6k}{N(N+1)} + \delta_{mk} \right] \\
&= \sum_m \left[\frac{12jm}{N(N-1)(N+1)} - \frac{6j}{N(N+1)} + \frac{2(2N-1)}{N(N+1)} - \frac{6m}{N(N+1)} \right] \\
&\quad \left[\frac{12mk}{N(N-1)(N+1)} - \frac{6}{N(N+1)} m + \frac{2(2N-1)}{N(N+1)} - \frac{6k}{N(N+1)} \right] \\
&\quad -2 \left[\frac{12jk}{N(N-1)(N+1)} - \frac{6j}{N(N+1)} + \frac{2(2N-1)}{N(N+1)} - \frac{6k}{N(N+1)} \right] + \delta_{jk} \\
&= \frac{6j+6k-6N+12jk-6jN-6kN+4N^2+2}{N(N-1)(N+1)} \\
&\quad + \frac{-24jk+12(N-1)j-4(N-1)(2N-1)+12(N-1)k}{N(N-1)(N+1)} + \delta_{jk} \\
&= \frac{-12jk+6j(N-1)-2(2N-1)(N-1)+6k(N-1)}{N(N-1)(N+1)} + \delta_{jk} \\
&= -\frac{12jk}{N(N-1)(N+1)} + \frac{6j}{N(N+1)} - \frac{2(2N-1)}{N(N+1)} + \frac{6k}{N(N+1)} + \delta_{jk} \\
&= B_{j,k}
\end{aligned}$$

- **Fact 4**: The matrix $\mathbf{B}$ is positive semi-definite.

  Proof: By definition the eigenvalues of a projection matrix are all equal to either zero or one. Further, one test for positive semi-definiteness is having all eigenvalues being non-negative. Hence, $\mathbf{B}$ is positive semi-definite.

- **Related Fact 5**: The matrix $\mathbf{A}$ is negative semi-definite.

  Proof: By definition of being positive semi-definite, $\mathbf{B}$ satisfies

$$\mathbf{a}^T \mathbf{B} \mathbf{a} \geq 0$$

  for all vectors $\mathbf{a}$. Substituting for $\mathbf{B}$

$$\mathbf{a}^T (-\mathbf{A})\, \mathbf{a} \geq 0 \qquad \text{or} \qquad \mathbf{a}^T \mathbf{A} \mathbf{a} \leq 0$$

  and $\mathbf{A}$ is negative semi-definite.

- **Related Fact 6**:

$$\mathbf{A}\mathbf{A} = -\mathbf{A}$$

  Proof: Since $\mathbf{B}$ is a projection we have $\mathbf{B}\mathbf{B} = \mathbf{B}$. Substituting the definition that $\mathbf{B} = -\mathbf{A}$, this is

$$(-\mathbf{A})\,(-\mathbf{A}) = -\mathbf{A} \qquad \text{or} \qquad \mathbf{A}\mathbf{A} = -\mathbf{A}$$

- **Fact 7**: Not only is $\mathbf{B}$ a projection, but exactly $N-2$ of its eigenvalues are equal to unity; the two zero eigenvalues correspond to the vectors $\mathbf{1}$ and $\mathbf{v}$.

  Proof: We do this in two steps: first, constructing a vector $\mathbf{x}$ that is orthogonal to both $\mathbf{1}$ and $\mathbf{v}$, and second, showing that $\mathbf{B}$ preserves this vector exactly.

  Defining $\mathbf{x}$

  Consider an arbitrary vector $\mathbf{x} = [x_0, x_1, \ldots x_{N-1}]^T$. So as to be perpendicular to both $\mathbf{1}$ and $\mathbf{v}$ the elements of the vector must satisfy two equations

$$\mathbf{1}^T \mathbf{x} = 0 \qquad \text{and} \qquad \mathbf{v}^T \mathbf{x} = 0$$

  Since these are linear equations, they constrain the values of two of the components of $\mathbf{x}$. The first equation is

$$\mathbf{1}^T \mathbf{x} = \sum_{s=0}^{N-1} x_s = x_0 + \sum_{s=1}^{N-1} x_s = 0$$

  so, without loss of generality we can write x_0 in terms of the other components

$$x_0 = -\sum_{s=1}^{N-1} x_s$$

  The second equation is

$$\begin{aligned} \mathbf{v}^T \mathbf{x} &= \sum_{s=0}^{N-1} s x_s = 0 \cdot x_0 + 1 \cdot x_1 + \sum_{s=2}^{N-1} s x_s \\ &= x_1 + \sum_{s=2}^{N-1} s x_s = 0 \end{aligned}$$

  so we have

$$x_1 = -\sum_{s=2}^{N-1} s x_s$$

  With these selections the other $N-2$ elements of $\mathbf{x}$ are arbitrary.

  Projecting $\mathbf{x}$

  Let's consider the application of $\mathbf{B}$ to this vector $\mathbf{x}$, calling the result $\mathbf{r}$

$$\mathbf{r} = \mathbf{B}\mathbf{x}$$

  and examine the individual elements of $\mathbf{r}$, the r_j.

$$\begin{aligned} r_j &= \sum_{k=0}^{N-1} B_{jk} x_k \\ &= B_{j0} x_0 + B_{j1} x_1 + \sum_{k=2}^{N-1} B_{jk} x_k \\ &= B_{j0} \left(-\sum_{s=1}^{N-1} x_s \right) + B_{j1} \left(-\sum_{s=2}^{N-1} s x_s \right) \\ &\quad + \sum_{k=2}^{N-1} B_{jk} x_k \\ &= B_{j0} \left(-\sum_{s=1}^{N-1} x_s \right) + B_{j1} \left(-\sum_{s=2}^{N-1} s x_s \right) \\ &\quad - \frac{12}{N(N-1)(N+1)} j \left(\sum_{k=2}^{N-1} k x_k \right) \\ &\quad + \frac{6}{N(N+1)} \left(\sum_{k=2}^{N-1} k x_k \right) \\ &\quad - \frac{2(2N-1)}{N(N+1)} \left(\sum_{k=2}^{N-1} x_k \right) \\ &\quad + \frac{6}{N(N+1)} j \left(\sum_{k=2}^{N-1} x_k \right) + \sum_{k=2}^{N-1} x_k \delta_{jk} \end{aligned}$$

Reordering terms

$$
\begin{aligned}
r_j \quad = \quad & B_{j0}\left(-\sum_{s=1}^{N-1} x_s\right) \\
& -\frac{2(2N-1)}{N(N+1)}\left(\sum_{k=2}^{N-1} x_k\right) \\
& +\frac{6}{N(N+1)} j\left(\sum_{k=2}^{N-1} x_k\right) \\
& +B_{j1}\left(-\sum_{s=2}^{N-1} s x_s\right) \\
& -\frac{12}{N(N-1)(N+1)} j\left(\sum_{k=2}^{N-1} k x_k\right) \\
& +\frac{6}{N(N+1)}\left(\sum_{k=2}^{N-1} k x_k\right)+\sum_{k=2}^{N-1} x_k \delta_{jk}
\end{aligned}
$$

or

$$
\begin{aligned}
r_j \quad = \quad & -B_{j0} x_1 \\
& +\left(\sum_{s=2}^{N-1} x_s\right)\left[-B_{j0}-\tfrac{2(2N-1)}{N(N+1)}+\tfrac{6}{N(N+1)} j\right] \\
& +\left(\sum_{s=2}^{N-1} s x_s\right)\left[-B_{j1}-\tfrac{12}{N(N-1)(N+1)} j+\tfrac{6}{N(N+1)}\right] \\
& +\sum_{k=2}^{N-1} x_k \delta_{jk}
\end{aligned}
$$

Filling in the values of B_{j0} and B_{j1} in the terms in brackets

$$
B_{j0}=\frac{6}{N(N+1)} j-\frac{2(2N-1)}{N(N+1)}+\delta_{j0}
$$

$$
B_{j1}=\frac{6(N-3)}{N(N-1)(N+1)} j-\frac{4(N-2))}{N(N+1)}+\delta_{j1}
$$

and simplifying

$$
\begin{aligned}
r_j \quad = \quad & -B_{j0} x_1+\left(\sum_{s=2}^{N-1} x_s\right)\left[-\delta_{j0}\right] \\
& +\left(\sum_{s=2}^{N-1} s x_s\right)\left[-\tfrac{6N-6}{N(N-1)(N+1)} j+\tfrac{4N-2}{N(N+1)}-\delta_{j1}\right] \\
& +\sum_{k=2}^{N-1} x_k \delta_{jk}
\end{aligned}
$$

Plugging in for the two components of the first product

$$
\begin{aligned}
r_j \quad = \quad & -\left[\tfrac{6}{N(N+1)} j-\tfrac{2(2N-1)}{N(N+1)}+\delta_{j0}\right]\left(-\sum_{s=2}^{N-1} s x_s\right) \\
& +\left(\sum_{s=2}^{N-1} x_s\right)\left[-\delta_{j0}\right] \\
& +\left(\sum_{s=2}^{N-1} s x_s\right)\left[-\tfrac{6N-6}{N(N-1)(N+1)} j+\tfrac{4N-2}{N(N+1)}-\delta_{j1}\right] \\
& +\sum_{k=2}^{N-1} x_k \delta_{jk}
\end{aligned}
$$

or

$$
\begin{aligned}
r_j \quad = \quad & -\left(\sum_{s=2}^{N-1} x_s\right) \delta_{j0}+\left(\sum_{s=2}^{N-1} s x_s\right)\left[\delta_{j0}-\delta_{j1}\right] \\
& +\sum_{k=2}^{N-1} x_k \delta_{jk}
\end{aligned}
$$

Interpreting this expression for the different values of j

$$
r_0=-\left(\sum_{s=2}^{N-1} x_s\right)+\left(\sum_{s=2}^{N-1} s x_s\right)
$$

$$
r_1=-\left(\sum_{s=2}^{N-1} s x_s\right)=x_1
$$

and

$$
r_j=x_j
$$

for $j>1$. This is almost done, $\mathbf{r}$ matches $\mathbf{x}$ for elements 1 through $N-1$; we need only further manipulate r_0.

$$
\begin{aligned}
r_0 \quad & = \quad -\left(\sum_{s=2}^{N-1} x_s\right)+\left(\sum_{s=2}^{N-1} s x_s\right) \\
& = \quad \left(x_1-x_1-\sum_{s=2}^{N-1} x_s\right)+\left(\sum_{s=2}^{N-1} s x_s\right) \\
& = \quad x_1-\left(\sum_{s=1}^{N-1} x_s\right)+\left(\sum_{s=2}^{N-1} s x_s\right) \\
& = \quad x_1+x_0-x_1=x_0
\end{aligned}
$$

and <u>all</u> of the components match, $\mathbf{Bx}=\mathbf{x}$.

Since $\mathbf{x}$ was an arbitrary vector orthogonal to both $\mathbf{1}$ and $\mathbf{v}$ (in other words, $\mathbf{x}$ falls within an $N-2$ dimensional subspace) and since $\mathbf{B}$ is an identity operator for any such $\mathbf{x}$, then $\mathbf{B}$ must have $N-2$ eigenvalues equal to unity.

APPENDIX 3 – Variance Computation

By definition the variance is the difference between the second moment and the square of the mean

$$\mathrm{Var}\left\{l\right\} = E\left\{l^2\right\} - E\left\{l\right\}^2$$

As we have already solved for the mean, let's concentrate on the second moment. In matrix form, this is

$$E\left\{l^2\right\} = E\left\{\left(\mathbf{z}^T\mathbf{A}\mathbf{y}\right)\left(\mathbf{z}^T\mathbf{A}\mathbf{y}\right)\right\}$$

Taking the transpose of the first term (a scalar), we can regroup matrix multiplications and focus on the term in $\mathbf{z}$

$$\begin{aligned} E\left\{l^2\right\} &= E\left\{\left(\mathbf{z}^T\mathbf{A}\mathbf{y}\right)^T\left(\mathbf{z}^T\mathbf{A}\mathbf{y}\right)\right\} \\ &= E\left\{\left(\mathbf{y}^T\mathbf{A}^T\mathbf{z}\right)\left(\mathbf{z}^T\mathbf{A}\mathbf{y}\right)\right\} \\ &= E\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{z}\mathbf{z}^T\mathbf{A}\mathbf{y}\right\} \\ &= E\left\{\mathbf{y}^T\mathbf{A}^T\left(\mathbf{z}\mathbf{z}^T\right)\mathbf{A}\mathbf{y}\right\} \end{aligned}$$

The combined expectation (over $\mathbf{y}$ and $\mathbf{z}$) can be accomplished by first taking a conditional expectation over $\mathbf{z}$ with $\mathbf{y}$ fixed and then taking the expectation with respect to $\mathbf{y}$

$$\begin{aligned} E\left\{l^2\right\} &= E_y\left\{E_{z|y}\left\{\mathbf{y}^T\mathbf{A}\left(\mathbf{z}\mathbf{z}^T\right)\mathbf{A}\mathbf{y}\right\}\right\} \\ &= E_y\left\{\mathbf{y}^T\mathbf{A}E_{z|y}\left\{\mathbf{z}\mathbf{z}^T\right\}\mathbf{A}\mathbf{y}\right\} \end{aligned}$$

We have

$$\begin{aligned} \mathbf{z}\mathbf{z}^T &= \left(\mathbf{a}+\mathbf{w}_z\right)\left(\mathbf{a}+\mathbf{w}_z\right)^T \\ &= \mathbf{a}\mathbf{a}^T + \mathbf{a}\mathbf{w}_z^T + \mathbf{w}_z\mathbf{a}^T + \mathbf{w}_z\mathbf{w}_z^T \end{aligned}$$

Taking the $\mathbf{z}$ expectation

$$\begin{aligned} E_{z|y}\left\{\mathbf{z}\mathbf{z}^T\right\} &= E_{z|y}\left\{\mathbf{a}\mathbf{a}^T + \mathbf{a}\mathbf{w}_z^T + \mathbf{w}_z\mathbf{a}^T + \mathbf{w}_z\mathbf{w}_z^T\right\} \\ &= \mathbf{a}\mathbf{a}^T + \sigma_z^2\mathbf{I} \end{aligned}$$

which uses the fact that the elements of $\mathbf{w}_z$ are independent and all have variance σ_z^2. Substituting back to the second moment

$$\begin{aligned} E\left\{l^2\right\} &= E_y\left\{\mathbf{y}^T\mathbf{A}^T\left(\mathbf{a}\mathbf{a}^T + \sigma_z^2\mathbf{I}\right)\mathbf{A}\mathbf{y}\right\} \\ &= E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{a}\mathbf{a}^T\mathbf{A}\mathbf{y}\right\} + \sigma_z^2 E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y}\right\} \end{aligned}$$

To simplify these two terms, we consider them separately.

The First Term

The goal of this subsection is to simplify

$$E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{a}\mathbf{a}^T\mathbf{A}\mathbf{y}\right\}$$

As a first step, recognize that we again have a product of scalars and can transpose both

$$\begin{aligned} E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{a}\mathbf{a}^T\mathbf{A}\mathbf{y}\right\} &= E_y\left\{\left(\mathbf{y}^T\mathbf{A}^T\mathbf{a}\right)\left(\mathbf{a}^T\mathbf{A}\mathbf{y}\right)\right\} \\ &= E_y\left\{\left(\mathbf{y}^T\mathbf{A}^T\mathbf{a}\right)^T\left(\mathbf{a}^T\mathbf{A}\mathbf{y}\right)^T\right\} \\ &= E_y\left\{\left(\mathbf{a}^T\mathbf{A}\mathbf{y}\right)\left(\mathbf{y}^T\mathbf{A}^T\mathbf{a}\right)\right\} \\ &= E_y\left\{\mathbf{a}^T\mathbf{A}\mathbf{y}\mathbf{y}^T\mathbf{A}^T\mathbf{a}\right\} \\ &= \mathbf{a}^T\mathbf{A}E_y\left\{\mathbf{y}\mathbf{y}^T\right\}\mathbf{A}^T\mathbf{a} \end{aligned}$$

Now

$$\begin{aligned} \mathbf{y}\mathbf{y}^T &= \left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y\right)\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y\right)^T \\ &= \left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T + \left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y\right)\mathbf{w}_y^T \\ &\quad + \mathbf{w}_y\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T + \mathbf{w}_y\mathbf{w}_y^T \end{aligned}$$

Taking the expectation

$$E_y\left\{\mathbf{y}\mathbf{y}^T\right\} = \left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T + \sigma_y^2\mathbf{I}$$

which uses the fact that the $\mathbf{w}_y$ are independent with individual variances all equal to σ_y^2. The first term is then

$$\begin{aligned} &E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{a}\mathbf{a}^T\mathbf{A}\mathbf{y}\right\} \\ &= \mathbf{a}^T\mathbf{A}^T\left[\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T + \sigma_y^2\mathbf{I}\right]\mathbf{A}\mathbf{a} \\ &= \mathbf{a}^T\mathbf{A}^T\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T\mathbf{A}\mathbf{a} \\ &\quad + \sigma_y^2\mathbf{a}^T\mathbf{A}^T\mathbf{I}\mathbf{A}\mathbf{a} \\ &= \mathbf{a}^T\underbrace{\left[\mathbf{A}^T\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)\right]}_{\text{first part}}\underbrace{\left[\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T\mathbf{A}\right]}_{\text{second part}}\mathbf{a} \\ &\quad + \sigma_y^2\mathbf{a}^T\mathbf{A}^T\mathbf{A}\mathbf{a} \end{aligned}$$

where we have added brackets for emphasis of the two parts. The first of these parts can be reduced since both $\mathbf{v}$ and $\mathbf{1}$ are in the null space of $\mathbf{A}$

$$\mathbf{A}^T\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right) = m\mathbf{A}^T\mathbf{v} + b\mathbf{A}^T\mathbf{1} + \mathbf{A}^T\mathbf{g} = \mathbf{A}^T\mathbf{g}$$

The second part is similar

$$\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}\right)^T\mathbf{A} = \mathbf{g}^T\mathbf{A}$$

So the first term in the second moment is

$$E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{a}\mathbf{a}^T\mathbf{A}\mathbf{y}\right\} = \mathbf{a}^T\mathbf{A}^T\mathbf{g}\mathbf{g}^T\mathbf{A}\mathbf{a} + \sigma_y^2\mathbf{a}^T\mathbf{A}^T\mathbf{A}\mathbf{a}$$

The Second Term

The goal of this subsection is to simplify

$$E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y}\right\}$$

Using the definition of $\mathbf{y}$

$$\begin{aligned} &\mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y} \\ &= \left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y\right)^T\mathbf{A}^T\mathbf{A}\left(m\mathbf{v}+b\mathbf{1}+\mathbf{g}+\mathbf{w}_y\right) \\ &= \left(\mathbf{g}+\mathbf{w}_y\right)^T\mathbf{A}^T\mathbf{A}\left(\mathbf{g}+\mathbf{w}_y\right) \end{aligned}$$

using the facts that both $\mathbf{1}$ and $\mathbf{v}$ are in $\mathbf{A}$'s null space. Expanding

$$\begin{aligned}\mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y} &= \mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{g} + \mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{g} \\ &\quad + \mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{g} + \mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\end{aligned}$$

Taking the expectation

$$E_y\left\{\mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y}\right\} = \mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{g} + E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\}$$

Back to the Variance

Putting things back together

$$\begin{aligned}E\left\{l^2\right\} &= \mathbf{a}^T\mathbf{A}^T\mathbf{g}\mathbf{g}^T\mathbf{A}\mathbf{a} + \sigma_y^2\mathbf{a}^T\mathbf{A}^T\mathbf{A}\mathbf{a} \\ &\quad + \sigma_z^2\mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{g} + \sigma_z^2 E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\}\end{aligned}$$

Recalling that the mean is $\mu = \mathbf{a}^T\mathbf{A}\mathbf{g}$, subtracting μ^2 yields the variance as

$$\begin{aligned}\mathrm{Var}\left\{l\right\} &= \sigma_y^2\mathbf{a}^T\mathbf{A}^T\mathbf{A}\mathbf{a} + \sigma_z^2\mathbf{g}^T\mathbf{A}^T\mathbf{A}\mathbf{g} \\ &\quad + \sigma_z^2 E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\}\end{aligned}$$

Recalling that $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A} = -\mathbf{A}$ (see Appendix 2 again)

$$\mathrm{Var}\left\{l\right\} = \sigma_z^2 E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\} - \sigma_y^2\mathbf{a}^T\mathbf{A}\mathbf{a} - \sigma_z^2\mathbf{g}^T\mathbf{A}\mathbf{g}$$

All that's left is one expectation.

The Final Expectation

To complete the variance calculation we need to compute $E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\}$. Grouping terms, the argument of the expectation is an inner product

$$\begin{aligned}\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y &= \left(\mathbf{w}_y^T\mathbf{A}^T\right)\left(\mathbf{A}\mathbf{w}_y\right) \\ &= \left(\mathbf{A}\mathbf{w}_y\right)^T\left(\mathbf{A}\mathbf{w}_y\right)\end{aligned}$$

which can be computed as the trace of the outer product

$$\begin{aligned}\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y &= \mathrm{Tr}\left(\left(\mathbf{A}\mathbf{w}_y\right)\left(\mathbf{A}\mathbf{w}_y\right)^T\right) \\ &= \mathrm{Tr}\left(\mathbf{A}\mathbf{w}_y\mathbf{w}_y^T\mathbf{A}^T\right)\end{aligned}$$

Taking the expectation

$$\begin{aligned}E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\} &= E_y\left\{\mathrm{Tr}\left(\mathbf{A}\mathbf{w}_y\mathbf{w}_y^T\mathbf{A}^T\right)\right\} \\ &= \mathrm{Tr}\left(\mathbf{A}E_y\left\{\mathbf{w}_y\mathbf{w}_y^T\right\}\mathbf{A}^T\right) \\ &= \mathrm{Tr}\left(\mathbf{A}\sigma_y^2\mathbf{I}\mathbf{A}^T\right) \\ &= \sigma_y^2\mathrm{Tr}\left(\mathbf{A}\mathbf{A}^T\right)\end{aligned}$$

Using the notation of Appendix 2, $\mathbf{A}\mathbf{A}^T = -\mathbf{A} = \mathbf{B}$ so

$$E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\} = \sigma_y^2\mathrm{Tr}\left(\mathbf{B}\right)$$

Since the trace of a matrix is equal to the sum of its eigenvalues (and from Appendix 2, $\mathbf{B}$ has $N-2$ equal to unity and two equal to zero

$$E_y\left\{\mathbf{w}_y^T\mathbf{A}^T\mathbf{A}\mathbf{w}_y\right\} = (N-2)\sigma_y^2$$

Combining this final result

$$\mathrm{Var}\left\{l\right\} = (N-2)\sigma_y^2\sigma_z^2 - \sigma_y^2\mathbf{a}^T\mathbf{A}\mathbf{a} - \sigma_z^2\mathbf{g}^T\mathbf{A}\mathbf{g}$$

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