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13. Ideal Quantum Gases I: Bosons

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Abstract

Part thirteen of course materials for Statistical Physics I: PHY525, taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

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Bose–Einstein functions [ts136]

$$g_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx \, x^{n-1}}{z^{-1}e^x - 1} = \sum_{l=1}^{\infty} \frac{z^l}{l^n}, \quad 0 \leq z \leq 1.$$

Special cases:

$$g_0(z) = \frac{z}{1-z}, \quad g_1(z) = -\ln(1-z), \quad g_\infty(z) = z.$$

Riemann zeta function:

$$g_n(1) = \zeta(n) \doteq \sum_{l=1}^{\infty} \frac{1}{l^n}.$$

Special values:

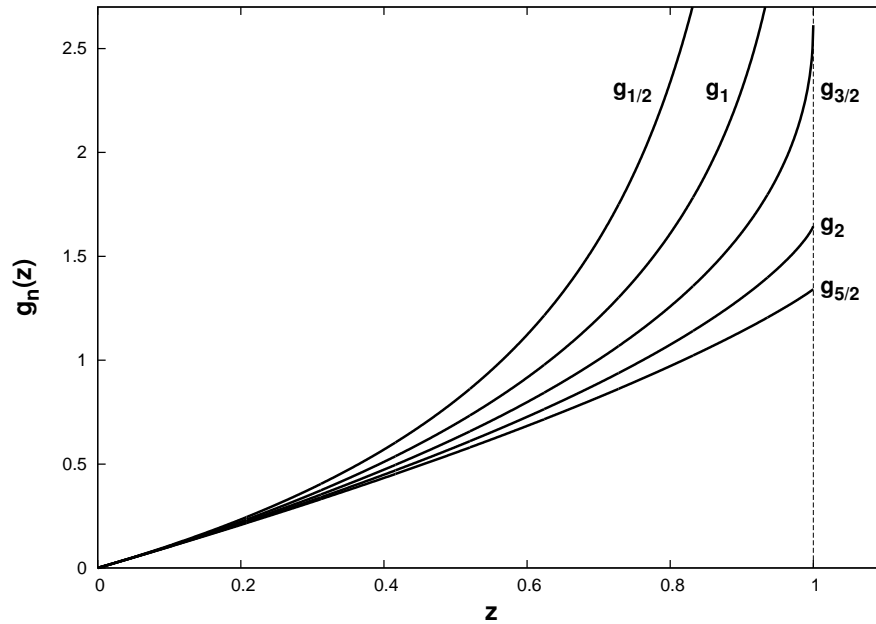
$$\zeta(1) \rightarrow \infty, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Recurrence relation:

$$z g'_n(z) = g_{n-1}(z), \quad n \geq 1.$$

Singularity at $z = 1$ for non-integer n :

$$g_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \zeta(n-\ell) \alpha^\ell, \quad \alpha \doteq -\ln z.$$



Ideal Bose-Einstein gas: equation of state and internal energy [tln67]

Conversion of sums into integrals by means of density of energy levels [tex113]:

$$D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)} \left(\frac{m}{2\pi\hbar^2} \right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}, \quad V = L^{\mathcal{D}}.$$

Fundamental thermodynamic relations for BE gas:

$$\frac{pV}{k_B T} = - \sum_k \ln(1 - ze^{-\beta\epsilon_k}) = - \int_0^\infty d\epsilon D(\epsilon) \ln(1 - ze^{-\beta\epsilon}) = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z),$$

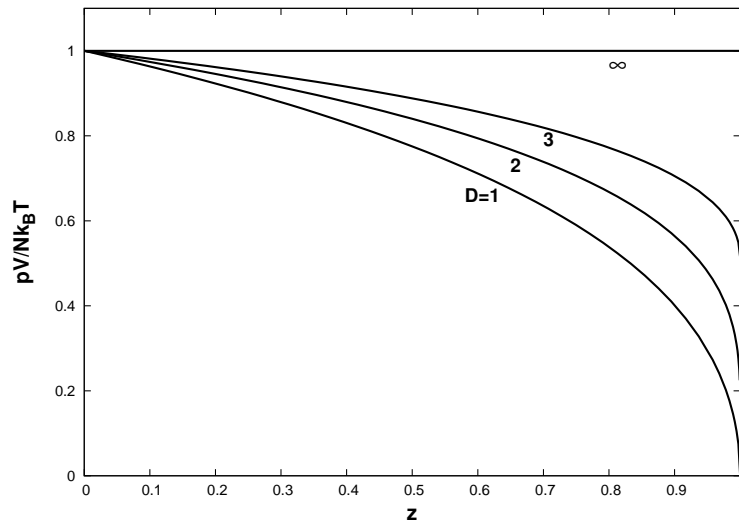
$$\mathcal{N} = \sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)}{z^{-1}e^{\beta\epsilon} - 1} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2}(z), \quad z < 1,$$

$$U = \sum_k \frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)\epsilon}{z^{-1}e^{\beta\epsilon} - 1} = \frac{\mathcal{D}}{2} k_B T \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z).$$

Warning: The range of fugacity is limited to the interval $0 \leq z \leq 1$. At $z = 1$, the expression for \mathcal{N} must be amended by an additive term $z/(1-z)$ to account for the possibility of a macroscopic population of the lowest energy level (at $\epsilon = 0$). This amendment is only necessary for dimensionalities $\mathcal{D} > 2$, i.e. for the cases with $\lim_{\epsilon \rightarrow 0} D(\epsilon) = 0$.

Equation of state (with fugacity z in the role of parameter):

$$\frac{pV}{\mathcal{N}k_B T} = \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}, \quad z < 1.$$



[tex113] BE gas in \mathcal{D} dimensions I: fundamental relations

From the expressions for the grand potential and the density of energy levels of an ideal Bose-Einstein gas in \mathcal{D} dimensions and confined to a box of volume $V = L^{\mathcal{D}}$ with rigid walls,

$$\Omega(T, V, \mu) = k_B T \sum_k \ln(1 - z e^{-\beta \epsilon_k}), \quad D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)} \left(\frac{m}{2\pi \hbar^2} \right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1},$$

derive the fundamental thermodynamic relations at fugacity $z < 1$ in terms of the Bose-Einstein functions $g_n(z)$ and the thermal wavelength $\lambda_T = \sqrt{h^2/2\pi m k_B T}$ as follows:

$$\frac{pV}{k_B T} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z), \quad \mathcal{N} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2}(z), \quad U = \frac{\mathcal{D}}{2} k_B T \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2+1}(z).$$

Solution:

Reference Values for T , V/\mathcal{N} , and p [tln71]

The reference values introduced here are based on

- (i) thermal wavelength: $\lambda_T \doteq \sqrt{\frac{h^2}{2\pi m k_B T}} = \sqrt{\frac{\Lambda}{k_B T}}, \quad \Lambda = \frac{h^2}{2\pi m}.$
- (ii) MB equation of state: $pv = k_B T, \quad v \doteq V/\mathcal{N}.$

The reference values for $k_B T$, v , and p in isochoric, isothermal, and isobaric processes are

$$\begin{aligned}
 k_B T_v &= \frac{\Lambda}{v^{2/\mathcal{D}}} & p_v &= \frac{\Lambda}{v^{2/\mathcal{D}+1}} & (v = \text{const.}) \\
 v_T &= \left(\frac{\Lambda}{k_B T} \right)^{\mathcal{D}/2} & p_T &= \Lambda \left(\frac{k_B T}{\Lambda} \right)^{\mathcal{D}/2+1} & (T = \text{const.}) \\
 k_B T_p &= \Lambda \left(\frac{p}{\Lambda} \right)^{2/(\mathcal{D}+2)} & v_p &= \left(\frac{\Lambda}{p} \right)^{\mathcal{D}/(\mathcal{D}+2)} & (p = \text{const.})
 \end{aligned}$$

These reference values are useful for bosons and fermions.

Universal curves for isochores, isotherms, and isobars:

- p/p_v versus T/T_v at $v = \text{const.}$
- p/p_T versus v/v_T at $T = \text{const.}$
- v/v_p versus T/T_p at $p = \text{const.}$

For fermions we will introduce alternative reference values based on the chemical potential (Fermi energy).

Bose-Einstein condensation [ts138]

Particles in the gas phase and in the Bose-Einstein condensate (BEC):

$$\mathcal{N} = \frac{V}{\lambda_T^{\mathcal{D}}} g_{\mathcal{D}/2}(z) + \frac{z}{1-z} = \mathcal{N}_{gas} + \mathcal{N}_{BEC}.$$

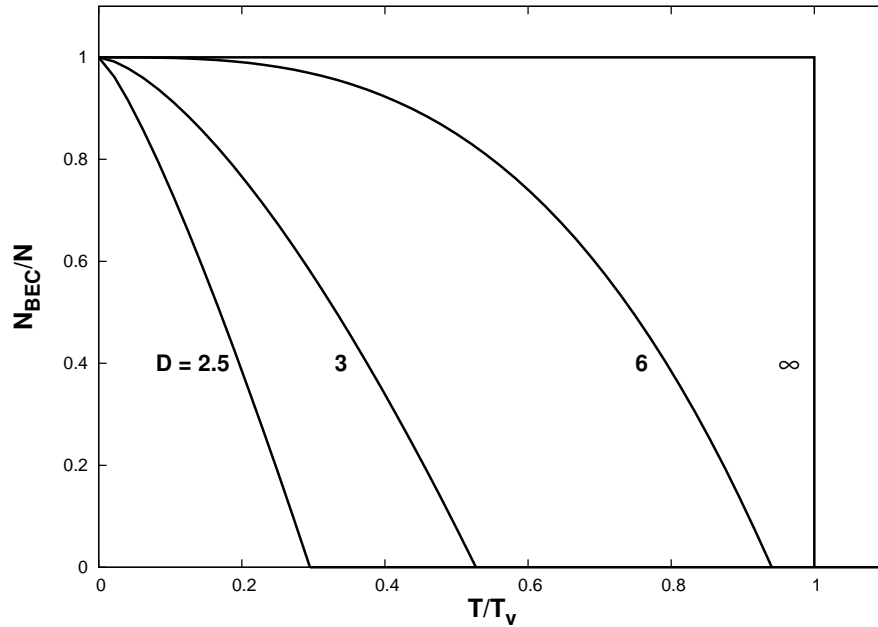
Consider process at $v = \text{const.}$

Onset of macroscopic population of the lowest energy level begins when the fugacity locks in to the value $z = 1$:

$$\frac{z}{1-z} = \begin{cases} \text{O}(1), & z < 1, \\ \text{O}(\mathcal{N}), & z = 1. \end{cases}$$

$$T \geq T_c : \quad \frac{\mathcal{N}_{gas}}{\mathcal{N}} = 1, \quad \frac{\mathcal{N}_{BEC}}{\mathcal{N}} = 0.$$

$$T \leq T_c : \quad \begin{cases} \frac{\mathcal{N}_{gas}}{\mathcal{N}} = \frac{[V/\lambda_T^{\mathcal{D}}]\zeta(\mathcal{D}/2)}{[V/\lambda_{T_c}^{\mathcal{D}}]\zeta(\mathcal{D}/2)} = \left(\frac{T}{T_c}\right)^{\mathcal{D}/2}, \\ \frac{\mathcal{N}_{BEC}}{\mathcal{N}} = 1 - \frac{\mathcal{N}_{gas}}{\mathcal{N}} = 1 - \left(\frac{T}{T_c}\right)^{\mathcal{D}/2}. \end{cases}$$



Ideal Bose-Einstein gas: isochores [tsl39]

Isochore at $T \geq T_c$ [tex114]:

$$\frac{p}{p_v} = \frac{g_{\mathcal{D}/2+1}(z)}{[g_{\mathcal{D}/2}(z)]^{2/\mathcal{D}+1}}, \quad \frac{T}{T_v} = [g_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}}.$$

Isochore at $T \leq T_c$ (also valid asymptotically for $T \ll T_v$ in $\mathcal{D} \leq 2$):

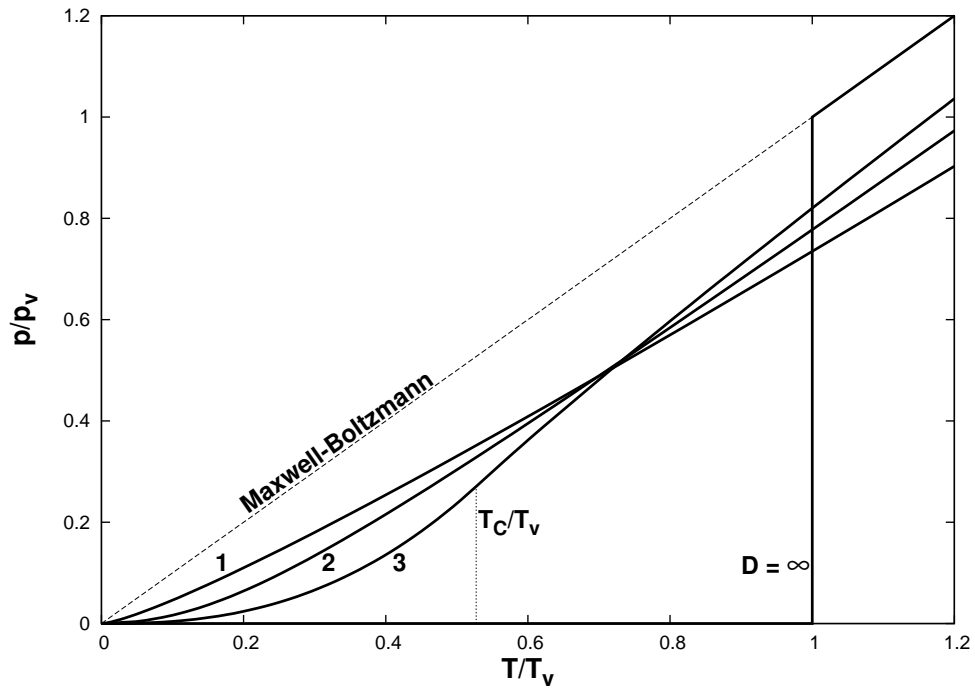
$$\frac{p}{p_v} = \left(\frac{T}{T_v}\right)^{\mathcal{D}/2+1} \zeta(\mathcal{D}/2 + 1).$$

Critical temperature:

$$\frac{T_c}{T_v} = [\zeta(\mathcal{D}/2)]^{-2/\mathcal{D}} = \begin{cases} 0 & \mathcal{D} = 1 \\ 0 & \mathcal{D} = 2 \\ 0.527 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$

High-temperature asymptotic behavior:

$$\frac{p}{p_v} \sim \frac{T}{T_v} \left[1 - \frac{1}{2^{\mathcal{D}/2+1}} \left(\frac{T_v}{T}\right)^{\mathcal{D}/2} \right].$$



[tex114] BE gas in \mathcal{D} dimensions II: isochore

(a) From the fundamental thermodynamic relations for the Bose-Einstein gas in \mathcal{D} dimensions (see [tln67]), derive the following parametric expression for the isochore at $T \geq T_c$:

$$\frac{p}{p_v} = \frac{g_{\mathcal{D}/2+1}(z)}{[g_{\mathcal{D}/2}(z)]^{2/\mathcal{D}+1}}, \quad \frac{T}{T_v} = [g_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}},$$

where $k_B T_v = \Lambda v^{-2/\mathcal{D}}$ and $p_v = \Lambda v^{-2/\mathcal{D}+1}$ with $\Lambda \doteq h^2/2\pi m$ are convenient reference values.

(b) Calculate the leading correction to the Maxwell-Boltzmann result at high temperature. (c) Calculate the exact dependence of p/p_v on T/T_v at $T \leq T_c$ in $\mathcal{D} > 2$. Show that this result also holds asymptotically for $T \ll T_v$ in dimensions $\mathcal{D} = 1$ and $\mathcal{D} = 2$.

Solution:

[tex115] BE gas in \mathcal{D} dimensions III: isotherm and isobar

(a) From the fundamental thermodynamic relations for the Bose-Einstein gas in $\mathcal{D} > 2$ dimensions (see [tln67]), derive the following expressions for the isotherm at $v > v_c$ and the isobar at $T \leq T_c$:

$$\frac{p}{p_T} = g_{\mathcal{D}/2+1}(z), \quad \frac{v}{v_T} = [g_{\mathcal{D}/2}(z)]^{-1};$$
$$\frac{v}{v_p} = \frac{[g_{\mathcal{D}/2+1}(z)]^{\mathcal{D}/(\mathcal{D}+2)}}{g_{\mathcal{D}/2}(z)}, \quad \frac{T}{T_p} = [g_{\mathcal{D}/2+1}(z)]^{-2/(\mathcal{D}+2)}.$$

where $v_T = (\Lambda/k_B T)^{\mathcal{D}/2}$, $p_T = \Lambda(k_B T/\Lambda)^{\mathcal{D}/2+1}$, $k_B T_p = \Lambda(p/\Lambda)^{2/(\mathcal{D}+2)}$, $v_p = (\Lambda/p)^{\mathcal{D}/(\mathcal{D}+2)}$ with $\Lambda \doteq h^2/2\pi m$ are convenient reference values for temperature and pressure and reduced volume. (b) Calculate the leading correction to the Maxwell-Boltzmann result for the isotherm at low density and for the isobar at high temperature.

Solution:

Ideal Bose-Einstein gas: isotherms [tsl40]

For $\mathcal{D} > 2$ we must again distinguish two regimes. At $v > v_c$, all bosons are in the gas phase. At $v < v_c$, a BEC is present. Only the bosons in the gas phase contribute to the pressure.

Isotherm at $v \geq v_c = \lambda_T^{\mathcal{D}}/\zeta(\mathcal{D}/2)$:

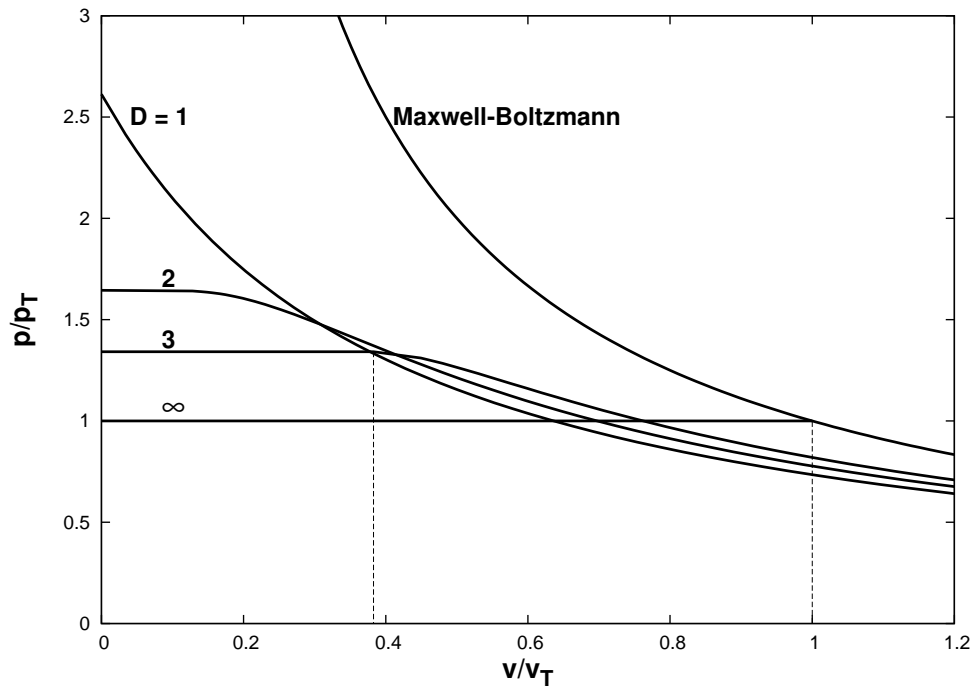
$$\frac{p}{p_T} = g_{\mathcal{D}/2+1}(z), \quad \frac{v}{v_T} = [g_{\mathcal{D}/2}(z)]^{-1}.$$

Isotherm at $v \leq v_c$:

$$\frac{p}{p_T} = \frac{p_c}{p_T} = \zeta(\mathcal{D}/2 + 1) = \begin{cases} 2.612 & \mathcal{D} = 1 \\ 1.645 & \mathcal{D} = 2 \\ 1.341 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$

Critical (reduced) volume:

$$\frac{v_c}{v_T} = [\zeta(\mathcal{D}/2)]^{-1} = \begin{cases} 0 & \mathcal{D} = 1 \\ 0 & \mathcal{D} = 2 \\ 0.383 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$



Ideal Bose-Einstein gas: isobars [tsl48]

A phase transition at $T_c > 0$ takes place in all dimensions $\mathcal{D} \geq 1$. However, the existence of a BEC requires $v_c > 0$, which is realized only for $\mathcal{D} > 2$.

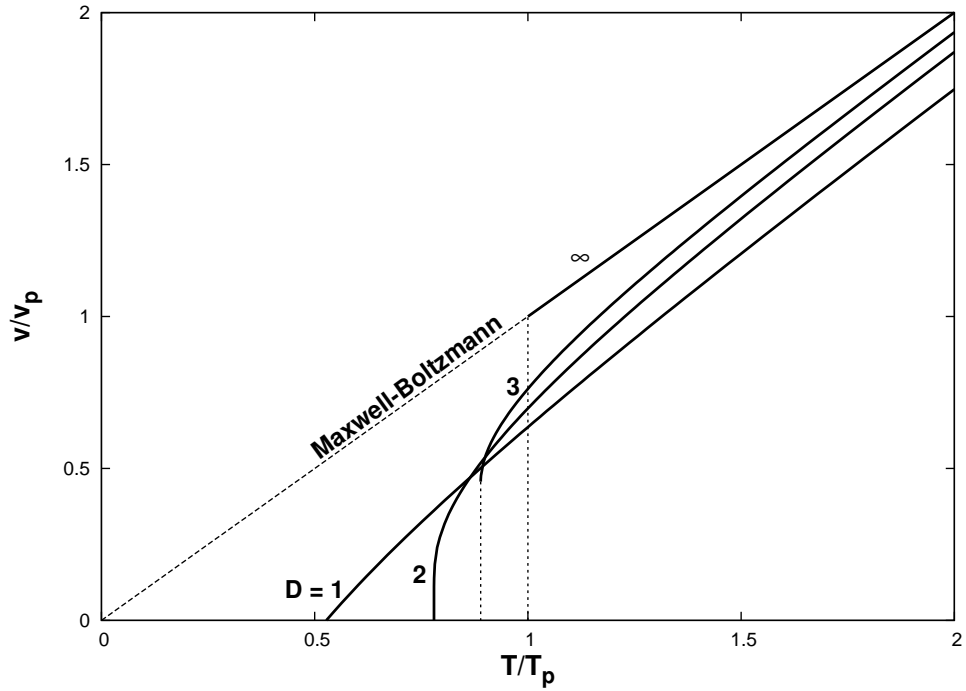
Isobar at $T > T_c$:

$$\frac{v}{v_p} = \frac{[g_{\mathcal{D}/2+1}(z)]^{\mathcal{D}/(\mathcal{D}+2)}}{g_{\mathcal{D}/2}(z)}, \quad \frac{T}{T_p} = [g_{\mathcal{D}/2+1}(z)]^{-2/(\mathcal{D}+2)}.$$

Critical point:

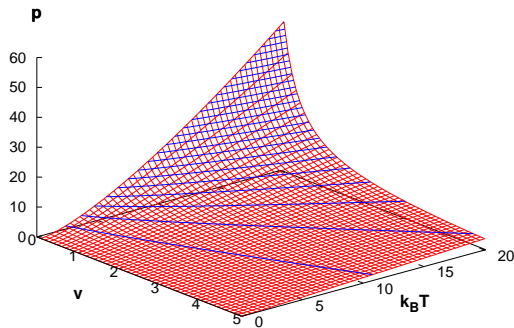
$$\frac{v_c}{v_p} = \frac{[\zeta(\mathcal{D}/2 + 1)]^{\mathcal{D}/(\mathcal{D}+2)}}{\zeta(\mathcal{D}/2)} = \begin{cases} 0 & \mathcal{D} = 1 \\ 0 & \mathcal{D} = 2 \\ 0.383 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$

$$\frac{T_c}{T_p} = [\zeta(\mathcal{D}/2 + 1)]^{-2/(\mathcal{D}+2)} = \begin{cases} 0.527 & \mathcal{D} = 1 \\ 0.779 & \mathcal{D} = 2 \\ 0.884 & \mathcal{D} = 3 \\ 1 & \mathcal{D} = \infty \end{cases}$$

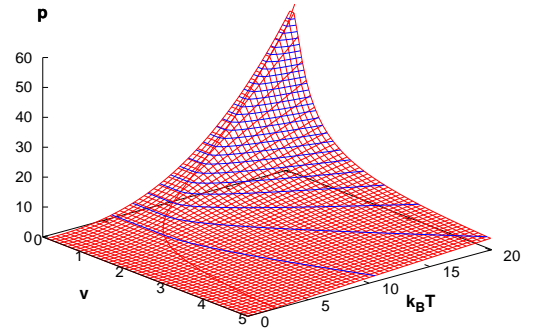


Ideal Bose-Einstein gas: phase diagram [tln72]

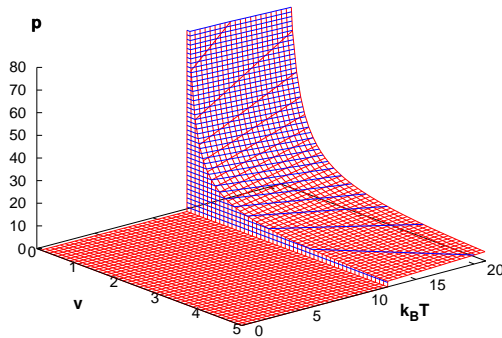
$$\mathcal{D} = 1$$



$$\mathcal{D} = 3$$



$$\mathcal{D} = \infty$$



$$pv = \begin{cases} k_B T, & T > T_c \\ 0, & T < T_c \end{cases}$$

$$k_B T_c = \Lambda \doteq \frac{h^2}{2\pi m}.$$

- $\mathcal{D} = 1$: Transition at $T \geq 0$ and $v = 0$ (transition line = isochore).
- $\mathcal{D} = 3$: Transition at $T > 0$ and $v > 0$.
- $\mathcal{D} = \infty$: Transition at $T > 0$ and $v > 0$ (transition line = isotherm).

Ideal Bose-Einstein gas: heat capacity [tsl41]

Internal energy:

$$\frac{U}{\mathcal{N}k_B T_v} = \begin{cases} \frac{\mathcal{D}}{2} \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)} \frac{T}{T_v}, & T \geq T_c, \\ \frac{\mathcal{D}}{2} \zeta(\mathcal{D}/2 + 1) \left(\frac{T}{T_v} \right)^{\mathcal{D}/2+1}, & T \leq T_c. \end{cases}$$

Heat capacity at $T \geq T_c$ [use $z g'_n(z) = g_{n-1}(z)$ for $n \geq 1$]:

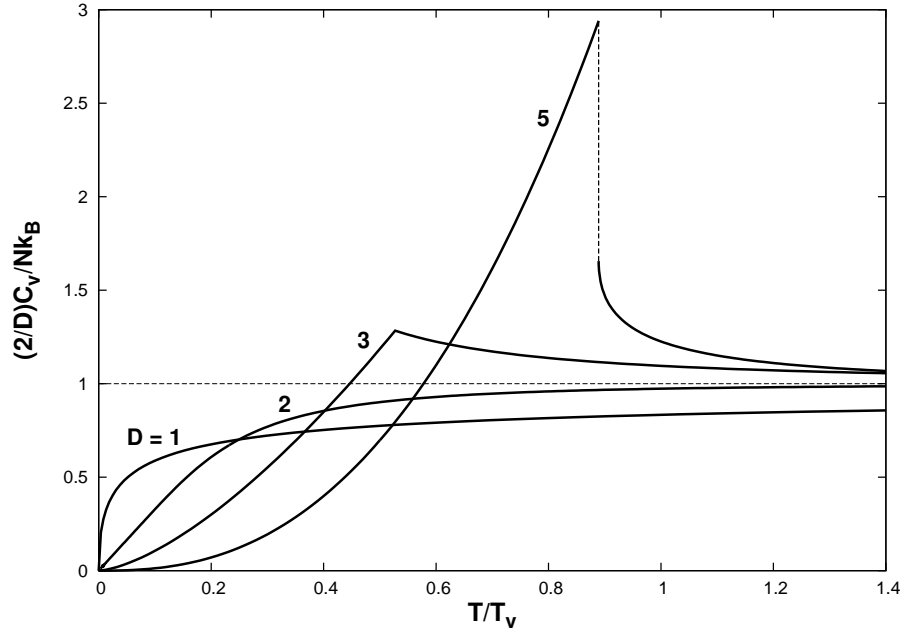
$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4} \frac{g'_{\mathcal{D}/2+1}(z)}{g'_{\mathcal{D}/2}(z)}.$$

Heat capacity at $T \leq T_c$:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \zeta \left(\frac{\mathcal{D}}{2} + 1 \right) \left(\frac{T}{T_v} \right)^{\mathcal{D}/2} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{\zeta \left(\frac{\mathcal{D}}{2} + 1 \right)}{\zeta \left(\frac{\mathcal{D}}{2} \right)} \left(\frac{T}{T_c} \right)^{\mathcal{D}/2}.$$

High-temperature asymptotic behavior:

$$\frac{C_V}{\mathcal{N}k_B} \sim \frac{\mathcal{D}}{2} \left[1 + \frac{\mathcal{D}/2 - 1}{2^{\mathcal{D}/2+1}} \left(\frac{T_v}{T} \right)^{\mathcal{D}/2} \right].$$



[tex97] BE gas in \mathcal{D} dimensions IV: heat capacity at high temperature

The internal energy of the ideal Bose-Einstein gas in \mathcal{D} dimensions and at $T \geq T_c$ is given by the following expression:

$$U = \mathcal{N}k_B T \frac{\mathcal{D}}{2} \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}.$$

Use this result to derive the following expression for the heat capacity $C_V = (\partial U / \partial T)_{V\mathcal{N}}$:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)} - \frac{\mathcal{D}^2}{4} \frac{g'_{\mathcal{D}/2+1}(z)}{g'_{\mathcal{D}/2}(z)}.$$

Use the derivative $\partial/\partial T$ of the result $g_{\mathcal{D}/2}(z) = \mathcal{N}\lambda_T^{\mathcal{D}}/V$ with $V = L^{\mathcal{D}}$ to calculate any occurrence of $(\partial z/\partial T)_{V\mathcal{N}}$ in the derivation. Use the recursion relation $z g'_n(z) = g_{n-1}(z)$ for $n \geq 1$ to further simplify the results pertaining to $\mathcal{D} \geq 2$.

Solution:

[tex116] BE gas in \mathcal{D} dimensions V: heat capacity at low temperature

The internal energy of the ideal Bose-Einstein gas in $\mathcal{D} > 2$ dimensions and at $T \leq T_c$ is given by the following expression:

$$\frac{U}{\mathcal{N}k_B T_v} = \frac{\mathcal{D}}{2} \zeta(\mathcal{D}/2 + 1) \left(\frac{T}{T_v} \right)^{\mathcal{D}/2+1}$$

(a) Use this result to derive the following expression for the heat capacity $C_V = (\partial U / \partial T)_{V\mathcal{N}}$:

$$\frac{C_V}{\mathcal{N}k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{\zeta(\frac{\mathcal{D}}{2} + 1)}{\zeta(\frac{\mathcal{D}}{2})} \left(\frac{T}{T_c} \right)^{\mathcal{D}/2},$$

where $T_c = T_v [\zeta(\mathcal{D}/2)]^{-2/\mathcal{D}}$ is the critical temperature and $k_B T_v = \Lambda / v^{2/\mathcal{D}}$ with $v \doteq V/\mathcal{N}$ and $\Lambda \doteq h^2/2\pi m$ a convenient reference temperature. (b) Show that the heat capacity is continuous at $T = T_c$ if $\mathcal{D} \leq 4$ and discontinuous if $\mathcal{D} > 4$. Find the discontinuity $\Delta C_V / \mathcal{N}k_B$ as a function of \mathcal{D} for $\mathcal{D} > 4$. (c) Infer from the result of [tex97] the leading singularity of $C_V / \mathcal{N}k_B$ at $T/T_v \ll 1$ for $\mathcal{D} = 1$ and $\mathcal{D} = 2$. Then show that these singularities are consistent with the expression for $C_V / \mathcal{N}k_B$ obtained here in part (a) provided we substitute $(T_v/T_c)^{\mathcal{D}/2} = \zeta(\mathcal{D}/2)$.

Solution:

[tex128] BE gas in \mathcal{D} dimensions VI: isothermal compressibility

(a) Show that the isothermal compressibility, $\kappa_T = -(1/V)(\partial V/\partial p)_{T\mathcal{N}}$, of the ideal BE gas in \mathcal{D} dimensions at $T > T_c$ is

$$p_T \kappa_T = \frac{g'_{\mathcal{D}/2}(z)}{g_{\mathcal{D}/2}(z)g'_{\mathcal{D}/2+1}(z)}, \quad \frac{v}{v_T} = \frac{1}{g_{\mathcal{D}/2}(z)},$$

where $v \doteq V/\mathcal{N}$, $v_T \doteq (\Lambda/k_B T)^{\mathcal{D}/2}$, $p_T \doteq k_B T/v_T$, $\Lambda \doteq h^2/2\pi m$, and $g_n(z)$ are BE functions. Use $z g'_n(z) = g_{n-1}(z)$ for $n \geq 1$ to simplify the results in $\mathcal{D} \geq 2$. (b) Sketch $p_T \kappa_T$ versus v/v_T for $v \geq 0$ in $\mathcal{D} = 1$ and for $v \geq v_c$ in $\mathcal{D} = 3$, where $v_c/v_T = [\zeta(\mathcal{D}/2)]^{-1}$ marks the onset of BEC. (c) Determine the nature of the singularity of κ_T as $v/v_T \rightarrow 0$ in $\mathcal{D} = 1, 2$. Determine the critical compressibility $p_T \kappa_T$ at $v = v_c$ in $\mathcal{D} = 3, 5$.

Solution:

[tex129] BE gas in \mathcal{D} dimensions VII: isobaric expansivity

To derive the parametric expression of the isobaric expansivity of the ideal BE gas at $T > T_c$,

$$T_p \alpha_p = \frac{T_p}{T} \left[\left(\frac{\mathcal{D}}{2} + 1 \right) \frac{g_{\mathcal{D}/2+1}(z) g'_{\mathcal{D}/2}(z)}{g_{\mathcal{D}/2}(z) g'_{\mathcal{D}/2+1}(z)} - \frac{\mathcal{D}}{2} \right], \quad \frac{T_p}{T} = [g_{\mathcal{D}/2+1}(z)]^{\mathcal{D}/2+1},$$

where $k_B T_p = \Lambda(p/\Lambda)^{2/(\mathcal{D}+2)}$, $\Lambda \doteq h^2/2\pi m$, and $g_n(z)$ are BE functions, establish first the general thermodynamic relation $\alpha_p = \kappa_T (\partial p / \partial T)_v$ with $v \doteq V/\mathcal{N}$, the BE-specific relation $C_V = \mathcal{N}(\mathcal{D}/2)v(\partial p / \partial T)_v$, and the results for C_V and κ_T calculated in [tex97] and [tex128].

Solution:

[tex130] BE gas in \mathcal{D} dimensions VIII: speed of sound

(a) Start from the relation $c = (\rho\kappa_S)^{-1/2}$ for the speed of sound as established in [tex18], where $\rho = m/v$ is the mass density and κ_S the adiabatic compressibility. Use general thermodynamic relations between response functions to derive the following expression for c in terms of dimensionless quantities:

$$\frac{mc^2}{k_B T} = \frac{(v/v_T)}{(p_T \kappa_T)} \left[1 + \frac{(T/T_p)^2 (v/v_T) (T_p \alpha_p)^2}{(p_T \kappa_T) (C_V / \mathcal{N} k_B)} \right],$$

where v_T, p_T, T_p are defined in [tln71]. (b) Use the expressions derived in [tex129] for α_p , in [tex128] for κ_T , and in [tex97] for C_V to derive the result

$$\frac{mc^2}{k_B T} = \gamma \frac{g_{\mathcal{D}/2+1}(z)}{g_{\mathcal{D}/2}(z)}, \quad \gamma = 1 + \frac{2}{\mathcal{D}}.$$

(c) Relate the T -dependence of mc^2 to that of the isochore for $v = \text{const}$ and to that of the isobar for $p = \text{const}$.

Solution:

[tex98] Ultrarelativistic Bose–Einstein gas

Consider a Bose-Einstein gas with ultrarelativistic one-particle energy $\epsilon_k = c\hbar k = cp$ in the grandcanonical ensemble at temperature T and chemical potential $\mu = 0$.

- (a) Show that the one-particle density of states is $D(\epsilon) = (4\pi V/h^3 c^3)\epsilon^2$.
- (b) Calculate the pressure $p(T)$, the internal energy $U(T, V)$, and the average number of particles in excited states $\mathcal{N}_\epsilon(T, V)$.
- (c) Show that the heat capacity is $C_V/k_B = [16\pi^5/15h^3 c^3]V(k_B T)^3$.

Solution:

Blackbody radiation [tln68]

Electromagnetic radiation inside cavity in thermal equilibrium at temperature T . Grandcanonical ensemble of photons ($\epsilon = \hbar\omega = cp$, $\mathbf{p} = \hbar\mathbf{k}$, spin $s = 1$, bosonic, purely transverse).

Density of states: $D(\epsilon) = g \frac{4\pi V}{h^3 c^3} \epsilon^2$ with $g = 2$ independent polarizations.

Average occupation number: $\langle n_\epsilon \rangle_{BE} = \frac{1}{e^{\beta\epsilon} - 1}$.

Number of photons with energies between ϵ and $\epsilon + d\epsilon$:

$$dN(\epsilon) = \langle n_\epsilon \rangle_{BE} D(\epsilon) d\epsilon = \frac{8\pi V \epsilon^2}{h^3 c^3} \frac{1}{e^{\beta\epsilon} - 1} d\epsilon.$$

Spectral density inside cavity: [use $dN(\epsilon) = V dn(\omega)$ and $\epsilon = \hbar\omega$]:

$$\frac{dn(\omega)}{d\omega} = \frac{\hbar}{V} \frac{dN(\epsilon)}{d\epsilon} = \frac{\omega^2}{\pi^2 c^3} \frac{1}{e^{\beta\hbar\omega} - 1}.$$

Spectral energy density inside cavity: $du = \hbar\omega dn = \rho(\omega) d\omega$.

$$\rho(\omega) d\omega = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} d\omega = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{\beta h\nu} - 1} d\nu.$$

Rate (per unit area) at which particles with (average) speed c escape from cavity through small opening [tex62]: $dN/dt = \frac{1}{4}(N/V)c$.

Spectral density of radiation: $R(\omega) = \frac{c}{4} \frac{dn(\omega)}{d\omega} = \frac{\omega^2}{4\pi^2 c^2} \frac{1}{e^{\beta\hbar\omega} - 1}$.

Spectral energy density of radiation:

$$Q(\omega) = \hbar\omega R(\omega) = \frac{\omega^2}{4\pi^2 c^2} \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \quad (\text{Planck radiation law}).$$

High frequencies: ultrarelativistic MB particles [use $\langle n_\epsilon \rangle_{MB} = e^{-\beta\epsilon}$]:

$$Q(\omega) = \frac{\hbar\omega^3}{4\pi^2 c^2} e^{-\beta\hbar\omega} \quad (\text{Wien radiation law}).$$

Low frequencies: equipartition law applied to electromagnetic modes:

$$Q(\omega) = \frac{k_B T \omega^2}{4\pi^2 c^2} \quad (\text{Rayleigh-Jeans radiation law}).$$

[tex105] **Statistical mechanics of blackbody radiation**

Electromagnetic radiation inside a cavity is in thermal equilibrium with the walls at temperature T . This system can be described by a grandcanonical ensemble (with $\mu = 0$) of photons (massless bosonic particles) with energy $\epsilon = \hbar\omega$ and density of states $\bar{D}(\omega) = (V/\pi^2 c^3)\omega^2$.

(a) Show that the internal energy can be expressed in the form

$$U(T, V) = \sigma VT^4, \quad \sigma = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$$

as postulated in a previous thermodynamics problem [tex23].

(b) Show that the equation of state can be expressed in the form $pV = \frac{1}{3}U(T, V)$ as was also postulated in [tex23].

Solution: