11-5-2015

15. Hamiltonian Mechanics

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Abstract
Part fifteen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc15]

Legendre transform [tln77]

Given is a function $f(x)$ with monotonic derivative $f'(x)$. The goal is to replace the independent variable x by $p = f'(x)$ with no loss of information.

Note: The function $G(p) = f(x)$ with $p = f'(x)$ is, in general, not invertible.

The Legendre transform solves this task elegantly.

- Forward direction: $g(p) = f(x) - xp$ with $p = f'(x)$.
- Reverse direction: $f(x) = g(p) + px$ with $x = -g'(p)$

Example 1: $f(x) = x^2 + 1$.

- $f(x) = x^2 + 1 \quad \Rightarrow \quad f'(x) = 2x \quad \Rightarrow \quad x = \dfrac{p}{2} \quad \Rightarrow \quad g(p) = 1 - \dfrac{p^2}{4}$.
- $g(p) = 1 - \dfrac{p^2}{4} \quad \Rightarrow \quad g'(p) = -\dfrac{p}{2} \quad \Rightarrow \quad p = 2x \quad \Rightarrow \quad f(x) = x^2 + 1$.

Example 2: $f(x) = e^{2x}$.

- $f(x) = e^{2x} \quad \Rightarrow \quad f'(x) = 2e^{2x} = p \quad \Rightarrow \quad x = \dfrac{1}{2} \ln \dfrac{p}{2}$
  $\Rightarrow \quad g(p) = \dfrac{p}{2} - \dfrac{p}{2} \ln \dfrac{p}{2}$.
- $g(p) = \dfrac{p}{2} - \dfrac{p}{2} \ln \dfrac{p}{2} \quad \Rightarrow \quad g'(p) = -\dfrac{1}{2} \ln \dfrac{p}{2} = -x$
  $\Rightarrow \quad p = 2e^{2x} \quad \Rightarrow \quad f(x) = e^{2x}$.

Hamiltonian and Canonical Equations [mln82]

Hamiltonian from Lagrangian via Legendre transform:

- Given the Lagrangian of a mechanical system: $L(q_1,\ldots,q_n;\dot{q}_1,\ldots,\dot{q}_n;t)$.
- Introduce canonical coordinates: $q_i,\ p_i \doteq \dfrac{\partial L}{\partial \dot{q}_i},\ i=1,\ldots,n.$
- Construct Hamiltonian:

  $$H(q_1,\ldots,q_n;p_1,\ldots,p_n;t) = \sum_j \dot{q}_j p_j - L(q_1,\ldots,q_n;\dot{q}_1,\ldots,\dot{q}_n;t),$$

  where $\dot{q}_j = \dot{q}_j(q_1,\ldots,q_n;p_1,\ldots,p_n;t)$ is inferred from $p_i = \partial L/\partial \dot{q}_i$.

Canonical equations from total differential of H:

- $dH = \sum_j \left[\dfrac{\partial H}{\partial q_j}\, dq_j + \dfrac{\partial H}{\partial p_j}\, dp_j\right] + \dfrac{\partial H}{\partial t}\, dt.$
- $d\left(\sum_j \dot{q}_j p_j - L\right) = \sum_j \left[\dot{q}_j dp_j + p_j d\dot{q}_j - \dfrac{\partial L}{\partial q_j}\, dq_j - \dfrac{\partial L}{\partial \dot{q}_j}\, d\dot{q}_j\right] - \dfrac{\partial L}{\partial t}\, dt;$

  use $\dfrac{\partial L}{\partial q_j} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = \dot{p}_j, \quad \dfrac{\partial L}{\partial \dot{q}_j} = p_j;$

  $\Rightarrow\ d\left(\sum_j \dot{q}_j p_j - L\right) = \sum_j \left[\dot{q}_j dp_j - \dot{p}_j dq_j\right] - \dfrac{\partial L}{\partial t}\, dt;$
- comparison of coefficients yields
  - $\dot{q}_j = \dfrac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}, \quad j=1,\ldots,n$ (canonical equations),
  - $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t}.$

Comments:

- The inversion of $p_i = \partial L/\partial \dot{q}_i$ as used above requires that $\det\left(\dfrac{\partial^2 L}{\partial \dot{q}_i \dot{q}_j}\right) \neq 0$ [mex189].
- Lagrangian from Hamiltonian: [mex188].

[mex188] Lagrangian from Hamiltonian via Legende transform

Given a Hamiltonian system $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ and the associated canonical equations $\dot{q}_i = \partial H/\partial p_i,\ \dot{p}_i = -\partial H/\partial q_i,\ i = 1, \ldots, n$, find the Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ of the same system via Legendre transform, derive the Lagrange equations for the generalized coordinates $q_1, \ldots, q_n$ and establish the relation $\partial L/\partial t = -\partial H/\partial t$.

Solution:

[mex189] Can you find the Hamiltonian of this system?

Consider the Lagrangian system

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2}m(\dot{q}_1 + \dot{q}_2)^2 - \frac{1}{2}k(q_1^2 + q_2^2).$$

(a) Find the most general solution $q_1(t), q_2(t)$ of the associated Lagrange equations. (b) Find the Hamiltonian $H(q_1, q_2, p_1, p_2)$ such that the associated canonical equations have the same solution $q_1(t), q_2(t)$. (c) Find the most general solution of $H(q_1, q_2, p_1, p_2)$.

Solution:

Variational Principle in Phase Space [mln83]

Hamilton's principle: variations in configuration space.

$$\delta J \doteq \delta \int_{t_1}^{t_2} dt\, L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0,$$

where $\delta q_i = 0$ at t_1 and t_2.

$\Rightarrow$ Lagrange equations: $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \ldots, n.$

Derivation: [mln78], [msl20].

Modified Hamilton's principle: variations in phase space.

$$\delta J \doteq \delta \int_{t_1}^{t_2} dt \left[\sum_{i=1}^{n} p_i \dot{q}_i - H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) \right] = 0,$$

where $\delta q_i = 0$ and $\delta p_i = 0$ at t_1 and t_2.

$\Rightarrow$ Canonical equations: $\dot{q}_i = \dfrac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n.$

Derivation:

$$\delta J = \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i}\,\delta q_i - \frac{\partial H}{\partial p_i}\,\delta p_i \right] = 0;$$

use $\displaystyle \int_{t_1}^{t_2} dt \sum_{i=1}^{n} p_i \delta \dot{q}_i = \underbrace{\left[\sum_{i=1}^{n} p_i \delta q_i \right]_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \dot{p}_i \delta q_i;$

$$\Rightarrow \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right] = 0.$$

Properties of the Hamiltonian [mln87]

How is the Hamiltonian constructed from kinetic energy and potential energy? When does it represents the total energy? When is it conserved?

Here is a list of some answers:

- If the Hamiltonian does not depend explicitly on time then it is a conserved quantity: $H(q_1, \ldots, q_n; p_1, \ldots, p_n) = \text{const.}$

$$\frac{dH}{dt} = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right) = \sum_{j=1}^{n} (-\dot{p}_j \dot{q}_j + \dot{p}_j \dot{q}_j) = 0.$$

- If $T(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ is the kinetic energy and $V(q_1, \ldots, q_n; t)$ the potential energy of a Lagrangian $L = T - V$, then the Hamiltonian is equal to the total energy:

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = T + V = E(t), \quad \text{where } p_j \doteq \frac{\partial L}{\partial \dot{q}_j}.$$

- Suppose that some of the generalized coordinates $q_1, \ldots, q_n$ are subject to holonomic constraints. Then $H = T + V$ only holds if all those constraints are scleronomic, i.e. time-independent [mex81].
- Depending on the nature of the dynamical system and the choice of coordinates, the Hamiltonian may represent the total energy or a conserved quantity or both or neither [mex77].
- The property $H \neq T + V$ occurs in the presence of velocity-dependent potentials [mln85]. The motion of a charged particle in a static magnetic field is a prominent example [mln86].
- In the presence of time-dependent fields, the conceptual framework used here quickly shows its limitations, because such fields themselves can transport momentum and energy.

[mex81] When does the Hamiltonian represent the total energy?

Consider a dynamical system with $3N$ degrees of freedom subject to k holonomic constraints: $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_n, t), i = 1, \ldots, N, n = 3N - k$. The kinetic and potential energies are given by the expressions

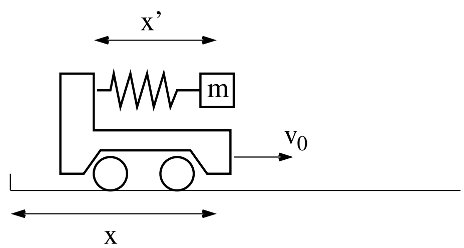
$$T = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2, \quad V = V(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t).$$

Show that the Hamiltonian $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ derived from these specifications is equal to the total energy, $E = T + V$, only if (i) the potential energy does not depend on the velocities $\dot{\mathbf{r}}_i$ and (ii) if the holonomic constraints are not explicitly time-dependent .

Solution:

[mex77] Hamiltonian: conserved quantity or total energy?

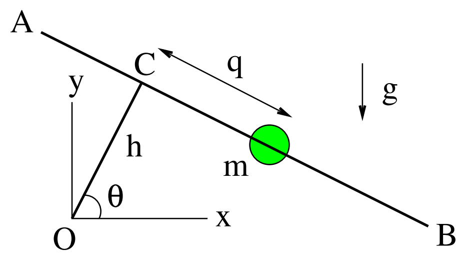
A harmonic oscillator (mass m, spring constant k) is attached to a cart that moves with constant velocity $\mathbf{v}_0$. Describe the dynamics in the coordinate system (x) that is at rest and in the coordinate system (x') that is moving with the cart.
(a) Construct the Lagrangian L of the oscillator in the rest frame and derive the associated Lagrange equation. Construct the Hamiltonian H from L.
(b) Construct the Lagrangian L' of the oscillator in the moving frame and derive the associated Lagrange equation. Construct the Hamiltonian H' from L'.
(c) Show that the Lagrange equations obtained in (a) and (b) are equivalent.
(d) Which of the two quantities H, H', if any, represents the total energy of the oscillator?
(e) Which of the two quantities H, H', if any, represents a conserved quantity?



Solution:

[mex78] Bead sliding on rotating rod in vertical plane

The rod AB rotates with constant angular velocity $\dot{\theta} = \omega$ at fixed perpendicular distance h about point O in a vertical plane. A bead of mass m is free to slide along the rod. Its position (relative to point C) on the rod is described by the variable q. (a) Construct the Lagrangian $L(q, \dot{q}, t)$ and derive the Lagrange equation for the variable $q(t)$. (b) Solve the Lagrange equation for the following initial conditions: $\theta(0) = q(0) = \dot{q}(0) = 0$. (c) Construct the Hamiltonian $H(q, p, t)$ from L. Determine whether or not H represents the total energy of the bead.



Solution:

Use of Cyclic Coordinates [mln84]

Lagrangian mechanics:

Lagrangian: $L(q_1,\ldots,q_{n-1};\dot{q}_1,\ldots,\dot{q}_n)$.

Cyclic coordinate q_n: $\Rightarrow\ \dfrac{\partial L}{\partial q_n}=0 \quad \Rightarrow\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_n}=0.$

Conserved quantity: $\dfrac{\partial L}{\partial \dot{q}_n} \doteq \beta_n(q_1,\ldots,q_{n-1};\dot{q}_1,\ldots,\dot{q}_n)=\text{const.}$

Eliminate $\dot{q}_n=\dot{q}_n(q_1,\ldots,q_{n-1};\dot{q}_1,\ldots,\dot{q}_{n-1};\beta_n)$ as independent variable.

Do not substitute $\dot{q}_n(q_1,\ldots,q_{n-1};\dot{q}_1,\ldots,\dot{q}_{n-1};\beta_n)$ into Lagrangian.
Substitute $\dot{q}_n(q_1,\ldots,q_{n-1};\dot{q}_1,\ldots,\dot{q}_{n-1};\beta_n)$ into Routhian instead.

Routhian: $R(q_1,\ldots,q_{n-1};\dot{q}_1,\ldots,\dot{q}_{n-1};\beta_n)=L-\beta_n\dot{q}_n$.

Equations of motion: $\dfrac{\partial R}{\partial q_i}-\dfrac{d}{dt}\dfrac{\partial R}{\partial \dot{q}_i}=0, \quad i=1,\ldots,n-1.$

Supplement: $q_n(t)=-\displaystyle\int dt\,\frac{\partial R}{\partial \beta_n}.$

Hamiltonian mechanics:

Hamiltonian: $H(q_1,\ldots,q_{n-1};p_1,\ldots,p_n)$.

Cyclic coordinate q_n: $\Rightarrow\ \dfrac{\partial H}{\partial q_n}=0 \quad \Rightarrow\ \dot{p}_n=0.$

Conserved quantity: $p_n \doteq \alpha_n=\text{const.}$

Reduced Hamiltonian: $H(q_1,\ldots,q_{n-1};p_1,\ldots,p_{n-1};\alpha_n)$.

Angular frequency: $\omega_n \doteq \dot{q}_n(q_1,\ldots,q_{n-1};p_1,\ldots,p_{n-1};\alpha_n)=\dfrac{\partial H}{\partial \alpha_n}.$

Equations of motion: $\dot{q}_i=\dfrac{\partial H}{\partial p_i}, \quad \dot{p}_i=-\dfrac{\partial H}{\partial q_i}, \quad i=1,\ldots,n-1.$

Supplement: $q_n(t)=\displaystyle\int dt\,\omega_n(t).$

Velocity-Dependent Potential Energy [mln85]

Lagrange equations in raw form from [mln8]:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1, \ldots, n,$$

- kinetic energy: $T(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$,
- generalized forces: $Q_j(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$.

The standard form of the Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \ldots, 3n,$$

can be inferred from a Lagrangian $L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$ under the following circumstances:

(a) If the generalized forces can be derived from a position-dependent potential energy$V(q_1, \ldots, q_n; t)$,

$$Q_j(q_1, \ldots, q_n; t) = -\frac{\partial V}{\partial q_j},$$

then the Lagrangian is $L = T - V$.

(b) If the generalized forces can be derived from a velocity-dependent potential energy $U(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$,

$$Q_j(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j},$$

then the Lagrangian is $L = T - U$.

Hamiltonian derived from the Lagrangian via Legendre transform:

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = \sum_{j=1}^{n} p_j \dot{q}_j - L, \quad p_j = \frac{\partial L}{\partial \dot{q}_j}.$$

Examples of velocity-dependent potential energy:

- Lorentz force [mln86],
- velocity-dependent central force [mex76].

Charged Particle in Electromagnetic Field [mln86]

Lorentz force: $\mathbf{F} = e\,\mathbf{E} + \frac{e}{c}\,\mathbf{v}\times\mathbf{B}$.

Electric field: $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$.

Magnetic field: $\mathbf{B} = \nabla\times\mathbf{A}$.

Velocity-dependent potential energy: $U(\mathbf{r},\mathbf{v},t) = e\phi(\mathbf{r},t) - \frac{e}{c}\,\mathbf{v}\cdot\mathbf{A}(\mathbf{r},t)$.

Lagrangian: $L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m|\mathbf{v}|^2 - U(\mathbf{r},\mathbf{v},t)$.

Lagrange equations for $\mathbf{r} = (x_1,x_2,x_3)$, $\mathbf{v} = (\dot{x}_1,\dot{x}_2,\dot{x}_3)$:

$$m\ddot{x}_1 + \frac{e}{c}\frac{dA_1}{dt} = e\left(-\frac{\partial\phi}{\partial x_1} + \frac{\dot{x}_1}{c}\frac{\partial A_1}{\partial x_1} + \frac{\dot{x}_2}{c}\frac{\partial A_2}{\partial x_1} + \frac{\dot{x}_3}{c}\frac{\partial A_3}{\partial x_1}\right) \qquad \text{etc.}$$

Use $\frac{dA_1}{dt} = \frac{\partial A_1}{\partial t} + \mathbf{v}\cdot\nabla A_1 = \frac{\partial A_1}{\partial t} + \dot{x}_1\,\frac{\partial A_1}{\partial x_1} + \dot{x}_2\,\frac{\partial A_1}{\partial x_2} + \dot{x}_3\,\frac{\partial A_1}{\partial x_3}$.

$$\Rightarrow\ m\ddot{x}_1 = e\left[-\frac{\partial\phi}{\partial x_1} - \frac{1}{c}\frac{\partial A_1}{\partial t} + \frac{\dot{x}_2}{c}\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) + \frac{\dot{x}_3}{c}\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right)\right]$$
$$= eE_x + \frac{e}{c}\left(\mathbf{v}\times\mathbf{B}\right)_x \qquad \text{etc.}$$

Generalized momenta: $p_i = \frac{\partial L}{\partial\dot{x}_i} = m\dot{x}_i + \frac{e}{c}\,A_i$.

- p_i: canonical momenta.
- $m\dot{x}_i$: kinetic momenta

Hamiltonian: $H(\mathbf{r},\mathbf{p},t) = \sum_{i=1}^{3} p_i\dot{x}_i - L = \frac{1}{2m}\left|\mathbf{p} - \frac{e}{c}\,\mathbf{A}(\mathbf{r},t)\right|^2 + e\phi(\mathbf{r},t)$.

Relativistic mechanics:

- Momenta: [mln63]
- Hamiltonian: $H(\mathbf{r},\mathbf{p},t) = \sqrt{\left|\mathbf{p} - \frac{e}{c}\,\mathbf{A}(\mathbf{r},t)\right|^2 + m_0^2c^4} + e\phi(\mathbf{r},t)$.

[mex76] Velocity-dependent central force

A particle moves under the influence of a velocity-dependent central force

$$F(r,\dot{r}) = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right),$$

where c is a constant. (a) Show that the Lagrangian and Hamiltonian of this system can be expressed as follows:

$$L(r,\dot{r},\dot{\vartheta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2) - \frac{1}{r}\left(1 + \frac{\dot{r}^2}{c^2}\right), \quad H(r,p,\ell) = \frac{p^2}{2(m - 2/c^2 r)} + \frac{\ell^2}{2mr^2} + \frac{1}{r}.$$

(b) Derive the Lagrange equations from L and the canonical equations from H and show that they are equivalent.

Solution:

[mex190] Charged particle in a uniform magnetic field

Consider a particle with mass m and electric charge q moving in a magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$. (a) Find the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z})$ and derive the Lagrange equations from it. (b) Find the Hamiltonian $H(x, y, z, p_x, p_y, p_z)$ and derive the canonical equations from it. (c) Show that both sets of equations of motion can be brought into the form $\ddot{x} - \omega\dot{y} = 0,\ \ddot{y} + \omega\dot{x} = 0,\ \ddot{z} = 0$, where $\omega = qB/mc$ is the cyclotron frequency.

Solution:

[mex88] Particle with position-dependent mass moving in 1D potential

Consider a dynamical system with one degree of freedom specified by the equation of motion

$$\ddot{q} + G(q)\dot{q}^2 - F(q) = 0,$$

for arbitrary functions of $G(q)$ and $F(q)$. Show that any such system can be brought into canonical form, i.e. expressed as a pair of canonical equations by choosing the canonical momentum conjugate to q as follows: $p = m(q)\dot{q}$, $m(q) \equiv \exp[2\int dq\, G(q)]$. Express the associated Hamiltonian $H(q,p)$ in terms of the quantities p (momentum), $m(q)$ (position-dependent mass) and $V(q) \equiv -\int dq\, F(q)m(q)$ (potential energy).

Solution:

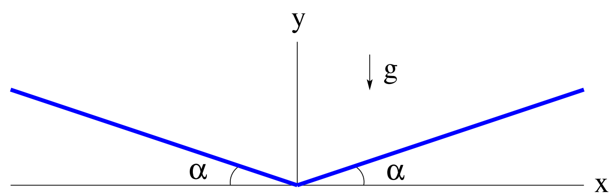
[mex89] Pendulum with string of slowly increasing length

Consider a plane pendulum consisting of a point mass m attached to a string of slowly increasing length $\ell = \ell_0 + \alpha t$. (a) Determine the Lagrangian $L(\phi, \dot{\phi}, t)$ and the Hamiltonian $H(\phi, p, t)$ of this dynamical system. (b) Evaluate the equation of motion for the variable ϕ in the form of a 2^{nd} order ODE from both L and H. Compare this equation of motion with that of a damped pendulum.

Solution:

[mex259] Libration between inclines

A particles of mass m and energy $E = T + V$ is sliding back and forth without friction along the two inclines shown under the influence of a uniform gravitational field g.
(a) Construct the Lagrangian $L(x, \dot{x})$.
(b) Construct the Hamiltonian $H(x, p_x)$.
(c) Derive the Lagrange equation from L.
(d) Derive the canonical equations from H.
(e) Calculate the period of oscillation τ as a function E.



Solution: