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13. Analysis of capacitor circuits at equilibrium

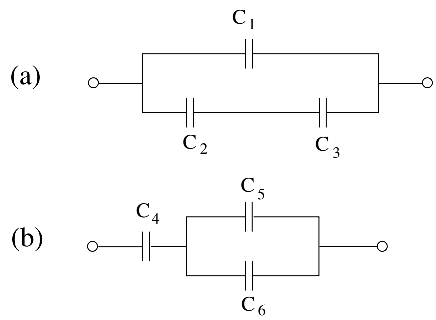
Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

PHY204 Lecture 13 [rln13]

Capacitor Circuit (1)

Find the equivalent capacitances of the two capacitor networks.
All capacitors have a capacitance of $1\mu F$.



tsl114

Reading electric circuits is a skill, like reading music. It takes practice to become proficient. In this course, we just take a few elementary steps toward acquiring the skill of reading circuits.

It is important that we recognize devices, any devices, that are connected in parallel or in series. Here we practice this skill for capacitors. We can then employ with confidence the rules established in the previous lecture for series and parallel connections.

We have learned that when two or more capacitors are connected in parallel or in series we can simplify the circuit by replacing a series connection or a parallel connection by a single capacitor with equivalent capacitance.

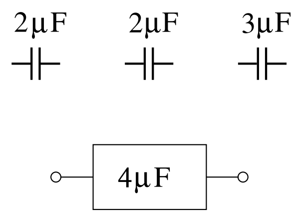
By this method it is possible to reduce some capacitor circuits to a single capacitor. Doing that is a bit like solving a Sudoku. The principal value of such exercises is the practice of reading circuits.

(a) We recognize that the two 1μF capacitors on the lower branch are in series. We can replace them by a $\frac{1}{2}\mu$F capacitor, which then is in parallel with the 1μF capacitor in the upper branch. We can replace them by a $\frac{3}{2}\mu$F equivalent capacitor.

(b) We recognize the two 1μF capacitors on the right to be in parallel. We replace them by a 2μF capacitor, which then is in series with the 1μF capacitor on the left, yielding a $\frac{2}{3}\mu$F equivalent capacitance.

Capacitor Circuit (3)

Connect the three capacitors in such a way that the equivalent capacitance is $C_{eq} = 4\mu\text{F}$. Draw the circuit diagram.



tsl116

There are six different ways of connecting three capacitors between two terminals if two have the same capacitance. They can all be reduced to a single capacitor in one or two steps.

(a) $\left(\frac{1}{2\mu\text{F}} + \frac{1}{2\mu\text{F}} + \frac{1}{3\mu\text{F}}\right)^{-1} = \frac{3}{4}\mu\text{F}$ (three in series)

(b) $2\mu\text{F} + 2\mu\text{F} + 3\mu\text{F} = 7\mu\text{F}$ (three in parallel)

(c) $\left(\frac{1}{2\mu\text{F}} + \frac{1}{2\mu\text{F} + 3\mu\text{F}}\right)^{-1} = \frac{10}{7}\mu\text{F}$ (two in parallel, in series with third)

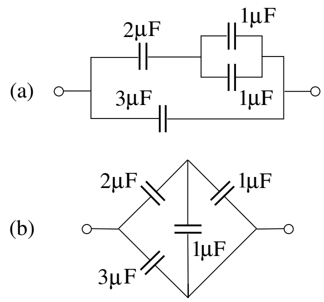
(d) $\left(\frac{1}{3\mu\text{F}} + \frac{1}{2\mu\text{F} + 2\mu\text{F}}\right)^{-1} = \frac{12}{7}\mu\text{F}$ (two in parallel, in series with third)

(e) $\left(\frac{1}{2\mu\text{F}} + \frac{1}{3\mu\text{F}}\right)^{-1} + 2\mu\text{F} = \frac{16}{5}\mu\text{F}$ (two in series, in parallel with third)

(f) $\left(\frac{1}{2\mu\text{F}} + \frac{1}{2\mu\text{F}}\right)^{-1} + 3\mu\text{F} = 4\mu\text{F}$ (two in series, in parallel with third)

Capacitor Circuit (5)

Find the equivalent capacitances of the following circuits.



tsl118

(a) This circuits can be reduced to a single capacitor in three steps. There is only one way to do it.

First we recognize that the two 1μF capacitors are in parallel. We replace them by a 2μF capacitor, which then is in series with the other 2μF. This series connection can be replaced by a 1μF capacitor, which then is in parallel with the 3μF. This parallel connection yields a 4μF equivalent capacitance.

(b) This circuit is harder to read. An experienced practitioner recognizes at once that it is the same circuit as the one above.

The functionality of a circuit remains unchanged if wires are stretched or bent or if devices and junctions are moved along wires. The only prohibition is that devices must not be moved across junctions.

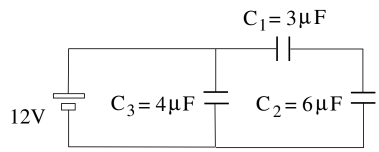
If we bend a few wires and move the junction that leads to the terminal on the right past the junction at the bottom, we readily recognize the equivalence with the other circuit.

Not all capacitor circuits can be reduced to a single capacitor by this method of sequential simplification. There are circuits where none of the capacitors are in parallel or in series. This does not mean that there is no equivalent capacitance. There is. However, finding it requires a different method of analyis. We will analyze one such case near the end of this lecture.

Capacitor Circuit (9)

The circuit of capacitors connected to a battery is at equilibrium.

(a) Find the equivalent capacitance C_{eq}.
(b) Find the total energy U stored in the circuit (excluding the battery).
(c) Find the the charge Q_3 on capacitor C_3.
(d) Find the voltage V_2 across capacitor C_2.



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Analyzing capacitor circuits at equilibrium is not all about equivalent capacitances. Of interest are the charges on individual capacitors, the voltages across them, and the energies stored on them, when the circuit is connected to a power source (battery). In the following we discuss a few simple applications ending with a more complex one.

(a) We recognize that capacitors C_1 and C_2 are in series. We replace them by C_{12}, which then is in parallel with C_3.

$$C_{12} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = 2\mu\text{F}, \quad C_{eq} = C_{12} + C_3 = 6\mu\text{F}.$$

(b) Here we take advantage of the concept of equivalence. The equivalent capacitor is connected to a 12V battery:

$$U = \frac{1}{2}C_{eq}(12\text{V})^2 = 432\mu\text{J}.$$

(c) The voltage across capacitor C_3 is that supplied by the battery.

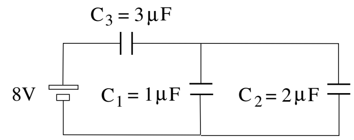
$$Q_3 = C_3(12\text{V}) = 48\mu\text{C}.$$

(d) Capacitor C_{12} is in parallel to C_3 as used earlier. The voltage across it is also 12V. The charge on it: $Q_{12} = C_{12}(12\text{V}) = 24\mu\text{C}$. C_{12} is a series combination of C_1 and C_2. Both have the same charge on it: $Q_1 = Q_2 = Q_{12} = 24\mu\text{C}$. The voltage cross C_2 then is $V_2 = Q_2/C_2 = 4\text{V}$.

Intermediate Exam II: Problem #1 (Spring '05)

The circuit of capacitors connected to a battery is at equilibrium.

(a) Find the equivalent capacitance C_{eq}.
(b) Find the voltage V_3 across capacitor C_3.
(c) Find the the charge Q_2 on capacitor C_2.



Solution:

(a) $C_{12} = C_1 + C_2 = 3\mu\text{F}, \quad C_{eq} = \left(\frac{1}{C_{12}} + \frac{1}{C_3}\right)^{-1} = 1.5\mu\text{F}.$

(b) $Q_3 = Q_{12} = Q_{eq} = C_{eq}(8\text{V}) = 12\mu\text{C}$

$\Rightarrow \ V_3 = \frac{Q_3}{C_3} = \frac{12\mu\text{C}}{3\mu\text{F}} = 4\text{V}.$

(c) $Q_2 = V_2 C_2 = 8\mu\text{C}.$

tsl336

This circuit looks similar to the one on the previous page. But watch out. Moving one capacitor from right to left across a junction produces an entirely different circuit.

(a) We recognize that capacitors C_1 and C_2 are in parallel. We replace them by C_{12}, which then is in series with C_3.

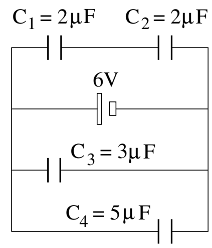
(b) The parallel combination of C_{12} and the capacitor C_3 are in series and thus acquire the same charge, equal to the charge on C_{eq}, which has 8V across. Once we know the charge Q_3, the voltage V_3 follows directly.

(c) The voltages across the parallel combination C_{12} and the capacitor C_3 must add up to the 8V provided by the battery. We know $V_3 = 4\text{V}$. Therefore, given that C_1 and C_2 are in parallel, we have $V_{12} = V_1 = V_2 = 4\text{V}$. The charge Q_2 then follows directly.

Unit Exam II: Problem #1 (Spring '07)

Consider the configuration of two point charges as shown.

(a) Find the energy U_3 stored on capacitor C_3.
(b) Find the voltage V_4 across capacitor C_4.
(c) Find the voltage V_2 across capacitor C_2.
(d) Find the charge Q_1 on capacitor C_1.



Solution:

(a) $U_3 = \frac{1}{2}(3\mu\text{F})(6\text{V})^2 = 54\mu\text{J}$.
(b) $V_4 = 6\text{V}$.
(c) $V_2 = \frac{1}{2}6\text{V} = 3\text{V}$.
(d) $Q_1 = (2\mu\text{F})(3\text{V}) = 6\mu\text{C}$.

tsl362

The only challenge in this circuit is to read it properly. There is little calculation involved in answering the questions.

We recognize that the voltages across capacitors 3 and 4 are 6V, equal to the voltage supplied by the battery, to which they are connected in parallel.

Capacitors 1 and 2 are in series. Hence they carry the same charge, $Q_1 = Q_2$. Since they have equal capacitance, $C_1 = C_2$, they also have equal voltage across, $V_1 = V_2$.

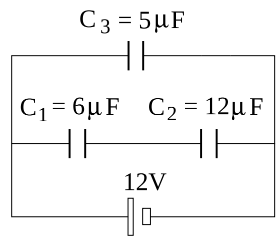
We also see that voltage across the series combination is equal to the voltage supplied by the battery: $V_1 + V_2 = 6\text{V}$. We thus conclude that $V_1 = V_2 = 3\text{V}$.

The rest is straightforward. The results are on the slide.

Unit Exam II: Problem #1 (Spring '08)

The circuit of capacitors is at equilibrium.

(a) Find the charge Q_1 on capacitor 1 and the charge Q_2 on capacitor 2.
(b) Find the voltage V_1 across capacitor 1 and the voltage V_2 across capacitor 2.
(c) Find the charge Q_3 and the energy U_3 on capacitor 3.



Solution:

(a) $C_{12} = \left(\frac{1}{6\mu\text{F}} + \frac{1}{12\mu\text{F}}\right)^{-1} = 4\mu\text{F}$,
$Q_1 = Q_2 = Q_{12} = (4\mu\text{F})(12\text{V}) = 48\mu\text{C}$.

(b) $V_1 = \frac{Q_1}{C_1} = \frac{48\mu\text{C}}{6\mu\text{F}} = 8\text{V}$,

$V_2 = \frac{Q_2}{C_2} = \frac{48\mu\text{C}}{12\mu\text{F}} = 4\text{V}$.

(c) $Q_3 = (5\mu\text{F})(12\text{V}) = 60\mu\text{C}$,

$U_3 = \frac{1}{2}(5\mu\text{F})(12\text{V})^2 = 360\mu\text{J}$.

tsl377

The circuit on this page looks simpler, at first glance, than the one on the previous page. However, there is one complication that requires some thought.

The two capacitors in series do not have the same capacitance. Each still carries the same charge, which is the charge on the equivalent of the series combination with capacitance C_{12}.

Once we have the charges $Q_1 = Q_2$, it is straightforward to calculate the voltages V_1 and V_2. The two voltages must add up to the 12V supplied by the battery. They do.

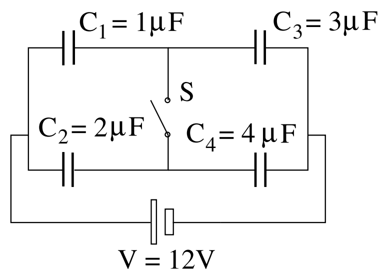
All other questions are elementary. The answers are on the slide.

There are plenty of additional exercises involving simple capacitor circuits among the previous unit-2 exam slides.

Capacitor Circuit (8)

In the circuit shown find the charges Q_1, Q_2, Q_3, Q_4 on each capacitor and the voltages V_1, V_2, V_3, V_4 across each capacitor

(a) when the switch S is open,
(b) when the switch S is closed.



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This application is more complex. It is a showcase for the full analysis of any capacitor circuit that is reducible sequentially as we have practiced earlier.

With the switch S open we find the equivalent capacitance in two steps: we first reduce two series connections, then one parallel connection:

$$C_{13} = \left(\frac{1}{C_1} + \frac{1}{C_3}\right)^{-1} = \frac{3}{4}\,\mu\text{F}, \quad C_{24} = \left(\frac{1}{C_2} + \frac{1}{C_4}\right)^{-1} = \frac{4}{3}\,\mu\text{F},$$

$$C_{eq} = C_{13} + C_{24} = \frac{25}{12}\,\mu\text{F}.$$

Next we calculate voltage and charge for capacitors C_{13} and C_{24}:

$$V_{13} = V_{24} = 12\text{V}, \quad Q_{13} = V_{13}C_{13} = 9\mu\text{C}, \quad Q_{24} = V_{24}C_{24} = 16\mu\text{C}.$$

Recalling that C_{13} and C_{24} are both series combinations, we infer the individual charges and then the individual voltages:

$$Q_1 = Q_3 = Q_{13} = 9\mu\text{C}, \quad Q_2 = Q_4 = Q_{24} = 16\mu\text{C},$$

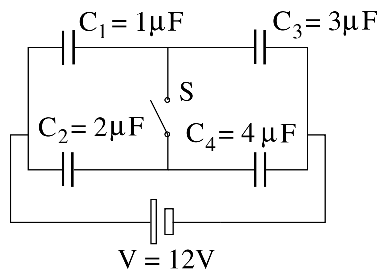
$$V_1 = \frac{Q_1}{C_1} = 9\text{V}, \quad V_3 = \frac{Q_3}{C_3} = 3\text{V}, \quad V_2 = \frac{Q_2}{C_2} = 8\text{V}, \quad V_4 = \frac{Q_4}{C_4} = 4\text{V}.$$

A consistency check requires that $V_1 + V_3 = V_2 + V_4 =$12V, which comes out as it should.

Capacitor Circuit (8)

In the circuit shown find the charges Q_1, Q_2, Q_3, Q_4 on each capacitor and the voltages V_1, V_2, V_3, V_4 across each capacitor

(a) when the switch S is open,
(b) when the switch S is closed.



tsl122

With the switch S closed we again find the equivalent capacitance in two steps: we first reduce two parallel connections, then one series connection:

$$C_{12} = C_1 + C_2 = 3\mu\text{F}, \quad C_{34} = C_3 + C_4 = 7\mu\text{F},$$

$$C_{eq} = \left(\frac{1}{C_{12}} + \frac{1}{C_{34}}\right)^{-1} = \frac{21}{10}\,\mu\text{F}.$$

Capacitors in series acquire the same charge when charged up:

$$Q_{12} = Q_{34} = C_{eq}(12\text{V}) = \frac{126}{5}\,\mu\text{C}.$$

These results determine the voltages across each parallel combinaton:

$$V_{12} = \frac{Q_{12}}{C_{12}} = \frac{42}{5}\,\text{V}, \quad V_{34} = \frac{Q_{34}}{C_{34}} = \frac{18}{5}\,\text{V}.$$

A consistency check requires that $V_{12} + V_{34}$ =12V, which pans out. The voltage across each parallel combination is the voltage across each capacitor of the combination:

$$V_1 = V_2 = V_{12} = \frac{42}{5}\,\text{V}, \quad V_3 = V_4 = V_{34} = \frac{18}{5}\,\text{V}.$$

The charges on all individual capacitors then follow directly:

$$Q_1 = V_1C_1 = \frac{42}{5}\,\mu\text{C}, \quad Q_2 = V_2C_2 = \frac{84}{5}\,\mu\text{C},$$

$$Q_3 = V_3C_3 = \frac{54}{5}\,\mu\text{C}, \quad Q_4 = V_4C_4 = \frac{72}{5}\,\mu\text{C}.$$

More Complex Capacitor Circuit

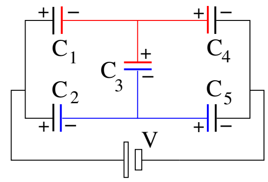
No two capacitors are in parallel or in series.
Solution requires different strategy:

- zero charge on each conductor (here color coded),
- zero voltage around any closed loop.

Specifications: $C_1, \ldots, C_5, V$.
Five equations for unknowns $Q_1, \ldots, Q_5$:

- $Q_1 + Q_2 - Q_4 - Q_5 = 0$
- $Q_3 + Q_4 - Q_1 = 0$
- $\frac{Q_5}{C_5} + \frac{Q_3}{C_3} - \frac{Q_4}{C_4} = 0$
- $\frac{Q_2}{C_2} - \frac{Q_1}{C_1} - \frac{Q_3}{C_3} = 0$
- $V - \frac{Q_1}{C_1} - \frac{Q_4}{C_4} = 0$

Equivalent capacitance: $C_{eq} = \frac{Q_1 + Q_2}{V}$



(a) $C_m = 1\text{pF}, m = 1, \ldots, 5$ and $V = 1\text{V}$:

$$C_{eq} = 1\text{pF},\ Q_3 = 0,$$

$$Q_1 = Q_2 = Q_4 = Q_5 = \frac{1}{2}\text{pC}.$$

(b) $C_m = m\,\text{pF}, m = 1, \ldots, 5$ and $V = 1\text{V}$:

$$C_{eq} = \frac{159}{71}\text{pF},\ Q_1 = \frac{55}{71}\text{pC},\ Q_2 = \frac{104}{71}\text{pC},$$

$$Q_3 = -\frac{9}{71}\text{pC},\ Q_4 = \frac{64}{71}\text{pC},\ Q_5 = \frac{95}{71}\text{pC}.$$

tsl511

We conclude this lecture with a capacitor circuit that cannot be analyzed by sequential reduction of units that are either series or parallel combinations. The method we use here looks ahead to techniques we shall employ for the analysis of resistor circuits later.

We wish to know the charge on each capacitor for a given set of capacitances and a given voltage supplied by the power source.

The circuit consists of three conductors (color coded black, red, and blue). Each conductor carries zero net charge. When the circuit is connected to the battery, charge moves within each conductor, but no charge is added to or removed from any of them.

The first two of the five equations for the five unknowns, state that there is no net charge on the black and red conductors. A third equation, stating that the net charge on the blue conductor is zero as well, is redundant.

The remaining three equations state that the sum of potential differences across devices around a loop must vanish. The voltage across a capacitor is Q/C. Across the battery it is V.

The third equation uses the square loop on the right. Starting in the bottom right corner and going clockwise we go up $+Q_5/C_5$, then up again $+Q_3/C_3$, then down $-Q_4/C_4$ to arrive at the point of origin.

The fourth equation uses the square loop on the left. The fifth equation uses the loop that goes around the circuit the long way.

More Complex Capacitor Circuit

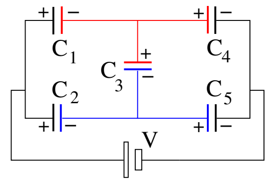
No two capacitors are in parallel or in series.
Solution requires different strategy:

- zero charge on each conductor (here color coded),
- zero voltage around any closed loop.

Specifications: $C_1, \ldots, C_5, V$.
Five equations for unknowns $Q_1, \ldots, Q_5$:

- $Q_1 + Q_2 - Q_4 - Q_5 = 0$
- $Q_3 + Q_4 - Q_1 = 0$
- $\frac{Q_5}{C_5} + \frac{Q_3}{C_3} - \frac{Q_4}{C_4} = 0$
- $\frac{Q_2}{C_2} - \frac{Q_1}{C_1} - \frac{Q_3}{C_3} = 0$
- $V - \frac{Q_1}{C_1} - \frac{Q_4}{C_4} = 0$

Equivalent capacitance: $C_{eq} = \frac{Q_1 + Q_2}{V}$



(a) $C_m = 1\text{pF}, m = 1, \ldots, 5$ and $V = 1\text{V}$:

$$C_{eq} = 1\text{pF},\ Q_3 = 0,$$

$$Q_1 = Q_2 = Q_4 = Q_5 = \frac{1}{2}\text{pC}.$$

(b) $C_m = m\,\text{pF}, m = 1, \ldots, 5$ and $V = 1\text{V}$:

$$C_{eq} = \frac{159}{71}\text{pF},\ Q_1 = \frac{55}{71}\text{pC},\ Q_2 = \frac{104}{71}\text{pC},$$

$$Q_3 = -\frac{9}{71}\text{pC},\ Q_4 = \frac{64}{71}\text{pC},\ Q_5 = \frac{95}{71}\text{pC}.$$

tsl511

The five linear equations are readily solved for arbitrary specifications. The general solutions are too lengthy to be reproduced here. The slide gives the solutions for two sets of specifications.

How do we find the equivalent capacitance in this case? We use the definition of capacitance, $C = Q/V$ and consider the circuit to be a single capacitor in a black box with two wires sticking out left and right.

The voltage applied is that supplied by the power source, namely V. The charge that goes into the box through the wire on the left is the sum of the charges that go onto capacitors 1 and 2. The same charge but with opposite sign flows through the wire on the right onto the capacitors 3 and 4. Hence we have

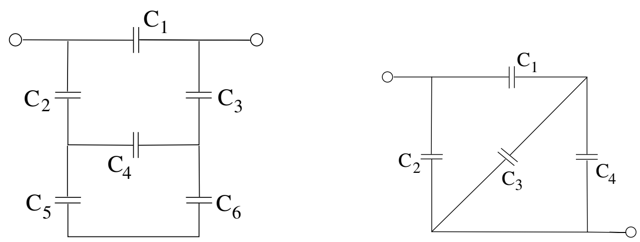
$$C_{eq} = \frac{Q_1 + Q_2}{V} = \frac{Q_3 + Q_4}{V}.$$

The slide states the numerical value of C_{eq} for the two sets of circuit specifications.

Capacitor Circuit (6)

(a) Name two capacitors from the circuit on the **left** that are connected in **series**.

(b) Name two capacitors from the circuit on the **right** that are connected in **parallel**.



tsl119

This is the quiz for lecture 13.

There is only one pair of capacitors that qualify as answer to either question.