15. Hamiltonian Mechanics

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Abstract
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Legendre transform

Given is a function \( f(x) \) with monotonic derivative \( f'(x) \). The goal is to replace the independent variable \( x \) by \( p = f'(x) \) with no loss of information.

Note: The function \( G(p) = f(x) \) with \( p = f'(x) \) is, in general, not invertible.

The Legendre transform solves this task elegantly.

- **Forward direction:** \( g(p) = f(x) - xp \) with \( p = f'(x) \).
- **Reverse direction:** \( f(x) = g(p) + px \) with \( x = -g'(p) \)

Example 1: \( f(x) = x^2 + 1 \).

\[
\begin{align*}
\text{• } f(x) &= x^2 + 1 \quad \Rightarrow \quad f'(x) = 2x \quad \Rightarrow \quad x = \frac{p}{2} \quad \Rightarrow \quad g(p) = 1 - \frac{p^2}{4}.
\text{• } g(p) &= 1 - \frac{p^2}{4} \quad \Rightarrow \quad g'(p) = -\frac{p}{2} \quad \Rightarrow \quad p = 2x \quad \Rightarrow \quad f(x) = x^2 + 1.
\end{align*}
\]

Example 2: \( f(x) = e^{2x} \).

\[
\begin{align*}
\text{• } f(x) &= e^{2x} \quad \Rightarrow \quad f'(x) = 2e^{2x} = p \quad \Rightarrow \quad x = \frac{1}{2} \ln \frac{p}{2} \\
&\quad \Rightarrow \quad g(p) = \frac{p}{2} - \frac{p}{2} \ln \frac{p}{2}.
\text{• } g(p) &= \frac{p}{2} - \frac{p}{2} \ln \frac{p}{2} \quad \Rightarrow \quad g'(p) = -\frac{1}{2} \ln \frac{p}{2} = -x \\
&\quad \Rightarrow \quad p = 2e^{2x} \quad \Rightarrow \quad f(x) = e^{2x}.
\end{align*}
\]
Hamiltonian and Canonical Equations

Hamiltonian from Lagrangian via Legendre transform:

- Given the Lagrangian of a mechanical system: \( L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) \).
- Introduce canonical coordinates: \( q_i, p_i = \frac{\partial L}{\partial \dot{q}_i}, i = 1, \ldots, n \).
- Construct Hamiltonian:
  \[
  H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = \sum_j \dot{q}_j p_j - L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t),
  \]
  where \( \dot{q}_j = \dot{q}_j(q_1, \ldots, q_n; p_1, \ldots, p_n; t) \) is inferred from \( p_i = \frac{\partial L}{\partial \dot{q}_i} \).

Canonical equations from total differential of \( H \):

- \[
  \frac{dH}{dt} = \sum_j \left[ \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j \right] + \frac{\partial H}{\partial t} dt.
  \]
- \[
  \frac{d}{dt} \left( \sum_j \dot{q}_j p_j - L \right) = \sum_j \left[ \dot{q}_j dp_j + p_j d\dot{q}_j - \frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right] - \frac{\partial L}{\partial t} dt;
  \]
  use \( \frac{\partial L}{\partial q_j} \bigg|_{\dot{q}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = \ddot{p}_j, \frac{\partial L}{\partial \dot{q}_j} = p_j; \)
  \[
  \Rightarrow \frac{d}{dt} \left( \sum_j \dot{q}_j p_j - L \right) = \sum_j \left[ \ddot{q}_j dp_j + \dot{p}_j dq_j \right] - \frac{\partial L}{\partial t} dt;
  \]
  comparison of coefficients yields
  \[
  \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \ldots, n \quad \text{(canonical equations)},
  \]
  \[
  \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.
  \]

Comments:

- The inversion of \( p_i = \partial L/\partial \dot{q}_i \) as used above requires that \( \det \left( \frac{\partial^2 L}{\partial \dot{q}_i \dot{q}_j} \right) \neq 0 \) [mex189].
- Lagrangian from Hamiltonian: [mex188].
[mex188] Lagrangian from Hamiltonian via Legendre transform

Given a Hamiltonian system $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ and the associated canonical equations $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n$, find the Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ of the same system via Legendre transform, derive the Lagrange equations for the generalized coordinates $q_1, \ldots, q_n$ and establish the relation $\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$.

Solution:
Can you find the Hamiltonian of this system?

Consider the Lagrangian system

\[ L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} m (\dot{q}_1 + \dot{q}_2)^2 - \frac{1}{2} k (q_1^2 + q_2^2). \]

(a) Find the most general solution \( q_1(t), q_2(t) \) of the associated Lagrange equations. (b) Find the Hamiltonian \( H(q_1, q_2, p_1, p_2) \) such that the associated canonical equations have the same solution \( q_1(t), q_2(t) \). (c) Find the most general solution of \( H(q_1, q_2, p_1, p_2) \).

Solution:
Variational Principle in Phase Space

Hamilton’s principle: variations in configuration space.

\[ \delta J \doteq \delta \int_{t_1}^{t_2} dt \, L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0, \]

where \( \delta q_i = 0 \) at \( t_1 \) and \( t_2 \).

\[ \Rightarrow \quad \text{Lagrange equations: } \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \ldots, n. \]

Derivation: [mln78], [msl20].

Modified Hamilton’s principle: variations in phase space.

\[ \delta J \doteq \delta \int_{t_1}^{t_2} dt \left[ \sum_{i=1}^{n} p_i \dot{q}_i - H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) \right] = 0, \]

where \( \delta q_i = 0 \) and \( \delta p_i = 0 \) at \( t_1 \) and \( t_2 \).

\[ \Rightarrow \quad \text{Canonical equations: } \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n. \]

Derivation:

\[ \delta J = \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[ p_i \delta \dot{q}_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right] = 0; \]

use \( \int_{t_1}^{t_2} dt \sum_{i=1}^{n} p_i \delta \dot{q}_i = \left[ \sum_{i=1}^{n} p_i \delta q_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \dot{p}_i \delta q_i; \)

\[ \Rightarrow \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[ (\dot{q}_i - \frac{\partial H}{\partial p_i}) \delta p_i - (\dot{p}_i + \frac{\partial H}{\partial q_i}) \delta q_i \right] = 0. \]
Properties of the Hamiltonian

How is the Hamiltonian constructed from kinetic energy and potential energy? When does it represent the total energy? When is it conserved?

Here is a list of some answers:

- If the Hamiltonian does not depend explicitly on time then it is a conserved quantity: \( H(q_1, \ldots, q_n; p_1, \ldots, p_n) = \text{const.} \)

\[
\frac{dH}{dt} = \sum_{j=1}^{n} \left( \frac{\partial H}{\partial q_j} \dot{q}_j + \frac{\partial H}{\partial p_j} \dot{p}_j \right) = \sum_{j=1}^{n} \left( -\dot{p}_j \dot{q}_j + \dot{p}_j \dot{q}_j \right) = 0.
\]

- If \( T(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) \) is the kinetic energy and \( V(q_1, \ldots, q_n; t) \) the potential energy of a Lagrangian \( L = T - V \), then the Hamiltonian is equal to the total energy:

\[
H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = T + V = E(t), \quad \text{where } p_j = \frac{\partial L}{\partial \dot{q}_j}.
\]

- Suppose that some of the generalized coordinates \( q_1, \ldots, q_n \) are subject to holonomic constraints. Then \( H = T + V \) only holds if all those constraints are scleronomic, i.e. time-independent [mex81].

- Depending on the nature of the dynamical system and the choice of coordinates, the Hamiltonian may represent the total energy or a conserved quantity or both or neither [mex77].

- The property \( H \neq T + V \) occurs in the presence of velocity-dependent potentials [mln85]. The motion of a charged particle in a static magnetic field is a prominent example [mln86].

- In the presence of time-dependent fields, the conceptual framework used here quickly shows its limitations, because such fields themselves can transport momentum and energy.
Consider a dynamical system with $3N$ degrees of freedom subject to $k$ holonomic constraints: $r_i = r_i(q_1, \ldots, q_n, t), i = 1, \ldots, N, n = 3N - k$. The kinetic and potential energies are given by the expressions

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{r}_i|^2, \quad V = V(r_1, \ldots, r_N, \dot{r}_1, \ldots, \dot{r}_N, t).$$

Show that the Hamiltonian $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ derived from these specifications is equal to the total energy, $E = T + V$, only if (i) the potential energy does not depend on the velocities $\dot{r}_i$ and (ii) if the holonomic constraints are not explicitly time-dependent.

Solution:
A harmonic oscillator (mass $m$, spring constant $k$) is attached to a cart that moves with constant velocity $v_0$. Describe the dynamics in the coordinate system ($x$) that is at rest and in the coordinate system ($x'$) that is moving with the cart.

(a) Construct the Lagrangian $L$ of the oscillator in the rest frame and derive the associated Lagrange equation. Construct the Hamiltonian $H$ from $L$.

(b) Construct the Lagrangian $L'$ of the oscillator in the moving frame and derive the associated Lagrange equation. Construct the Hamiltonian $H'$ from $L'$.

(c) Show that the Lagrange equations obtained in (a) and (b) are equivalent.

(d) Which of the two quantities $H, H'$, if any, represents the total energy of the oscillator?

(e) Which of the two quantities $H, H'$, if any, represents a conserved quantity?

Solution:
Bead sliding on rotating rod in vertical plane

The rod AB rotates with constant angular velocity $\dot{\theta} = \omega$ at fixed perpendicular distance $h$ about point O in a vertical plane. A bead of mass $m$ is free to slide along the rod. Its position (relative to point C) on the rod is described by the variable $q$. (a) Construct the Lagrangian $L(q, \dot{q}, t)$ and derive the Lagrange equation for the variable $q(t)$. (b) Solve the Lagrange equation for the following initial conditions: $\theta(0) = q(0) = \dot{q}(0) = 0$. (c) Construct the Hamiltonian $H(q, p, t)$ from $L$. Determine whether or not $H$ represents the total energy of the bead.

Solution:
Use of Cyclic Coordinates

Lagrangian mechanics:

Lagrangian: \( L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n). \)

Cyclic coordinate \( q_n: \Rightarrow \frac{\partial L}{\partial q_n} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} = 0. \)

Conserved quantity: \( \frac{\partial L}{\partial \dot{q}_n} \equiv \beta_n(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n) = \text{const}. \)

Eliminate \( \dot{q}_n = \dot{q}_n(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; \beta_n) \) as independent variable.

Do not substitute \( \dot{q}_n(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; \beta_n) \) into Lagrangian.

Substitute \( \dot{q}_n(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; \beta_n) \) into Routhian instead.

Routhian: \( R(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_{n-1}; \beta_n) = L - \beta_n \dot{q}_n. \)

Equations of motion: \( \frac{\partial R}{\partial q_i} - d \frac{\partial R}{dt} \frac{\partial q_i}{\partial \dot{q}_i} = 0, \quad i = 1, \ldots, n-1. \)

Supplement: \( q_n(t) = -\int dt \frac{\partial R}{\partial \beta_n}. \)

Hamiltonian mechanics:

Hamiltonian: \( H(q_1, \ldots, q_{n-1}; p_1, \ldots, p_n). \)

Cyclic coordinate \( q_n: \Rightarrow \frac{\partial H}{\partial q_n} = 0 \Rightarrow \dot{p}_n = 0. \)

Conserved quantity: \( p_n \equiv \alpha_n = \text{const}. \)

Reduced Hamiltonian: \( H(q_1, \ldots, q_{n-1}; p_1, \ldots, p_{n-1}; \alpha_n). \)

Angular frequency: \( \omega_n \equiv \dot{q}_n(q_1, \ldots, q_{n-1}; p_1, \ldots, p_{n-1}; \alpha_n) = \frac{\partial H}{\partial \alpha_n}. \)

Equations of motion: \( \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n-1. \)

Supplement: \( q_n(t) = \int dt \omega_n(t). \)
Velocity-Dependent Potential Energy

Lagrange equations in raw form from [mln8]:
\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1, \ldots, n,
\]

- kinetic energy: \(T(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)\),
- generalized forces: \(Q_j(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)\).

The standard form of the Lagrange equations,
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \ldots, 3n,
\]
can be inferred from a Lagrangian \(L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)\) under the following circumstances:

(a) If the generalized forces can be derived from a position-dependent potential energy \(V(q_1, \ldots, q_n; t)\),
\[
Q_j(q_1, \ldots, q_n; t) = -\frac{\partial V}{\partial q_j},
\]
then the Lagrangian is \(L = T - V\).

(b) If the generalized forces can be derived from a velocity-dependent potential energy \(U(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)\),
\[
Q_j(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_j},
\]
then the Lagrangian is \(L = T - U\).

Hamiltonian derived from the Lagrangian via Legendre transform:
\[
H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = \sum_{j=1}^{n} p_j \dot{q}_j - L, \quad p_j = \frac{\partial L}{\partial \dot{q}_j}.
\]

Examples of velocity-dependent potential energy:
- Lorentz force [mln86],
- velocity-dependent central force [mex76].
Charged Particle in Electromagnetic Field

Lorentz force: \( \mathbf{F} = e \mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B} \).

Electric field: \( \mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \).

Magnetic field: \( \mathbf{B} = \nabla \times \mathbf{A} \).

Velocity-dependent potential energy: \( U(\mathbf{r}, \mathbf{v}, t) = e \phi(\mathbf{r}, t) - \frac{e}{c} \mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t) \).

Lagrangian: \( L(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2} m |\mathbf{v}|^2 - U(\mathbf{r}, \mathbf{v}, t) \).

Lagrange equations for \( \mathbf{r} = (x_1, x_2, x_3) \), \( \mathbf{v} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) \):

\[
m\ddot{x}_1 + \frac{e}{c} \frac{dA_1}{dt} = e \left( -\frac{\partial \phi}{\partial x_1} + \frac{\dot{x}_1}{c} \frac{\partial A_1}{\partial x_1} + \frac{\dot{x}_2}{c} \frac{\partial A_2}{\partial x_1} + \frac{\dot{x}_3}{c} \frac{\partial A_3}{\partial x_1} \right) \quad \text{etc.}
\]

Use \( \frac{dA_1}{dt} = \frac{\partial A_1}{\partial t} + \mathbf{v} \cdot \nabla A_1 = \frac{\partial A_1}{\partial t} + \dot{x}_1 \frac{\partial A_1}{\partial x_1} + \dot{x}_2 \frac{\partial A_1}{\partial x_2} + \dot{x}_3 \frac{\partial A_1}{\partial x_3} \).

\[
\Rightarrow m\ddot{x}_1 = e \left[ -\frac{\partial \phi}{\partial x_1} - \frac{1}{c} \frac{dA_1}{dt} + \frac{\dot{x}_2}{c} \left( \frac{\partial A_2}{\partial x_1} \right) + \frac{\dot{x}_3}{c} \left( \frac{\partial A_3}{\partial x_1} \right) \right] \\
= eE_x + \frac{e}{c} (\mathbf{v} \times \mathbf{B})_x \quad \text{etc.}
\]

Generalized momenta: \( p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c} A_i \).

- \( p_i \): canonical momenta.
- \( m\dot{x}_i \): kinetic momenta

Hamiltonian: \( H(\mathbf{r}, \mathbf{p}, t) = \sum_{i=1}^{3} p_i \dot{x}_i - L = \frac{1}{2m} \left| \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, t) \right|^2 + e\phi(\mathbf{r}, t) \).

Relativistic mechanics:

- Momenta: [mln63]
- Hamiltonian: \( H(\mathbf{r}, \mathbf{p}, t) = \sqrt{\left| \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, t) \right|^2 + m_0^2 c^4} + e\phi(\mathbf{r}, t) \).
A particle moves under the influence of a velocity-dependent central force
\[ F(r, \dot{r}) = \frac{1}{r^2} \left( 1 - \dot{r}^2 - 2r \dot{r} c^2 \right), \]
where \( c \) is a constant. (a) Show that the Lagrangian and Hamiltonian of this system can be expressed as follows:

\[ L(r, \dot{r}, \dot{\vartheta}) = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\vartheta}^2 \right) - \frac{1}{r} \left( 1 + \frac{\dot{r}^2}{c^2} \right), \quad H(r, p, \ell) = \frac{p^2}{2(m - 2/c^2 r)} + \frac{\ell^2}{2mr^2} + \frac{1}{r}. \]

(b) Derive the Lagrange equations from \( L \) and the canonical equations from \( H \) and show that they are equivalent.

Solution:
[mex190] Charged particle in a uniform magnetic field

Consider a particle with mass $m$ and electric charge $q$ moving in a magnetic field $\mathbf{B} = B\hat{e}_z$. (a) Find the Lagrangian $L(x, y, z, \dot{x}, \dot{y}, \dot{z})$ and derive the Lagrange equations from it. (b) Find the Hamiltonian $H(x, y, z, p_x, p_y, p_z)$ and derive the canonical equations from it. (c) Show that both sets of equations of motion can be brought into the form $\ddot{x} - \omega \dot{y} = 0$, $\ddot{y} + \omega \dot{x} = 0$, $\ddot{z} = 0$, where $\omega = qB/mc$ is the cyclotron frequency.

Solution:
Particle with position-dependent mass moving in 1D potential

Consider a dynamical system with one degree of freedom specified by the equation of motion
\[ \ddot{q} + G(q)\dot{q}^2 - F(q) = 0, \]
for arbitrary functions of $G(q)$ and $F(q)$. Show that any such system can be brought into canonical form, i.e. expressed as a pair of canonical equations by choosing the canonical momentum conjugate to $q$ as follows: $p = m(q)\dot{q}$, $m(q) \equiv \exp\left[2\int dq \, G(q)\right]$. Express the associated Hamiltonian $H(q, p)$ in terms of the quantities $p$ (momentum), $m(q)$ (position-dependent mass) and $V(q) \equiv -\int dq \, F(q)m(q)$ (potential energy).

Solution:
Consider a plane pendulum consisting of a point mass \( m \) attached to a string of slowly increasing length \( \ell = \ell_0 + \alpha t \). (a) Determine the Lagrangian \( L(\phi, \dot{\phi}, t) \) and the Hamiltonian \( H(\phi, p, t) \) of this dynamical system. (b) Evaluate the equation of motion for the variable \( \phi \) in the form of a 2\textsuperscript{nd} order ODE from both \( L \) and \( H \). Compare this equation of motion with that of a damped pendulum.

Solution:
[mex259] Libration between inclines

A particle of mass \( m \) and energy \( E = T + V \) is sliding back and forth without friction along the two inclines shown under the influence of a uniform gravitational field \( g \).

(a) Construct the Lagrangian \( L(x, \dot{x}) \).
(b) Construct the Hamiltonian \( H(x, p_x) \).
(c) Derive the Lagrange equation from \( L \).
(d) Derive the canonical equations from \( H \).
(e) Calculate the period of oscillation \( \tau \) as a function \( E \).

Solution: