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14. Ideal Quantum Gases II: Fermions

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Abstract

Part fourteen of course materials for Statistical Physics I: PHY525, taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

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Fermi-Dirac functions [tsl42]

$$f_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx x^{n-1}}{z^{-1}e^x + 1}, \quad 0 \leq z < \infty$$

Series expansion:

$$f_n(z) = \sum_{l=1}^{\infty} (-1)^{l-1} \frac{z^l}{l^n}, \quad 0 \leq z \leq 1.$$

Special cases:

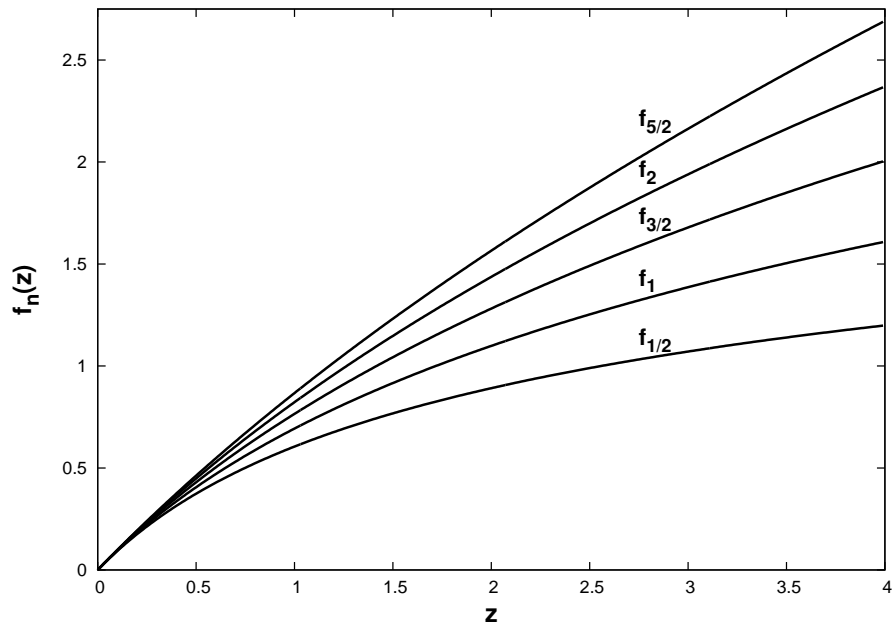
$$f_0(z) = \frac{z}{1+z}, \quad f_1(z) = \ln(1+z), \quad f_\infty(z) = z.$$

Recurrence relation:

$$z f'_n(z) = f_{n-1}(z), \quad n \geq 1.$$

Asymptotic expansion for $z \gg 1$:

$$\begin{aligned} f_n(z) &= \frac{(\ln z)^n}{\Gamma(n+1)} \left[1 + \sum_{k=2,4,\dots} 2n(n-1)\cdots(n-k+1) \left(1 - \frac{1}{2^{k-1}}\right) \frac{\zeta(k)}{(\ln z)^k} \right] \\ &= \frac{(\ln z)^n}{\Gamma(n+1)} \left[1 + n(n-1) \frac{\pi^2}{6} (\ln z)^{-2} \right. \\ &\quad \left. + n(n-1)(n-3) \frac{7\pi^4}{360} (\ln z)^{-4} + \dots \right] \end{aligned}$$



Ideal Fermi-Dirac gas: equation of state and internal energy [tIn69]

Conversion of sums into integrals by means of density of energy levels:

$$D(\epsilon) = \frac{gV}{\Gamma(\mathcal{D}/2)} \left(\frac{m}{2\pi\hbar^2} \right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}, \quad V = L^{\mathcal{D}}.$$

Fundamental thermodynamic relations for FD gas:

$$\frac{pV}{k_B T} = \sum_k \ln(1 + ze^{-\beta\epsilon_k}) = \int_0^\infty d\epsilon D(\epsilon) \ln(1 + ze^{-\beta\epsilon}) = \frac{gV}{\lambda_T^{\mathcal{D}}} f_{\mathcal{D}/2+1}(z),$$

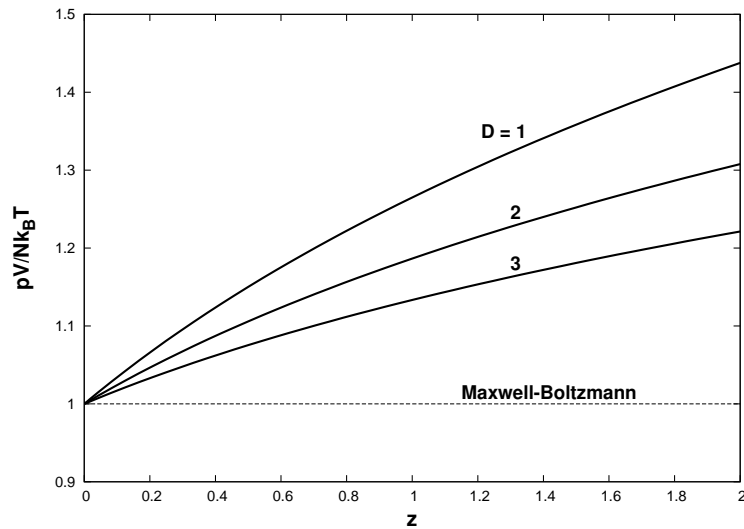
$$\mathcal{N} = \sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} + 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)}{z^{-1}e^{\beta\epsilon} + 1} = \frac{gV}{\lambda_T^{\mathcal{D}}} f_{\mathcal{D}/2}(z),$$

$$U = \sum_k \frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k} + 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)\epsilon}{z^{-1}e^{\beta\epsilon} + 1} = \frac{\mathcal{D}}{2} k_B T \frac{gV}{\lambda_T^{\mathcal{D}}} f_{\mathcal{D}/2+1}(z).$$

Note: The range of fugacity has no upper limit: $0 \leq z \leq \infty$. The chemical potential μ is unrestricted. The factor g is included to account for any existing level degeneracy due to internal degrees of freedom (e.g. spin) of the fermions.

Equation of state (with fugacity z in the role of parameter):

$$\frac{pV}{\mathcal{N}k_B T} = \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)}.$$

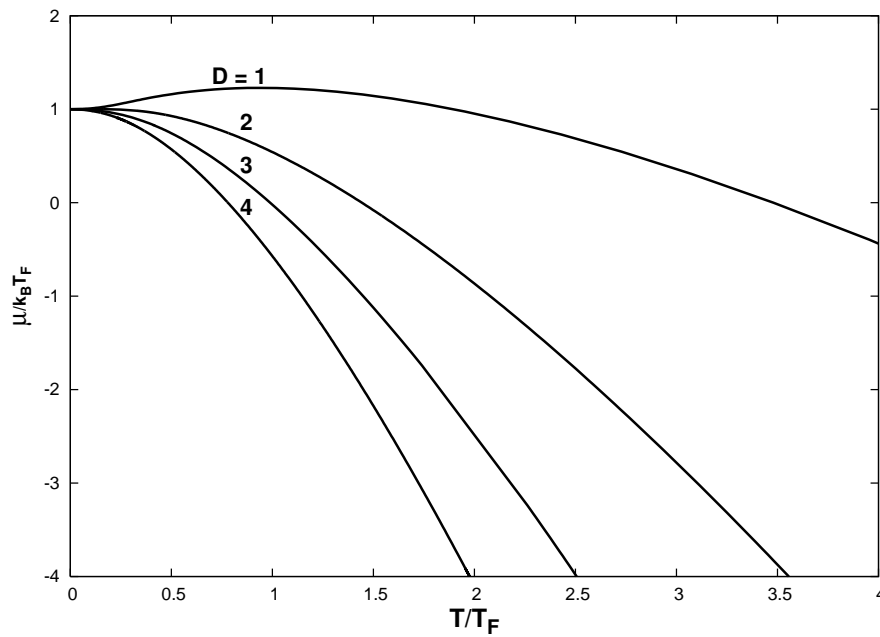


Ideal Fermi-Dirac gas: chemical potential [tsl43]

Fugacity z from $x = f_{\mathcal{D}/2}(z)$, where

$$x = \frac{\lambda_T^{\mathcal{D}}}{v}, \quad v \doteq \frac{gV}{\mathcal{N}}, \quad \lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}}.$$

Chemical potential [tex117]: $\frac{\mu}{k_B T_v} = \frac{T}{T_v} \ln z$, $\frac{T}{T_v} = [f_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}}$.



Reference temperature: $k_B T_v = \frac{\Lambda}{v^{2/\mathcal{D}}}$, $\Lambda \doteq \frac{h^2}{2\pi m}$.

For a complete list of reference values see [tln71].

Fermi energy: $\lim_{T \rightarrow 0} \mu = \epsilon_F = k_B T_F$.

Fermi temperature: $\frac{T_F}{T_v} = [\Gamma(\mathcal{D}/2 + 1)]^{2/\mathcal{D}} \stackrel{\mathcal{D} \gg 1}{\approx} \frac{\mathcal{D}}{2e}$.

[tex117] FD gas in \mathcal{D} dimensions: chemical potential I

(a) Start from the fundamental thermodynamic relation $\mathcal{N} = (gV/\lambda_T^{\mathcal{D}})f_{\mathcal{D}/2}(z)$ for the ideal Fermi-Dirac gas in \mathcal{D} dimensions and use the reference temperature $k_B T_v = \Lambda/v^{2/\mathcal{D}}$, $v \doteq gV/\mathcal{N}$, $\Lambda \doteq h^2/2\pi m$ to derive the following parametric expression for the dependence on temperature T of the chemical potential μ :

$$\frac{\mu}{k_B T_v} = \frac{T}{T_v} \ln z, \quad \frac{T}{T_v} = [f_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}}.$$

(b) Derive the following expression for the Fermi energy ϵ_F and the Fermi temperature T_F :

$$\lim_{T \rightarrow 0} \frac{\mu(T)}{k_B T_v} = \frac{\epsilon_F}{k_B T_v} = \frac{T_F}{T_v} = [\Gamma(\mathcal{D}/2 + 1)]^{2/\mathcal{D}}.$$

(c) Show that this result includes the familiar result, $\epsilon_F = (h^2/2m)(3\mathcal{N}/4\pi gV)^{2/3}$ for $\mathcal{D} = 3$.

Solution:

[tex118] FD gas in \mathcal{D} dimensions: chemical potential II

Start from the results derived in [tex117] to infer the following expressions for the fugacity z and the chemical potential μ at $T \ll T_F$:

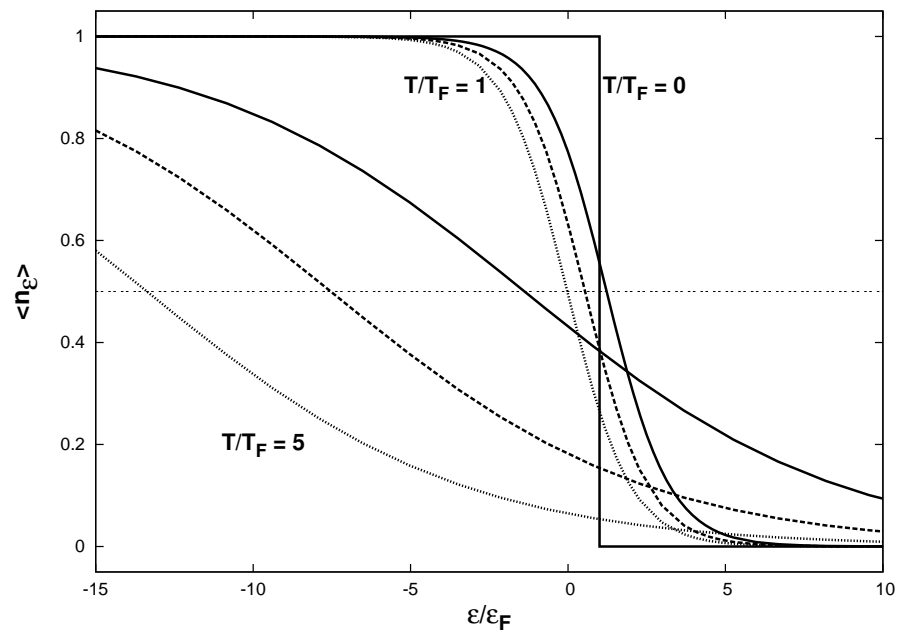
$$\ln z \sim \frac{T_F}{T}, \quad \frac{\mu}{k_B T_F} \sim 1 - \frac{\pi^2}{12} (\mathcal{D} - 2) \left(\frac{T}{T_F} \right)^2.$$

Solution:

Ideal Fermi-Dirac gas: average level occupancy [tsl44]

Average occupancy of 1-particle state at energy ϵ if system (with fixed \mathcal{N}, V) is at temperature T :

$$\langle n_\epsilon \rangle = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad \text{with } \mu(T) \text{ from [tex117].}$$



Note: $\langle n_\epsilon \rangle = \frac{1}{2}$ occurs at $\epsilon = \mu(T)$.

Limit $T \rightarrow 0$: $\mu(T) \rightarrow \epsilon_F$, $\langle n_\epsilon \rangle \rightarrow \Theta(\epsilon_F - \epsilon)$.

Ideal Fermi-Dirac gas: isochores I [tsl46]

Reference values for temperature and pressure:

$$k_B T_v = \frac{\Lambda}{v^{2/\mathcal{D}}}, \quad p_v = \frac{k_B T_v}{v}; \quad \Lambda \doteq \frac{h^2}{2\pi m}, \quad v \doteq \frac{gV}{\mathcal{N}}.$$

$$\frac{T_F}{T_v} = \frac{p_F}{p_v} = \left[\Gamma \left(\frac{\mathcal{D}}{2} + 1 \right) \right]^{2/\mathcal{D}} \stackrel{\mathcal{D} \gg 1}{\approx} \frac{\mathcal{D}}{2e}.$$

Isochore:

$$\frac{p}{p_F} = \frac{T}{T_F} \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)}, \quad \frac{T}{T_F} = \left[\Gamma \left(\frac{\mathcal{D}}{2} + 1 \right) f_{\mathcal{D}/2}(z) \right]^{-2/\mathcal{D}}.$$

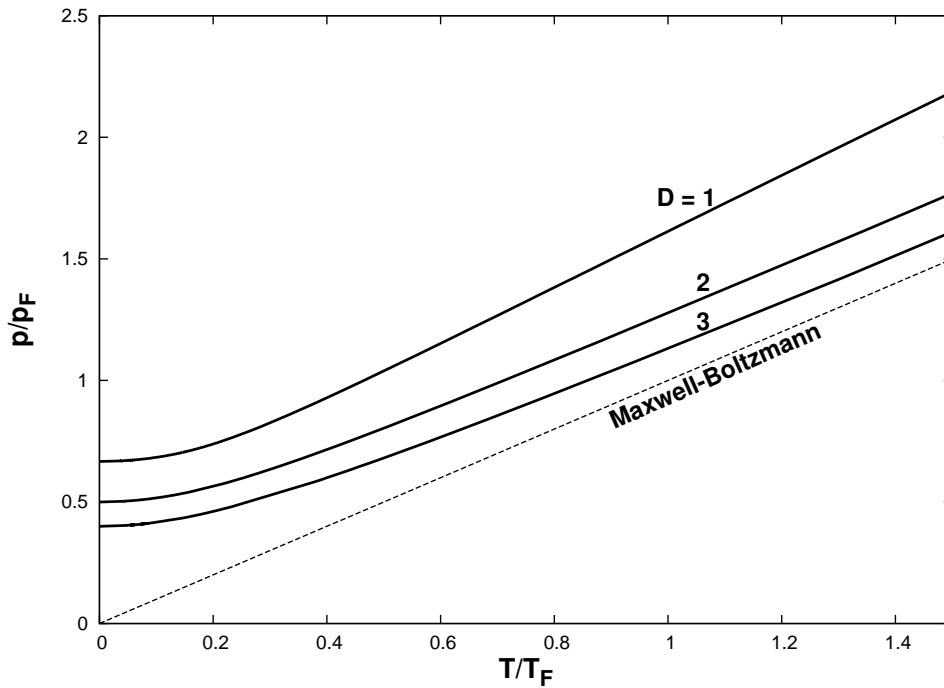
Low-temperature limit [tex119]:

$$\lim_{T \rightarrow 0} \frac{p}{p_F} = \left(\frac{\mathcal{D}}{2} + 1 \right)^{-1}.$$

High-temperature asymptotic regime [tex119]:

$$\frac{pV}{\mathcal{N}k_B T_F} \sim \frac{T}{T_F} \left[1 + \left[2^{\mathcal{D}/2+1} \Gamma \left(\frac{\mathcal{D}}{2} + 1 \right) \right]^{-1} \left(\frac{T_F}{T} \right)^{\mathcal{D}/2} \right].$$

The excess pressure relative to the Maxwell-Boltzmann line may be called a manifestation of *statistical interaction pressure*.



[tex119] FD gas in \mathcal{D} dimensions: statistical interaction pressure

Consider the isochore of an ideal Fermi-Dirac gas in \mathcal{D} dimensions, as given by the parametric relation

$$\frac{p}{p_v} = \frac{T}{T_v} \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)}, \quad \frac{T}{T_v} = [f_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}}.$$

where $k_B T_v = \Lambda/v^{2/\mathcal{D}}$, $p_v = k_B T/v$, $\Lambda \doteq h^2/2\pi m$, $v \doteq gV/\mathcal{N}$. The upward deviation of this result from the Maxwell-Boltzmann result, $p/p_v = T/T_v$, is a manifestation of repulsive statistical interaction between fermions. (a) Calculate the high- T asymptotic dependence of p/p_v on T/T_v including the leading correction to MB behavior. (b) Calculate the low- T limit of p/p_v . (c) Calculate the low- T limit of p/p_F , where $T_F = T_v[\Gamma(\mathcal{D}/2 + 1)]^{2/\mathcal{D}}$ is the Fermi temperature and $p_F = k_B T_F/v$ the associated reference pressure. (d) Compare the differently scaled statistical interaction pressures p/p_v and p/p_F at $T = 0$ in the limit $\mathcal{D} \rightarrow \infty$.

Solution:

Ideal Fermi-Dirac gas: isotherms [tln70]

Reference values for reduced volume $v \doteq gV/\mathcal{N}$ and pressure p :

$$v_T = \lambda_T^D, \quad p_T = gk_B T / \lambda_T^D.$$

Parametric expression for isotherm:

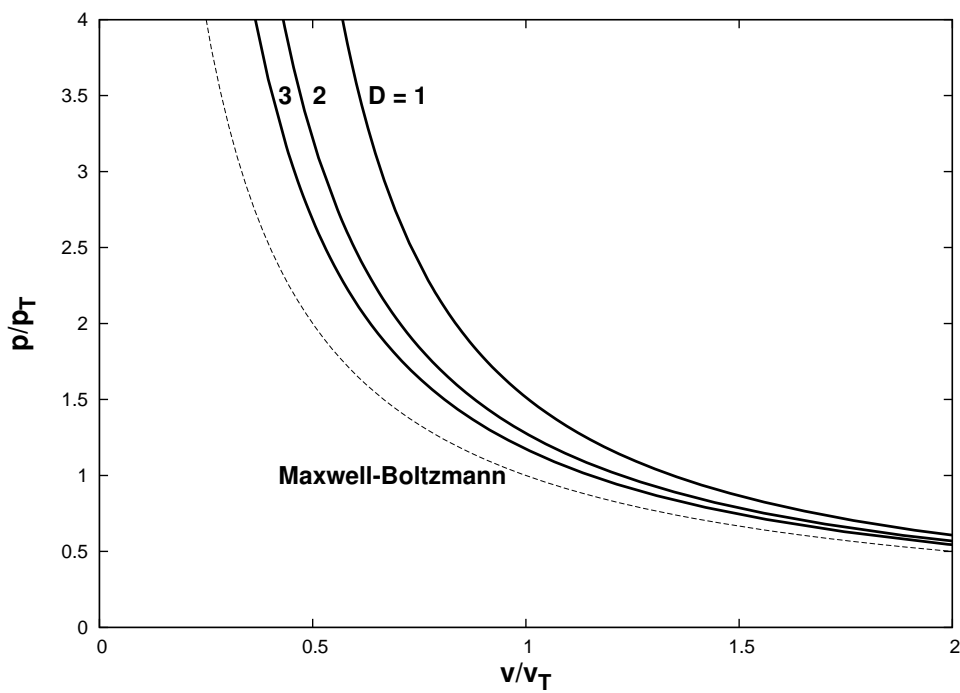
$$\frac{p}{p_T} = f_{D/2+1}(z), \quad \frac{v}{v_T} = [f_{D/2}(z)]^{-1}.$$

Isotherm at low density [tex120]:

$$pv = \text{const}, \quad v \gg v_T.$$

Isotherm at high density [tex120]:

$$pv^{(D+2)/D} = \text{const}, \quad v \ll v_T.$$



[tex120] FD gas in \mathcal{D} dimensions: isotherm and adiabat

(a) Show that the isotherm of an ideal Fermi-Dirac gas in \mathcal{D} dimensions is described by the parametric relation

$$\frac{p}{p_T} = f_{\mathcal{D}/2+1}(z), \quad \frac{v}{v_T} = [f_{\mathcal{D}/2}(z)]^{-1},$$

where $v_T = \lambda_T^{\mathcal{D}}$ and $p_T = gk_B T / \lambda_T^{\mathcal{D}}$ are convenient reference values and $v = gV/\mathcal{N}$ is the reduced volume. (b) Show that the adiabat is described by the relation $pv^{(\mathcal{D}+2)/\mathcal{D}} = \text{const}$ for all values of v/v_0 . (c) Show that the relation for the isotherm approaches Boyle's law, $pv = \text{const}$, for $v \gg v_T$ and that it approaches the adiabat, $pv^{(\mathcal{D}+2)/\mathcal{D}} = \text{const}$, for $v \ll v_T$.

Solution:

[tex102] FD gas in \mathcal{D} dimensions: ground-state energy

Given are the following expressions for the average number of particles, the average energy, the average occupation number at $T = 0$, and the density of states for an ideal Fermi-Dirac gas in \mathcal{D} dimensions:

$$\mathcal{N} = \sum_k \langle n_k \rangle, \quad U = \sum_k \langle n_k \rangle \epsilon_k, \quad \langle n_k \rangle = \Theta(\epsilon_F - \epsilon_k), \quad D(\epsilon) = \frac{gV}{\Gamma(\mathcal{D}/2)} \left(\frac{2\pi m}{h^2} \right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}.$$

Derive from these expressions the following results for the dependence of the ground-state energy per particle, U_0/\mathcal{N} , on the Fermi energy ϵ_F and for the dependence of the ground-state energy density U_0/V on the particle density \mathcal{N}/V :

$$\frac{U_0}{\mathcal{N}} = \frac{\mathcal{D}}{\mathcal{D}+2} \epsilon_F, \quad \frac{U_0}{V} \propto \left(\frac{\mathcal{N}}{V} \right)^{(\mathcal{D}+2)/\mathcal{D}}.$$

Solution:

Ideal Fermi-Dirac gas: heat capacity [ts145]

Internal energy:

$$U = \frac{\mathcal{D}}{2} \mathcal{N} k_B T \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)}.$$

Heat capacity [use $z g'_n(z) = g_{n-1}(z)$ for $n \geq 1$]:

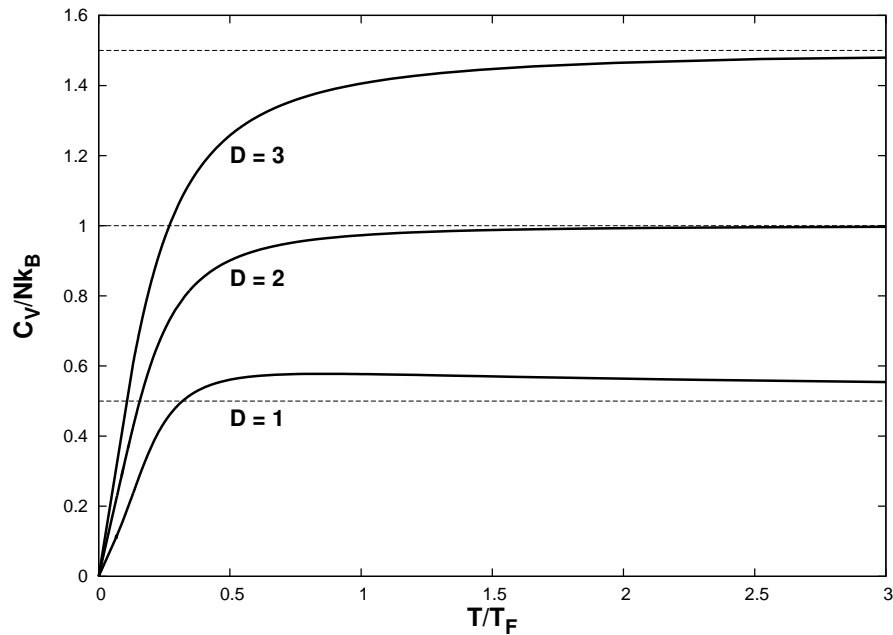
$$\frac{C_V}{\mathcal{N} k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4} \frac{f'_{\mathcal{D}/2+1}(z)}{f'_{\mathcal{D}/2}(z)}.$$

Low-temperature asymptotic behavior:

$$\frac{C_V}{\mathcal{N} k_B} \sim \mathcal{D} \frac{\pi^2}{6} \frac{T}{T_F}.$$

High-temperature asymptotic behavior:

$$\frac{C_V}{\mathcal{N} k_B} \sim \frac{\mathcal{D}}{2} \left[1 - \frac{\mathcal{D}/2 - 1}{2^{\mathcal{D}/2-1} \Gamma(\mathcal{D}/2)} \left(\frac{T_F}{T} \right)^{\mathcal{D}/2} \right].$$



[tex100] FD gas in \mathcal{D} dimensions: heat capacity at high temperature

The internal energy of the ideal Fermi-Dirac gas in \mathcal{D} dimensions is given by the expression,

$$U = \mathcal{N}k_B T \frac{\mathcal{D}}{2} \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)}.$$

(a) Use this result to derive the following expression for the heat capacity $C_V = (\partial U / \partial T)_{V, \mathcal{N}}$:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4} \frac{f'_{\mathcal{D}/2+1}(z)}{f'_{\mathcal{D}/2}(z)}.$$

Use the derivative $\partial / \partial T$ of the result $f_{\mathcal{D}/2}(z) = \mathcal{N} \lambda_T^{\mathcal{D}} / gV$ with $V = L^{\mathcal{D}}$ to calculate any occurrence of $(\partial z / \partial T)_{V, \mathcal{N}}$ in the derivation. Use the recursion relation $z f'_n(z) = f_{n-1}(z)$ for $n \geq 1$ to further simplify the results pertaining to $\mathcal{D} \geq 2$. (b) Infer from this result the leading correction to the Maxwell-Boltzmann result, $C_V = (\mathcal{D}/2) \mathcal{N}k_B$, at high temperature.

Solution:

[tex101] FD gas in \mathcal{D} dimensions: heat capacity at low temperature

Use the results of [tex118] and [tex100] to determine the low-temperature asymptotic behavior,

$$\frac{C_V}{\mathcal{N}k_B} \sim \mathcal{D} \frac{\pi^2}{6} \frac{T}{T_F},$$

of the heat capacity of the ideal Fermi-Dirac gas in \mathcal{D} dimensions.

Solution:

Ideal Fermi-Dirac gas: isochores II [tln73]

Reference values for temperature and pressure:

$$k_B T_v = \frac{\Lambda}{v^{2/\mathcal{D}}}, \quad p_v = \frac{k_B T_v}{v}; \quad \Lambda \doteq \frac{h^2}{2\pi m}, \quad v \doteq \frac{gV}{\mathcal{N}}.$$

Noncommuting limits $z \rightarrow \infty$, $\mathcal{D} \rightarrow \infty$:

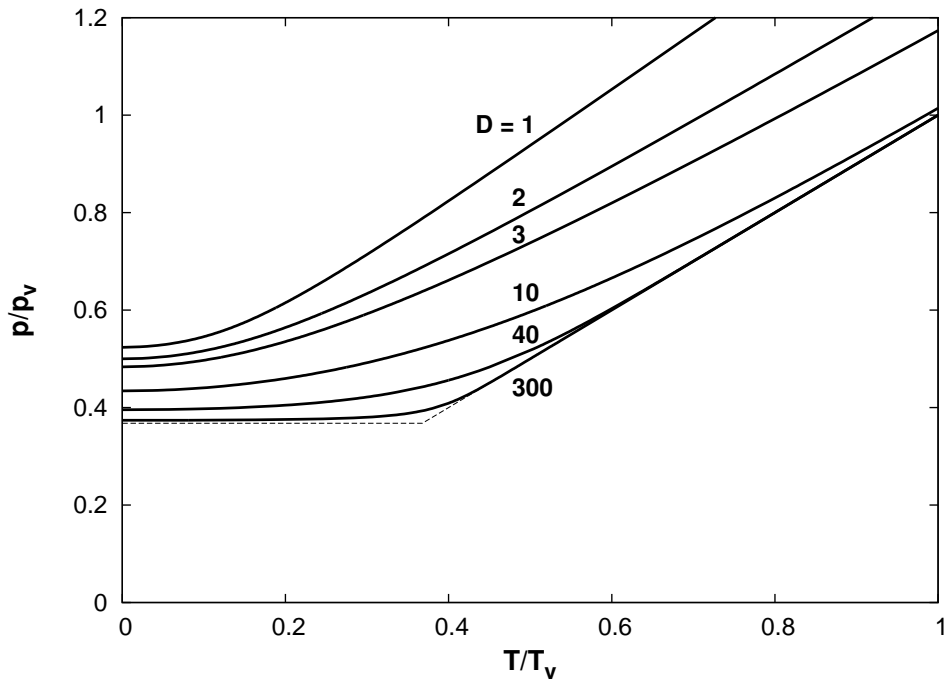
- $z < \infty$, $\mathcal{D} \rightarrow \infty$:

$$\frac{p}{p_v} = \frac{T}{T_v} \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)} \xrightarrow{\mathcal{D} \rightarrow \infty} \frac{T}{T_v} \quad (\text{ideal MB gas}).$$

- $\mathcal{D} \rightarrow \infty$, $z \rightarrow \infty$ with $\mathcal{D}/2 = r \ln z$, $r \geq 0$:

$$\frac{p}{p_v} = \frac{f_{\mathcal{D}/2+1}(z)}{[f_{\mathcal{D}/2}(z)]^{1+2/\mathcal{D}}} \xrightarrow{\mathcal{D} \gg 1} \frac{e^{-1}}{1+2/\mathcal{D}},$$

$$\frac{T}{T_v} = [f_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}} \xrightarrow{\mathcal{D} \gg 1} \frac{\mathcal{D} e^{-1}}{2 \ln z} \quad (\text{pure Fermi sea}).$$



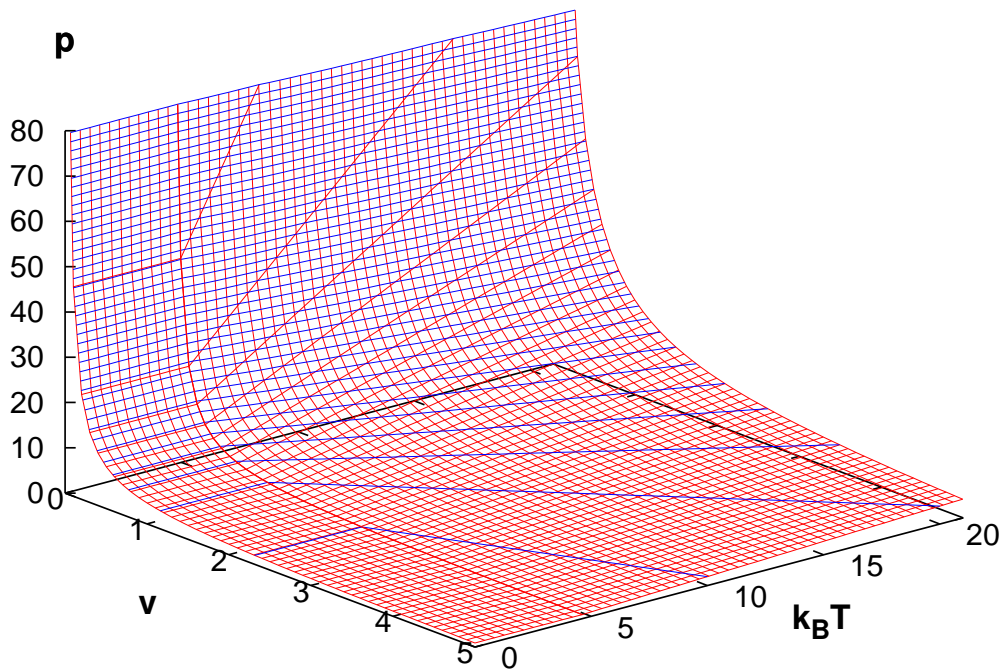
Ideal FD gas: phase diagram in $\mathcal{D} \rightarrow \infty$ [tln74]

Equation of state:

$$pv = \begin{cases} k_B T, & T > T_c \text{ (ideal MB gas)} \\ k_B T_c, & T < T_c \text{ (pure Fermi sea)} \end{cases}$$

Transition temperature:

$$k_B T_c = \frac{\Lambda}{e}, \quad \Lambda \doteq \frac{h^2}{2\pi m}.$$



[tex121] Stable white dwarf

Consider a burnt-out white dwarf star. For simplicity we assume that it consists of equal numbers N of electrons, protons, and neutrons. The electrons form a fully degenerate, nonrelativistic Fermi gas that prevents the star from collapsing into a neutron star or a black hole.

(a) Under the assumption that the kinetic energy is predominantly due to the electrons and that the potential energy is predominantly gravitational in nature, show that the total energy of the star depends on N and R (radius) as follows:

$$E = E_{kin} + E_{pot} = \frac{3\hbar^2}{10m_e} \left(\frac{9\pi}{4}\right)^{2/3} \frac{N^{5/3}}{R^2} - \frac{12}{5} m_n^2 G \frac{N^2}{R},$$

where m_e, m_n are the electron and neutron masses, and G is the universal gravitational constant.

(b) Using a star of solar mass, $m_\odot \simeq 1.99 \times 10^{30}$ kg, find the radius R_{wd} in units of the solar radius, $R_\odot \simeq 6.96 \times 10^8$ m.

Solution:

[tex122] Unstable white dwarf

Consider a burnt-out white dwarf star of the same composition as described in [tex121] but with N so large that most of the electrons are ultrarelativistic, $\epsilon \simeq cp = \hbar kc$, in the fully degenerate state.

(a) Under similar assumptions as in [tex121] show that the expression of the total energy now reads

$$E = E_{kin} + E_{pot} = \frac{\hbar c}{3\pi} \left(\frac{9\pi}{4}\right)^{4/3} \frac{N^{4/3}}{R} - \frac{12}{5} m_n^2 G \frac{N^2}{R},$$

where c is the speed of light.

(b) Find the critical mass in units of the solar mass, m_c/m_\odot , beyond which this star is unstable and thus prone to a gravitational collapse into a neutron star or a black hole.

Solution: