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14. Oscillations

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Abstract
Part fourteen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc14]

Linearly Damped Harmonic Oscillator [mln6]

Equation of motion: $m\ddot{x} = -kx - \gamma\dot{x} \;\Rightarrow\; \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0.$

Damping parameter: $\beta \equiv \gamma/2m$; characteristic frequency: $\omega_0 = \sqrt{k/m}$.

Ansatz: $x(t) = e^{rt} \;\Rightarrow\; (r^2 + 2\beta r + \omega_0^2)e^{rt} = 0 \;\Rightarrow\; r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}.$

Overdamped motion: $\Omega_1 \equiv \sqrt{\beta^2 - \omega_0^2} > 0$

Linearly independent solutions: $e^{r_+ t},\ e^{r_- t}$.

General solution: $x(t) = \left(A_+ e^{\Omega_1 t} + A_- e^{-\Omega_1 t}\right) e^{-\beta t}$.

Initial conditions: $A_+ = (\dot{x}_0 - r_- x_0)/2\Omega_1, \quad A_- = (r_+ x_0 - \dot{x}_0)/2\Omega_1$.

Critically damped motion: $\sqrt{\omega_0^2 - \beta^2} = 0, \quad r = -\beta$

Linearly independent solutions: $e^{rt},\ te^{rt}$.

General solution: $x(t) = (A_0 + A_1 t)e^{-\beta t}$.

Initial conditions: $A_0 = x_0, \quad A_1 = \dot{x}_0 + \beta x_0$.

Underdamped motion: $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2} > 0$

Linearly independent solutions: $e^{r_+ t},\ e^{r_- t}$.

General solution: $x(t) = (A\cos\omega_1 t + B\sin\omega_1 t)\, e^{-\beta t} = D\cos(\omega_1 t - \delta)e^{-\beta t}$.

$$D = \sqrt{A^2 + B^2}, \quad \delta = \arctan(B/A).$$

Initial conditions: $A = x_0, \quad B = (\dot{x}_0 + \beta x_0)/\omega_1$.

The dissipative force, $-\gamma\dot{x}$, effectively represents a coupling of one low-frequency oscillator to many high-frequency oscillators.

[mex150] Harmonic oscillator with friction

The equation of motion of a harmonic oscillator with Coulomb damping (friction) has the form

$$\ddot{x} + \alpha\,\mathrm{sgn}(\dot{x}) + \omega_0^2 x = 0,$$

where $\omega_0^2 = k/m$ is the angular frequency of the undamped oscillator and $\mathrm{sgn}(\dot{x})$ denotes the sign ($\pm$) of the instantaneous velocity.
(a) Show that the solution for $n\pi \leq \omega_0 t \leq (n+1)\pi, n = 0, 1, 2, \ldots$ and $x(0) = A_0 + \beta, \dot{x}(0) = 0$ has the form $x(t) = A_n \cos(\omega_0 t) + (-1)^n\beta$. Find the constant β, the maximum value of n, and the amplitudes A_n.
(b) For the case $\alpha = 1\text{cm/s}^2$, $\omega_0 = 1\text{rad/s}$, and $x(0) = 9\text{cm}$, find the time it takes the system to come to a halt and the total distance traveled. Plot the phase portrait in the $(x, \dot{x}/\omega_0)$-plane for this particular case.

Solution:

[mex261] Harmonic oscillator with attenuation

The equation of motion of a harmonic oscillator with attenuation is written in the form

$$\ddot{x} + \alpha_m \,\mathrm{sgn}(\dot{x})|\dot{x}|^m + \omega_0^2 x = 0,$$

where $\omega_0^2 = k/m$ is the angular frequency of the undamped oscillator and $\mathrm{sgn}(\dot{x})$ denotes the sign ($\pm$) of the instantaneous velocity. Here we consider the three cases of Coulomb damping ($m = 0$), linear damping ($m = 1$), and quadratic damping ($m = 2$). Use $\omega_0 = 1$ throughout.
Employ the Mathematica options of NDSolve and ParametricPlot to numerically solve the equation of motion for all three cases and to plot x versus $\dot{x}$ for initial conditions $x(0) = 9$ and $\dot{x}(0) = 0$. Vary the attenuation parameter α_m in each case and watch out for qualitative changes in the phase-plane trajectory. Present a collection of neat graphs that emphasize the differences between the three cases as well as the differences between parameter regimes for one or the other case. Describe the different types of trajectories.

Solution:

Driven Harmonic Oscillator I [mln28]

Equation of motion: $m\ddot{x} = -kx - \gamma\dot{x} + F_0 \cos\omega t \;\Rightarrow\; \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$.

Parameters: $\beta \doteq \gamma/2m, \quad \omega_0 \doteq \sqrt{k/m}, \quad A \doteq F_0/m$.

General solution: $x(t) = x_c(t) + x_p(t)$.

- $x_c(t)$: general solution of homogen. eq. (transients) $\Rightarrow$ [mln6].
- $x_p(t)$: particular solution of inhomogen. eq. (steady state) $\Rightarrow$ [mex180].

Steady-state oscillation: $x_p(t) = D\cos(\omega t - \delta)$

- amplitude: $D(\omega) = \dfrac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}$,
- phase angle: $\delta(\omega) = \arctan \dfrac{2\omega\beta}{\omega_0^2 - \omega^2}$.

Maximum amplitude realized at $\left.\dfrac{dD(\omega)}{d\omega}\right|_{\omega_R} = 0$.

Amplitude resonance frequency: $\omega_R = \sqrt{\omega_0^2 - 2\beta^2} \quad \text{if} \;\; 2\beta^2 < \omega_0^2$.

Average energy: $\langle E(\omega)\rangle = \langle T(\omega)\rangle + \langle V(\omega)\rangle = \dfrac{1}{4} mA^2 \dfrac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}$.

$\langle E(\omega)\rangle$, $\langle T(\omega)\rangle$, $\langle V(\omega)\rangle$ are resonant at different frequencies $\Rightarrow$ [mex181].

Average power input: $\langle P(\omega)\rangle \doteq \langle F_0 \cos\omega t \cdot \dot{x}(t)\rangle \quad \Rightarrow$ [mex182].

Quality factor: $\Rightarrow$ [mex183]

- driven oscillator: $Q \doteq 2\pi \, \dfrac{\text{average energy stored}}{\text{maximum energy input per period}}$,
- damped oscillator: $Q \doteq 2\pi \, \dfrac{\text{energy stored}}{\text{energy loss per period}}$.

For $\beta \ll \omega_0$ the width at half maximum of the power resonance curve is $\Delta\omega \simeq 2\beta$. Therefore, the quality factor is $Q \simeq \omega_0/\Delta\omega$.

[mex180] Driven harmonic oscillator: steady state solution

Consider the driven harmonic oscillator, $m\ddot{x} = -kx - \gamma\dot{x} + F_0 \cos \omega t$. Show that the steady-state solution has the form

$$x(t) = D\cos(\omega t - \delta), \quad D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}, \quad \delta(\omega) = \arctan \frac{2\omega\beta}{\omega_0^2 - \omega^2},$$

where we have used the parameters $\beta \doteq \gamma/2m,\ \omega_0 \doteq \sqrt{k/m},\ A \doteq F_0/m.$.

Solution:

[mex181] Driven harmonic oscillator: kinetic and potential energy

Consider the driven harmonic oscillator, $m\ddot{x} = -kx - \gamma\dot{x} + F_0 \cos\omega t$, in a steady-state motion. (a) Calculate the average kinetic energy $\langle T(\omega)\rangle$, the average potential energy $\langle V(\omega)\rangle$, and the average total energy $\langle E(\omega)\rangle = \langle T(\omega)\rangle + \langle V(\omega)\rangle$. Use the parameters $\beta \doteq \gamma/2m,\ \omega_0 \doteq \sqrt{k/m},\ A \doteq F_0/m.$.
(b) Each quantity assumes its maximum value at a different resonant frequency: $\omega_T, \omega_V, \omega_E$. Determine each resonant frequency.

Solution:

[mex182] Driven harmonic oscillator: power input

Consider the driven harmonic oscillator, $m\ddot{x} = -kx - \gamma\dot{x} + F_0 \cos\omega t$, in a steady-state motion. (a) Calculate the average power input, $\langle P(\omega)\rangle \doteq \langle F_0 \cos\omega t \cdot \dot{x}(t)\rangle$. Use the parameters $\beta \doteq \gamma/2m$, $\omega_0 \doteq \sqrt{k/m}$, $A \doteq F_0/m$. (b) Find the resonant frequency ω_P and the maximum (averaged) power input $P_{max} = \langle P(\omega_P)\rangle$.

Solution:

[mex183] Quality factor of damped harmonic oscillator

(a) Consider the driven harmonic oscillator, $m\ddot{x} = -kx - \gamma\dot{x} + F_0 \cos\omega t$, in a steady-state motion. Use the parameters $\beta \doteq \gamma/2m$, $\omega_0 \doteq \sqrt{k/m}$, $A \doteq F_0/m$. In [mex182] we have calculated the maximum (averaged) power input, $P_{max} = \langle P(\omega_P)\rangle$, and in [mex181] we have calculated the average energy $\langle E(\omega)\rangle$ stored in the oscillator. Determine the quality factor of the driven oscillator defined as $Q = 2\pi\langle E(\omega_P)\rangle/\langle P(\omega_P)\rangle\tau$ with $\tau = 2\pi/\omega_P$. Show that to leading or der in β/ω_0 the quality factor is equal to the amplitude ratio at resonance and at zero frequency: $Q = D(\omega_R)/D(0)$.
(b) Consider the harmonic oscillator, $m\ddot{x} = -kx - \gamma\dot{x}$, with weak damping ($\beta/\omega_0 \ll 1$) and no driving force. Determine the quality factor Q of the damped oscillator defined as 2π times the ratio of the instantaneous energy stored, $E(t)$, and the energy loss per period, $\tau|dE/dt|$. Evaluate the result to leading order in β/ω_0

Solution:

[mex262] Driven harmonic oscillator: runaway resonance

Consider the driven harmonic oscillator with no damping, $m\ddot{x} = -kx + F_0 \cos\omega t$. Take the general solution off resonance, $\omega \neq \omega_0 = \sqrt{k/m}$, and perform the limit $\omega \to \omega_0$ to show that the (runaway) solution at resonance with initial condition $x(0) = B\cos\beta$, $\dot{x}(0) = -\omega_0 B \sin\beta$ has the form

$$x(t) = B\cos(\omega_0 t + \beta) + \frac{F_0 t}{2m\omega_0}\sin(\omega_0 t).$$

Solution:

Driven Harmonic Oscillator II [mln29]

Equation of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A(t)$.

Parameters: $\beta \doteq \gamma/2m,\ \omega_0 \doteq \sqrt{k/m},\ A(t) \doteq F(t)/m$.

Periodic driving force: $F(t+\tau) = F(t)$.

Fourier series: $A(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega t + b_n \sin n\omega t\right], \quad \omega = \frac{2\pi}{\tau}$.

Fourier coefficients: $a_n = \frac{2}{\tau}\int_0^\tau dt\, A(t)\cos n\omega t, \quad b_n = \frac{2}{\tau}\int_0^\tau dt\, A(t)\sin n\omega t$.

Linear response of system to periodic driving force:

$$x(t) = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} d(\omega_n)\left[a_n\cos(\omega_n t + \delta_n) + b_n \sin(\omega_n t + \delta_n)\right],$$

$$\omega_n = \frac{2\pi n}{\tau}, \quad d(\omega_n) = \frac{1}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}}, \quad \delta_n = \arctan\left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right).$$

Aperiodic driving force: $F(t) = mA(t)$ with $\int_{-\infty}^{+\infty} dt\,|A(t)| < \infty$.

Fourier transform: $\tilde{x}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}x(t), \quad \tilde{A}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t}A(t)$.

Inverse transform: $x(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{x}(\omega), \quad A(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{A}(\omega)$.

Fourier transformed equation of motion is algebraic (not differential):

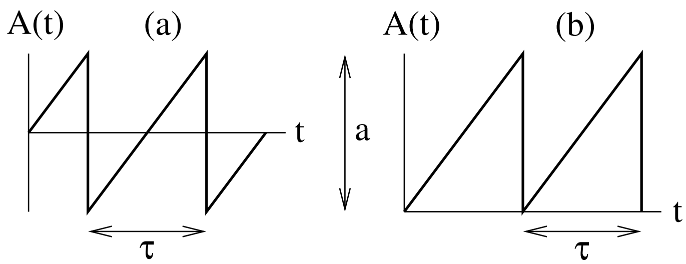
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A(t) \;\Rightarrow\; -\omega^2\tilde{x}(\omega) - 2i\beta\omega\tilde{x}(\omega) + \omega_0^2\tilde{x}(\omega) = \tilde{A}(\omega).$$

Linear response of system to aperiodic driving force:

$$\tilde{x}(\omega) = \frac{\tilde{A}(\omega)}{\omega_0^2 - \omega^2 - 2i\beta\omega} \quad \Rightarrow\; x(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{x}(\omega).$$

[mex184] Fourier coefficients of a sawtooth driving force

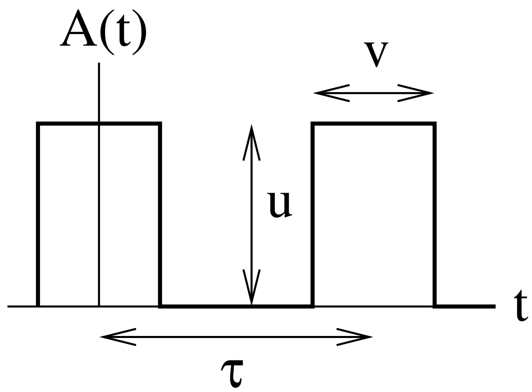
Find the Fourier coefficients a_n, b_n of a periodic sawtooth driving force $F(t) = mA(t)$ in two renditions with different symmetries. (a) $A(t) = at/\tau,\ |t| < \tau/2$; (b) $A(t) = at/\tau,\ 0 < t < \tau$.



Solution:

[mex185] Fourier coefficients of periodic sequence of rectangular pulses

(a) Find the Fourier coefficients a_n, b_n of a periodic driving force $F(t) = mA(t)$ in the shape of a sequence of rectangular pulses as shown. (b) Each pulse has an area $c = uv$. Find the Fourier coefficients in the limit $u \to \infty, v \to 0$ at fixed c.



Solution:

Driven Harmonic Oscillator III [mln107]

No damping and arbitrary driving force.

Equation of motion: $m\ddot{x} + \omega_0^2 x = F(t)$,

Complex variable: $\xi(t) \doteq \dot{x}(t) + \imath\omega_0 x(t) \quad \Rightarrow \dot{\xi}(t) - \imath\omega_0\xi(t) = F(t)/m.$

Ansatz: $\xi(t) = B(t)e^{\imath\omega_0 t} \quad \Rightarrow \dot{B}(t) = \frac{1}{m}F(t)e^{-\imath\omega_0 t}.$

Solution: $x(t) = \frac{1}{\omega_0}\Im[\xi(t)], \quad \xi(t) = e^{\imath\omega_0 t}\left[\frac{1}{m}\int_0^t dt' F(t')e^{-\imath\omega_0 t'} + \xi_0\right].$

Instantaneous energy at time t or total energy absorbed at time t if oscillator is initially at equilibrium ($x_0 = \dot{x}_0 = 0$):

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}m|\xi(t)|^2 = \frac{1}{2m}\left|\int_0^t dt' F(t')e^{-\imath\omega_0 t'}\right|^2.$$

Constant force switched on: $F(t) = F_0\Theta(t)$.

$$x(t) = \frac{F_0}{m\omega_0^2}\Big(1 - \cos\omega_0 t\Big), \quad E(t) = \frac{F_0^2}{m\omega_0^2}\Big(1 - \cos\omega_0 t\Big).$$

Force performs positive and negative work in alternation.

Fading force switched on: $F(t) = F_0 e^{-\alpha t}\Theta(t)$.

$$x(t) = \frac{F_0}{m(\omega_0^2 - \alpha^2)}\left[e^{-\alpha t} - \cos\omega_0 t + \frac{\alpha}{\omega_0}\sin\omega_0 t\right].$$

$$E(t) = \frac{F_0^2}{2m(\omega_0^2 + \alpha^2)}\Big[1 + e^{-2\alpha t} - 2e^{-\alpha t}\cos\omega_0 t\Big].$$

Constant force switched on and then off: $F(t) = F_0\Theta(t)\Theta(T - t)$.

$$x(t) = \frac{F_0}{m\omega_0^2}\Big[\cos\omega_0(t - T) - \cos\omega_0 t\Big] \qquad (t \geq T).$$

$$E(t) = \frac{2F_0^2}{m\omega_0^2}\sin^2\frac{\omega_0 T}{2} = \text{const} \qquad (t \geq T).$$

[mex263] Driven harmonic oscillator with Coulomb damping

The harmonic oscillator with Coulomb damping and harmonic driving force is described by the equation of motion,

$$\ddot{x} + \alpha\,\mathrm{sgn}(\dot{x}) + \omega_0^2 x = A\cos(\omega t), \tag{1}$$

where $\omega_0^2 = k/m$, $\alpha = \mu/m$, $A = F_0/m$. The function $\mathrm{sgn}(\dot{x})$ denotes the sign ($\pm$) of the instantaneous velocity. The oscillator has mass m and the spring has stiffness k. The coefficient of kinetic (and static) friction is μ. The natural angular frequency of oscillation is ω_0. The harmonic driving force has amplitude F_0 and angular frequency ω. In this project we consider an oscillator at resonance ($\omega = \omega_0 = 1$) launched from $x(0) = 0$ with initial velocity $\dot{x}(0) = v_0 > 0$.

(a) Use the DSolve option of Mathematica to determine the analytic solutions of (1) with the given initial conditions, valid over a time interval with $\dot{x} > 0$. Check whether the solution that Mathematica gives you can be further simplified by hand. Use the ParametricPlot option of Mathematica to plot this solution in the phase plane, i.e. x versus $\dot{x}$. Use $A = 1$, $v_0 = 9$ and various values of α (all in SI units).

(b) Use the NDSolve and ParametricPlot options of Mathematica to generate and plot data for the solution of (1) over a larger time interval. Use again $A = 1$, $v_0 = 9$ and various values of α. Tune α to a value that yields a periodic trajectory.

(c) Investigate the stability of the periodic trajectory thus found numerically. Is it a limit cycle? Vary the initial conditions and check whether the periodic trajectory attracts or repels nearby trajectories.

Solution:

Small Oscillations [mln43]

Consider undamped small-amplitude motion about a stable equilibrium.

Lagrangian: $L(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n) = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij} k_{ij}q_iq_j.$

Mass coefficients: $m_{ij} = \sum_{k=1}^{3N} m_k \left(\frac{\partial x_k}{\partial q_i}\right)_0 \left(\frac{\partial x_k}{\partial q_j}\right)_0.$

Stiffness coefficients: $k_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0.$

Lagrange equations are linear: $\sum_{j=1}^{n} m_{ij}\ddot{q}_j + \sum_{j=1}^{n} k_{ij}q_j = 0, \quad i = 1,\ldots,n.$

The matrices $\{m_{ij}\}$ and $\{k_{ij}\}$ are symmetric.

Ansatz for solution: $q_j(t) = A_j\cos(\omega t + \phi), \quad j = 1,\ldots,n.$

$\Rightarrow \sum_{j=1}^{n}(k_{ij} - \omega^2 m_{ij})A_j\cos(\omega t + \phi) = 0, \quad i = 1,\ldots,n.$

Linear homogeneous equations:

$$\sum_{j=1}^{n}(k_{ij} - \omega^2 m_{ij})A_j = 0, \quad i = 1,\ldots,n. \tag{1}$$

Characteristic equation (n^{th}-order polynomial in ω^2):

$$\begin{vmatrix} (k_{11} - \omega^2 m_{11}) & \cdots & (k_{1n} - \omega^2 m_{1n}) \\ \vdots & & \vdots \\ (k_{n1} - \omega^2 m_{n1}) & \cdots & (k_{nn} - \omega^2 m_{nn}) \end{vmatrix} = 0.$$

The n roots $\omega_1^2,\ldots,\omega_n^2$ of the characteristic equation are the eigenvalues associated with n *natural modes of vibration* (normal modes).

The normal mode with angular frequency ω_k is specified by a set of amplitudes $A_1^{(k)},\ldots,A_n^{(k)}$. The amplitude ratios for this normal mode are determined from Eqs. (1):

$$\sum_{j=1}^{n}(k_{ij} - \omega_k^2 m_{ij})A_j^{(k)} = 0, \quad i = 1,\ldots,n.$$

Transformation to Principal Axes [mln30]

Solving the equations $\displaystyle\sum_{j=1}^{n}(k_{ij} - \omega_r^2 m_{ij})A_{jr} = 0, \quad i, r = 1, \ldots, n$

for the amplitudes A_{jr} of the n normal modes amounts to finding an *orthogonal* matrix $\mathbf{A}$, which diagonalizes the *symmetric* matrices $\mathbf{m}$ and $\mathbf{k}$ simultaneously:

$$\mathbf{A}^T \cdot \mathbf{m} \cdot \mathbf{A} = \mathbf{1} \doteq \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$\mathbf{A}^T \cdot \mathbf{k} \cdot \mathbf{A} = \mathbf{\Omega}^2 \doteq \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 \end{pmatrix},$$

Normal coordinates: $\displaystyle Q_j = \sum_{i=1}^{n} A_{ij} q_i.$

Lagrangian: $\displaystyle L = \frac{1}{2}\sum_{ij}\left(m_{ij}\dot{q}_i\dot{q}_j - k_{ij}q_iq_j\right) = \frac{1}{2}\sum_{r=1}^{n}\left(\dot{Q}_r^2 - \omega_r^2 Q_r^2\right).$

Lagrange equations: $\displaystyle\sum_{j=1}^{n}\left(m_{ij}\ddot{q}_j + k_{ij}q_j\right) = 0, \quad i = 1, \ldots, n$ (coupled).

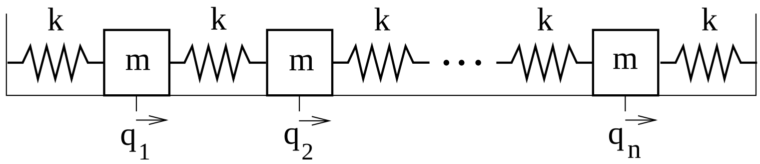
Lagrange equations: $\ddot{Q}_r + \omega_r^2 Q_r = 0, \quad r = 1, \ldots, n$ (decoupled).

Applications:

- Blocks and springs in series [mex123]
- Small oscillations of the double pendulum [mex124]
- Two coupled oscillators [mex186]

Elastic Chain [mln48]

Consider the elastic chain consisting of n blocks and $n+1$ springs as shown.



Equations of motion: $m\ddot{q}_j + k(2q_j - q_{j-1} - q_{j+1}) = 0, \quad j = 1, \ldots, n.$

Boundary conditions: $q_0(t) = q_{n+1}(t) = 0.$

Ansatz for solution: $q_j(t) = C\sin(\alpha j)\cos(\omega t + \phi), \quad j = 1, \ldots, n.$

Check boundary condition: $\sin[(n+1)\alpha] = 0 \quad \Rightarrow \alpha = \frac{\pi r}{n+1}, \; r = 1, \ldots, n.$

Check eqs. of motion: $\left[-m\omega^2 + 2k\left(1 - \cos\frac{\pi r}{n+1}\right)\right] C\sin(\alpha j) = 0.$

Normal mode frequencies: $\omega_r = 2\sqrt{\frac{k}{m}}\sin\frac{\pi r}{2(n+1)}, \quad r = 1, \ldots, n.$

Normal mode amplitudes: $A_{jr} = \sin\left(\frac{\pi r}{n+1}\,j\right), \quad j, r = 1, \ldots, n.$

General solution: superposition of normal modes

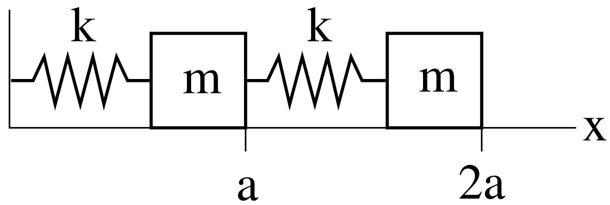
$$\begin{aligned} q_j(t) &= \sum_{r=1}^{n} C_r \sin\left(\frac{\pi r}{n+1}\,j\right)\cos(\omega_r t + \phi_r) \\ &= \sum_{r=1}^{n} \sin\left(\frac{\pi r}{n+1}\,j\right)\left[d_r\cos\omega_r t + e_r\sin\omega_r t\right], \end{aligned}$$

with

$$\begin{aligned} d_r &= \frac{2}{n+1}\sum_{r=1}^{n} q_j(0)\sin\left(\frac{\pi r}{n+1}\,j\right), \\ e_r &= \frac{2}{n+1}\frac{1}{\omega_r}\sum_{r=1}^{n} \dot{q}_j(0)\sin\left(\frac{\pi r}{n+1}\,j\right). \end{aligned}$$

[mex123] Blocks and springs in series

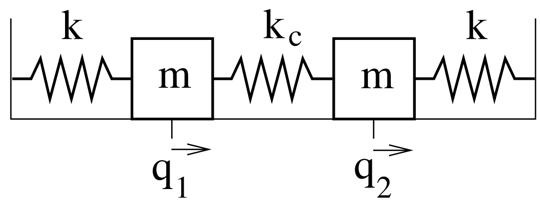
Consider a system of two blocks of mass m attached by springs of stiffness k to each other and to a rigid wall. The blocks can slide without friction along the x-axis. When the springs are relaxed, the blocks are at the positions $x_1 = a$ and $x_2 = 2a$. (a) Find the Lagrangian $L(q_1, q_2, \dot{q}_1, \dot{q}_2)$ for the system, where q_1, q_2 are the displacements of the two blocks from their equilibrium positions. (b) Find the angular frequencies ω_1, ω_2 of the two normal modes by solving the characteristic equation. (c) Find the amplitude ratios $A_1^{(k)}/A_2^{(k)}, k = 1, 2$ for the two normal modes.



Solution:

[mex186] Two coupled oscillators

Consider a system of two blocks of mass m attached by springs of stiffness k to rigid walls on two sides and by a spring of stiffness k_c to each other. The blocks can slide back and forth without friction. (a) Find the Lagrangian $L(q_1, q_2, \dot{q}_1, \dot{q}_2)$. (b) Write the equations of motion for q_1, q_2. (c) Find the normal mode frequencies ω_1, ω_2 by solving the characteristic equation. (d) Rewrite the Lagrangian in the form $L = \frac{1}{2}\sum_{ij}[m_{ij}\dot{q}_i\dot{q}_j + k_{ij}q_iq_j]$. Then find the orthogonal matrix A_{ij} which diagonalizes the symmetric matrices m_{ij} and k_{ij} simultaneously: $\sum_{lm} A^T_{il} m_{lm} A_{mj} = \delta_{ij}$, $\sum_{lm} A^T_{il} k_{lm} A_{mj} = \omega_i \delta_{ij}$. (e) Find the normal mode coordinates $Q_j = \sum_i A_{ij} q_i$.



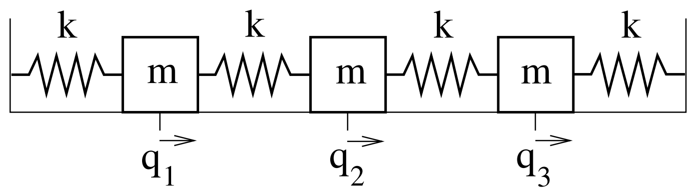
Solution:

[mex187] Three coupled oscillators

Consider the elastic chain consisting of three blocks and four springs as shown. (a) Show that the equations of motion for the generalized coordinates q_j can be brought into the form

$$m\ddot{q}_j + k(2q_j - q_{j-1} - q_{j+1}) = 0, \quad j = 1, 2, 3$$

with boundary conditions $q_0(t) = q_4(t) = 0$. (b) Use the ansatz $q_j(t) = A_j \cos(\omega t)$ and find the three normal-mode frequencies $\omega_r, r = 1, 2, 3$. (c) Find the normal coordinates $Q_j = \sum_i A_{ij} q_i, j = 1, 2, 3$. (d) Illustrate each normal mode Q_j modes by plotting $q_i(0)$ versus i.



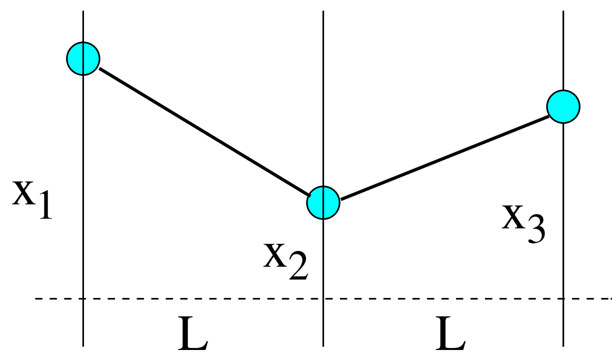
Solution:

[mex114] What is the physical nature of these modes?

Three beads of mass m each are constrained to slide without friction along parallel wires. The beads are connected to each other by rubber bands of negligible mass which are stretched considerably ($L \gg L_0$). (a) Describe the physical nature of the modes specified by the generalized coordinates q_1, q_2, q_3, where

$$x_1 = q_1 + q_2 + \frac{1}{2}q_3, \quad x_2 = q_1 - q_3, \quad x_3 = q_1 - q_2 + \frac{1}{2}q_3.$$

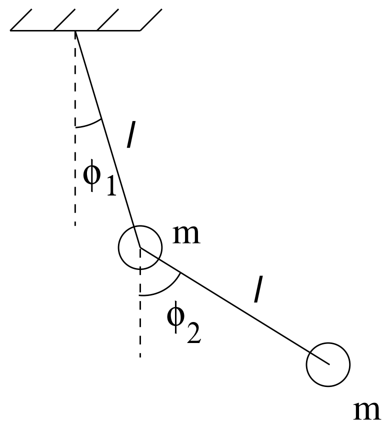
Give a quantitative description of the motion that ensues if the system is initially at rest with only one the generalized coordinates diplaced infinitesimally: (b) $0 < q_1^{(0)} \ll L,\ q_2^{(0)} = q_3^{(0)} = 0$, (c) $0 < q_2^{(0)} \ll L,\ q_1^{(0)} = q_3^{(0)} = 0$, (d) $0 < q_3^{(0)} \ll L,\ q_1^{(0)} = q_2^{(0)} = 0$.



Solution:

[mex124] Small oscillations of the double pendulum

Consider a plane double pendulum consisting of two equal point masses m and two rods of negligible mass and equal lengths ℓ. The Lagrangian $L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$ of this system is known from [mex20]. (a) Expand $L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$ to quadratic order in the dynamical variables and derive the Lagrange equations from it. They describe the small oscillations about the stable equilibrium position. (b) Find the angular frequencies ω_1, ω_2 of the two normal modes by solving the characteristic equation. (c) Find the amplitude ratios $A_1^{(k)}/A_2^{(k)}, k = 1, 2$ for the two normal modes.



Solution: