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12. Capacitance of and energy stored in capacitors. Parallel and series connections

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Robert Coyne
University of Rhode Island, robcoyne@uri.edu

PHY204 Lecture 12 [rln12]

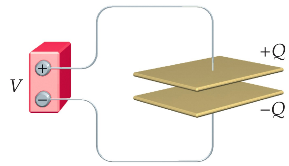
Capacitor and Capacitance

Capacitor (device):

- Two oppositely charged conductors separated by an insulator.
- The charges $+Q$ and $-Q$ on conductors generate an electric field $\vec{E}$ and a potential difference V (voltage).
- Only one conductor may be present. Then the relevant potential difference is between the conductor and a point at infinity.

Capacitance (device property):

- Definition: $C = \frac{Q}{V}$
- SI unit: 1F = 1C/V (one Farad)



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In this and the next two lectures we raise the discussion of electric potential and electric potential energy to the level of electric devices as used in electric circuits.

We focus our attention on a particular device, the *capacitor*, and restrict the discussion to electrostatics. Electric currents will be introduced later.

Two oppositely charged conductors of arbitrary shape are positioned near each other and are electrically insulated from each other. Hence they are at different electric potential. The conductor with charge $+Q$ is at potential V_+ and the conductor with charge $-Q$ at potential V_-.

When a capacitor is charged up, we say that the charge on it is Q, meaning $+Q$ on one conductor and $-Q$ on the other. We also say that the voltage across the capacitor is V, meaning the potential difference $V_+ - V_-$.

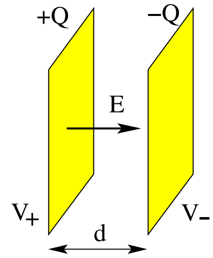
We can show, using the tools developed in the previous lectures, that the charge on a capacitor is proportional to the voltage across it. Hence the ratio $C \doteq Q/V$, named *capacitance*, is a constant.

The more charge a capacitor can hold at a given voltage, the larger its capacitance is. Note the SI unit Farad, [F]=[C/V], for capacitance.

All we need to know about a capacitor in a circuit analysis is its capacitance.

Parallel-Plate Capacitor

- A: area of each plate
- d: distance between plates
- Q: magnitude of charge on inside surface of each plate
- Charge per unit area (magnitude) on each plate: $\sigma = \frac{Q}{A}$
- Uniform electric field between plates:
  $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$
- Voltage between plates:
  $V \equiv V_+ - V_- = Ed = \frac{Qd}{\epsilon_0 A}$
- Capacitance for parallel-plate geometry:
  $C \equiv \frac{Q}{V} = \frac{\epsilon_0 A}{d}$



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For some capacitor designs, it is simple enough to determine the capacitance in terms of the geometric specifications.

The parallel-plate configuration is the prototypical design. The assumption here is that the linear dimensions of the plates are large compared to the distance between them.

The charge density on the inside surface of the plates and the electric field in the space between the plates are then close to uniform. Fringe fields and non-uniformities in the charge density around the edges are ignorable.

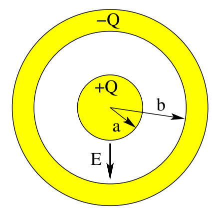
The slide then walks us through the calculation of the capacitance for a parallel-plate capacitor. We use tools developed earlier: (i) the relation between charge and charge density, (ii) the relation between electric field and charge density at the surface of a conductor, and (iii) the relation between (uniform) electric field and potential.

For this particular design, the capacitance C is proportional to the area A of each plate and inversely proportional to the distance d between them.

Cylindrical Capacitor

Conducting cylinder of radius a and length L surrounded concentrically by conducting cylindrical shell of inner radius b and equal length.

- Assumption: $L \gg b$.
- λ: charge per unit length (magnitude) on each cylinder
- $Q = \lambda L$: magnitude of charge on each cylinder
- Electric field between cylinders: use Gauss' law
  $E[2\pi r L] = \dfrac{\lambda L}{\epsilon_0} \Rightarrow E(r) = \dfrac{\lambda}{2\pi\epsilon_0 r}$
- Electric potential between cylinders: use $V(a) = 0$
  $V(r) = -\int_a^r E(r)dr = -\dfrac{\lambda}{2\pi\epsilon_0}\int_a^r \dfrac{dr}{r} = -\dfrac{\lambda}{2\pi\epsilon_0}\ln\dfrac{r}{a}$
- Voltage between cylinders:
  $V \equiv V_+ - V_- = V(a) - V(b) = \dfrac{Q}{2\pi\epsilon_0 L}\ln\dfrac{b}{a}$
- Capacitance for cylindrical geometry:
  $C \equiv \dfrac{Q}{V} = \dfrac{2\pi\epsilon_0 L}{\ln(b/a)}$



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A different design uses coaxial cylinders as shown here in cross section. The capacitance does not depend on which conductor is charged positively. However, a choice has been made for the calculation of C.

If the inner conductor has charge $+Q$ on it, then it is all on its surface, and there is a matching charge $-Q$ on the inner surface of the outer conductor. We arrive at this conclusion by reasoning developed in lecture 7.

The assumption underling the calculation worked out on the slide is that the length of the cylinders is much large than the diameter, which ensures that the surface charge density is close to uniform and the electric field radial almost everywhere.

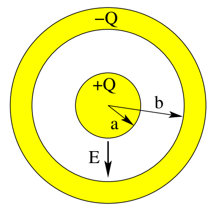
We use further expertise gained in lecture 7 to calculate the electric field $E(r)$ between the conductors and tools developed in lecture 11 to calculate the voltage V between the conductors. Substitution of these results into the definition of capacitance then yields the expression stated at the bottom of the slide.

The capacitance C of a cylindrical capacitor is proportional the length L of the cylinders. It depends logarithmically on the radii a and b of the surfaces where charge accumulates. Just as in the parallel-plate geometry, the capacitance goes up when the gap between the conductors is made narrower.

Spherical Capacitor

Conducting sphere of radius a surrounded concentrically by conducting spherical shell of inner radius b.

- Q: magnitude of charge on each sphere
- Electric field between spheres: use Gauss' law
  $E[4\pi r^2] = \frac{Q}{\epsilon_0} \Rightarrow E(r) = \frac{Q}{4\pi\epsilon_0 r^2}$
- Electric potential between spheres: use $V(a) = 0$
  $V(r) = -\int_a^r E(r)dr = -\frac{Q}{4\pi\epsilon_0}\int_a^r \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{a}\right]$
- Voltage between spheres:
  $V \equiv V_+ - V_- = V(a) - V(b) = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab}$
- Capacitance for spherical geometry:
  $C \equiv \frac{Q}{V} = 4\pi\epsilon_0 \frac{ab}{b-a}$



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The design of concentric spheres, as shown in cross section, has the advantage that the result is exact for all sizes. No assumptions regarding ratios of certain lengths are necessary.

For specificity we take the conducting sphere at the center to be positively charged and the surrounding conducting shell negatively. The capacitance does not depend on this choice.

The surface at radius a carries charge $+Q$ and the surface at radius b charge $-Q$. There is no excess charge anywhere else.

The calculation of the voltage between the two conductors follows the same chain of reasoning as in the case worked out on the previous page.

The resulting expression of the capacitance again depends on the two radii a and b. Unlike in the cylindrical capacitor, there is no third length entering the expression.

Once again, narrowing the gap between the conductors increases the capacitance.

Energy Stored in Capacitor

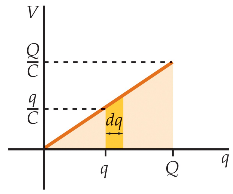
Charging a capacitor requires work.
The work done is equal to the potential energy stored in the capacitor.

While charging, V increases linearly with q:

$$V(q) = \frac{q}{C}.$$

Increment of potential energy:

$$dU = Vdq = \frac{q}{C}dq.$$



Potential energy of charged capacitor:

$$U = \int_0^Q Vdq = \frac{1}{C}\int_0^Q qdq = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV.$$

Q: where is the potential energy stored?
A: in the electric field.

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Placing a charge Q on a capacitor, i.e. charge $+Q$ on one conductor and charge $-Q$ on the other, is like lifting a weight or compressing a spring. The process requires work.

When an agent does work on a system quasistatically, i.e. slowly, without significant kinetic energy involved, and by exerting a conservative force, then the work done is equal to the change in potential energy of the system.

Charging a capacitor thus means storing energy in the device. This energy is retrievable, when the capacitor is being discharged.

We charge up a capacitor in increments dq. We are, effectively, moving dq from one plate to the other plate across the voltage between the plates. That voltage depends on the charge q already stored: $v(q) = q/C$.

With the addition of each charge increment dq we increase the energy stored on the capacitor by $dU = v(q)dq = (q/C)dq$. At the same time, the voltage increases by $dv = dq/C$.

The energy U of a capacitor that has charge Q on it and voltage V across it, is then the sum of such increments. In the limit of infinitesimal increments, this sum converts into an integral.

By using the definition of capacitance $C = Q/V$, we can write the expression for potential energy U in three equivalent ways as shown on the slide.

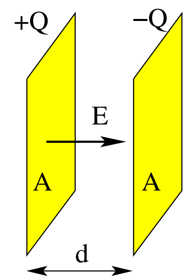
When we lift a weight it has gravitational potential energy. When we compress a spring, it contains elastic energy. What sort of energy does a capacitor contain and where is it?

The energy is not, as we might expect, on the conductors, where the excess charge is. It is in the electric field between the conductors.

Energy Density Between Parallel Plates

Energy is stored in the electric field between the plates of a capacitor.

- Capacitance: $C = \frac{\epsilon_0 A}{d}$.
- Voltage: $V = Ed$.
- Potential energy: $U = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0 E^2 (Ad)$.
- Volume between the plates: Ad.
- Energy density of the electric field: $u_E = \frac{U}{Ad} = \frac{1}{2}\epsilon_0 E^2$



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We use the parallel-plate geometry to determine the energy content of an electric field. Our choice is motivated by the uniformity of the electric field E in the space of volume Ad between the plates, where A is the area of each plate and d the distance between them.

We rewrite the expression for energy stored on the capacitor,

$$U = \frac{1}{2}CV^2,$$

as established on the previous page, by substituting the relation $V = Ed$ between voltage and (uniform) electric field, and the expression, $C = \epsilon_0 A/d$, for the capacitance as derived on page 2.

We conclude that the energy density, i.e. energy per unit volume, is

$$u_E = \frac{1}{2}\epsilon_0 E^2.$$

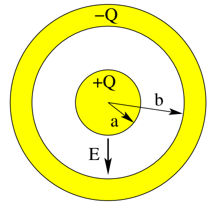
Wherever there is an electric field, there is energy. When we add charge to the capacitor, the voltage goes up, which implies that the electric field becomes stronger. Stronger fields carry more energy.

In this case of uniform electric field, the total energy stored in the device is simply the energy density u_E multiplied by the volume Ad of the space where the electric field is present.

The above result for electric energy density u_E, which we have derived for a very special situation, holds more generally for electric fields in vacuum. A slightly modified expression pertains to electric fields inside insulating (dielectric) materials, as will be discussed in lecture 14.

Integrating Energy Density in Spherical Capacitor

- Electric field: $E(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$
- Voltage: $V = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$
- Energy density: $u_E(r) = \frac{1}{2}\epsilon_0 E^2(r)$



- Energy stored in capacitor: $U = \int_a^b u_E(r)(4\pi r^2)dr$
- $\Rightarrow \; U = \int_a^b \frac{1}{2}\epsilon_0 \frac{Q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}(4\pi r^2)dr$
- $\Rightarrow \; U = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{1}{2}QV$

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If the energy-density expression derived on the previous page is indeed general, then we can use it to determine the energy stored in a capacitor where the field is not uniform.

Let us go ahead and do that for the case of a spherical capacitor. From page 4 we know the electric field $E(r)$. We substitute that result into the expression for energy density. Next we integrate the energy density across the space between the two conductors. That integration involves concentric spherical shells for area $4\pi r^2$ and width dr.

The integral is carried out in the last two items on the slide. The result can be written as $U = \frac{1}{2}QV$, which is one of the three general expression of energy stored on a capacitor derived on page 5.

This application confirms that the expression of energy density is not limited to the case of a parallel-plate configuration.

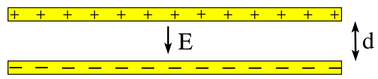
We shall see (much later) that half of the energy in an electromagnetic wave is carried by the electric field and the other half by the magnetic field. The part transported by the electric field uses the same expression, $u_E = \frac{1}{2}\epsilon_0 E^2$, for energy density. The part carried by the magnetic field uses a similar expression, which we will derive in the second half of this course.

Capacitor Problem (1)

Consider two oppositely charged parallel plates separated by a very small distance d.

What happens when the plates are pulled apart a fraction of d? Will the quantities listed below increase or decrease in magnitude or stay unchanged?

(a) Electric field $\vec{E}$ between the plates.
(b) Voltage V across the plates.
(c) Capacitance C of the device.
(d) Energy U stored in the device.



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This exercise is meant to give us further insight into key attributes of capacitors in the simplest possible context.

(a) The electric field between the plates is uniform and related to the (uniform) charge density on the inside surfaces of the plates: $E = \sigma/\epsilon_0$. Neither the charge Q on the capacitor nor the surface charge density σ change when we increase the width of the gap between the plates. Therefore, the electric field does not change either.

(b) The relation $V = Ed$ between voltage and (uniform) electric field tells us that the voltage increase proportional to the width d.

(c) Here we use the result $C = \epsilon_0 A/d$ for the capacitance of the parallel-plate capacitor to conclude that the capacitance decreases with increasing d.

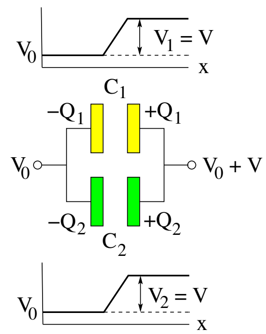
(d) For this question we use either the result $U = \frac{1}{2}QV$ or the equivalent result $U = Q^2/2C$ to reason that the energy stored in the capacitor increases as the capacitance C decreases and the voltage V increases, while the charge Q stays constant.

This raises the question about the origin of the extra energy. Where does it come from? We are not adding charge. The answer is that separating the plates requires mechanical work. That work is being converted into electrical potential energy.

Capacitors Connected in Parallel

Find the equivalent capacitance of two capacitors connected in parallel:

- Charge on capacitors: $Q_1 + Q_2 = Q$
- Voltage across capacitors: $V_1 = V_2 = V$
- Equivalent capacitance:
  $C \equiv \frac{Q}{V} = \frac{Q_1 + Q_2}{V} = \frac{Q_1}{V_1} + \frac{Q_2}{V_2}$
- $\Rightarrow \; C = C_1 + C_2$



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We now shift the emphasis from the physics of capacitors to circuits of capacitors. Circuits are devices connected by wires for specific purposes.

There are two distinct ways in which a pair of capacitors can be connected, as a unit, to other parts of a circuit. The slide on this page shows a *parallel* connection and the slide on the next page a *series* connection.

The terminals, indicated by little rings, are the points where the unit connects to other parts of a circuit. In both cases, the unit can be replaced by a single capacitor, named *equivalent capacitor*, with the same function in the circuit.

The hallmark of two capacitors connected in parallel is that the voltage across each is the same. The graph on the slide indicates the places where the potential changes, which is the space between the plates, where the electric field is. There are two conductors, one on the left at potential V_0 and the other on the right at potential $V_0 + V$.

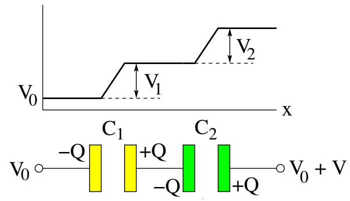
The equivalent capacitor must account for the charge $Q = Q_1 + Q_2$ that goes onto the unit of the two parallel capacitors, while the voltage across is $V = V_1 = V_2$. The equivalent capacitance is then readily calculated to be

$$C = C_1 + C_2.$$

Capacitors Connected in Series

Find the equivalent capacitance of two capacitors connected in series:

- Charge on capacitors: $Q_1 = Q_2 = Q$
- Voltage across capacitors: $V_1 + V_2 = V$
- Equivalent capacitance: $\frac{1}{C} \equiv \frac{V}{Q} = \frac{V_1 + V_2}{Q} = \frac{V_1}{Q_1} + \frac{V_2}{Q_2}$
- $\Rightarrow \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$



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A series connection of two capacitors consists of three conductors, one on the left at potential V_0, one on the right at potential $V_0 + V$, and one in the middle at a potential to be determined.

When the series unit is charged up, equal amounts of opposite charge flow onto the plates of both capacitors. The total charge on the middle conductor was zero initially and must remain zero because it is insulated from other conductors. Hence the charge on both capacitors must be equal and the charge on the series unit is $Q = Q_1 = Q_2$

The graph on the slide again indicates where the potential changes, namely across the two gaps between the three conductors. Going from the left to right there are two steps up in potential.

The first step is the voltage V_1 across the capacitor on the left and the second step the voltage V_2 across the capacitor on the right. The middle conductor is at potential $V_0 + V_1$. The voltage across the series unit is $V = V_1 + V_2$

If we wish to replace the series unit of two capacitors with capacitances C_1 and C_2 by an equivalent capacitor, what must its capacitance C be? The answer is worked out on the slide.

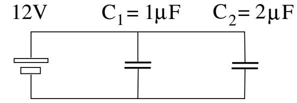
In this case, we have to add up the inverse capacitances to get the inverse equivalent capacitance:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}.$$

Capacitor Circuit (2)

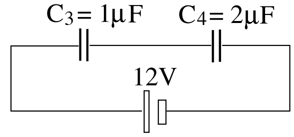
Consider the two capacitors connected in parallel.

(a) Which capacitor has the higher voltage?
(b) Which capacitor has more charge?
(c) Which capacitor has more energy?



Consider the two capacitors connected in series.

(d) Which capacitor has the higher voltage?
(e) Which capacitor has more charge?
(f) Which capacitor has more energy?



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Here we introduce a new device, the voltage source or battery, represented by two parallel rectangles. The tall and thin rectangle is at higher potential than the short and wide rectangle. The potential difference is the voltage V supplied by the battery to the circuit, here 12V.

We can best answer the six questions by calculating all the quantities in question. The sequence may not be the same for both circuits.

(a) Two capacitors in parallel have the same voltage across: $V_1 = V_2 = 12\text{V}$.

(b) The charges are $Q_1 = C_1V_1 = 12\mu\text{C}$ and $Q_2 = C_2V_2 = 24\mu\text{C}$. In a parallel connection, the capacitor with the larger capacitance carries more charge ...

(c) ... and stores more energy: $U_1 = \frac{1}{2}Q_1V_1 = 72\mu\text{J}$, $U_2 = \frac{1}{2}Q_2V_2 = 144\mu\text{J}$.

For the series connection, it helps to first calculate the equivalent capacitance:

$$C_{eq} = \left(\frac{1}{1\mu\text{F}} + \frac{1}{2\mu\text{F}}\right)^{-1} = \frac{2}{3}\mu\text{F}.$$

(e) The charge on both capacitors is the same: $Q_3 = Q_4 = C_{eq}(12\text{V}) = 8\mu\text{C}$.

(d) The voltages then follow directly: $V_3 = Q_3/C_3 = 8\text{V}$, $V_4 = Q_4/C_4 = 4\text{V}$. In a series connection, the capacitor with the smaller capacitance has the higher voltage across it ...

(f) ... and stores more energy: $U_3 = \frac{1}{2}Q_3V_3 = 32\mu\text{J}$, $U_4 = \frac{1}{2}Q_4V_4 = 16\mu\text{J}$.

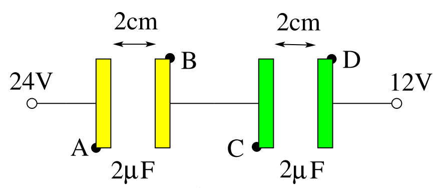
Capacitor Problem (2)

Consider two equal capacitors connected in series.

(a) Find the voltages $V_A - V_B, V_B - V_C, V_A - V_D$.

(b) Find the charge Q_A on plate A.

(c) Find the electric field E between plates C and D.



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This is the quiz for lecture 12.

Here are some relevant pieces of information:

- We have two capacitors of equal capacitance in series.
- There are three conductors.
- The electric potential is the same across all point of a conductor.
- Voltages are potential differences.
- The voltages across capacitors in series add up to the terminal voltage.
- Capacitors in series carry equal charge.
- The relation between voltage and uniform electric field is $V = Ed$.