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When Products are Sums

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When Products are Sums

When we open up the classroom to student thinking we run into discoveries and conjectures that adults would likely not see, blocked by the constraints and conventions we have accepted in our own thinking. It is this lack in established constraints and conventions that can enable young thinkers to explore the world of mathematics more unencumbered. The risk of content unfamiliarity in such learning environments is not for the students; it is for their teachers (Ball, Thames, & Phelps, 2008; Chazan, 1999; Liping Ma, 1999). We want to relay one such instance in which a third grade student saw a new connection that stimulated us to investigate the mathematics of the situation in greater depth. In this article we want to show how this novel idea can be explored by students at multiple grade levels in different ways. The tasks at hand are especially well suited to address several of the mathematical practices standards from the Common Core State Standards, such as reason abstractly and quantitatively, look for and make use of structure, and look for and make use of regularity in repeated reasoning.

Discovery of a Third Grade Student

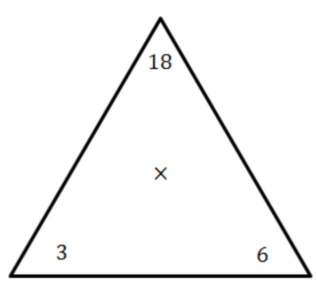

Figure 1. Fact Triangle Grade 3

While we were conducting a demonstration lesson in a third grade class, students explored the relationship between multiplication and division as represented by the fact triangle from Figure 1. We discussed the idea that if the quantity 3 was covered up we could prove that the missing number had to be 3 because "three times six equals eighteen" or, as some of the children said, "eighteen divided by six equals three." As this conversation was going on one girl stated that she could find 18 with 3 and 6 in another way. She claimed that if one adds 3 and 6 and then doubles this sum, the result is also 18. This created some "ahs" and "ohs" in the class. We asked if this was always possible and, with the class, checked to see if it also worked with 3 x 7 = 21. Alas, this did not work in this example: 2(3+7) = 20. Then one student found another example: 4 x 4 = 16 and 2(4+4) = 16. The class started buzzing until finally one student said that "it also works for 6 x 3, but that really the same as 3 x 6." In this article we want to share the mathematical richness of this idea and how it can be investigated at the different grade levels.

Investigating this idea in the elementary grades.

There are no other natural number factor pairs besides 3&6 and 4&4 that have the relationship that states that the product of two factors is equal to double their sum. Elementary students can discover this by creating tables and analyzing patterns, such as below:

Table 1

$6\times1=6$	$2(6+1)=14$
$6\times2=12$	$2(6+2)=16$
$6\times3=18$	$2(6+3)=18$
$6\times4=24$	$2(6+4)=20$
$6\times5=30$	$2(6+5)=22$

Table 2

$7\times1=7$	$2(7+1)=16$
$7\times2=14$	$2(7+2)=18$
$7\times3=21$	$2(7+3)=20$
$7\times4=28$	$2(7+4)=22$
$7\times5=35$	$2(7+5)=24$

In table 1, we can see that when 6 is chosen as the first factor, the factor partner that will satisfy the stated relationship will be 3. In table 2, we do not find a factor pair that has this relationship. Students can discover that double the sum (the second column in each table) will have to be even. Therefore, only factor pairs that have an even product can be considered as candidates for the property. In table 1, all factor partners of 6 will generate even products because 6 is even. In table 2, only even factor partners of 7 will yield an even product. Students can discover in the patterned table that the product can be less than, equal to, or greater than double the sum. In the case of table 1, all three of these instances occur, but in table 2 there is no partner of 7 that results in equality. Once the product is larger than double the sum, it cannot become equal. Therefore 7 has no natural number factor partner that satisfies the property. We encourage elementary teachers to allow students to investigate many tables like these, with one factor fixed and the other variable, to arrive at these possible patterns. There is a deeper underlying structure that teachers may want to address when students are ready for it. This structure will involve ideas of equivalence and the distributive property.

Let's first return to figure 1 and understand why the student's idea worked for the given example of 3 x 6 = 18. She stated that 18 can also be found as follows:

2 (3 + 6)

By applying the distributive property we can see that this is equivalent to:

2 x 3 + 2 x 6

Because 2 x 3 is equivalent to 1 x 6, we can write this expression as follows:

1 x 6 + 2 x 6,

And by applying the distributive property again we show that this is equivalent to 3 x 6.

So we have given a proof that 2 (3 + 6) = 3 x 6.

The crucial step here is to write 2 x 3 (double the first number) as a multiple of 6 (a multiple of the second number) so that the distributive property can be applied. *It is the distributive property that relates a product and a sum*. We will generalize this approach to the problem in our section on the high school grades below. But first we will take a look at how this relationship can be extended to the middle grades.

Investigating this idea in the middle grades.

In the middle school we can extend this idea to the domains of Integers and Rational numbers. We return to the tables 1 and 2 above. We will represent the first and second columns in each table as a linear function. We represent the pattern in the first column in table 1 as $y = 6x$ and the second column as $y = 2(6 + x)$. In these equations, 6 is the chosen factor that is kept constant, x is the factor that varies, and y represents the product or double the sum of the factor pairs. This then turns the problem of *finding factor pairs for which their product is equal to twice their sum* into solving a system of linear equations. Students can solve this algebraically (using substitution) or graphically. From the functions representing table 1 students can solve that x=3 is the factor partner of 6 such that y=18. When we apply the same idea to table 2, where 7 is the chosen factor, we will obtain a rational factor partner as follows:

$$7x = 2(7 + x)$$

$$7x = 14 + 2x$$

$$5x = 14$$

$$x = \frac{14}{5}$$

So in the example of table 2, we can see that $\frac{14}{5}$ is the rational factor partner of 7 such that y=$\frac{98}{5}$.

This demonstrates that there are factor pairs beyond the natural numbers for which this relationship holds. Teachers can expand this investigation by asking students if there are also factor pairs with negative rational numbers that have this relationship. For example, choose the first factor to be -1 and vary the second factor. This will lead to solving $-x = 2(-1 + x)$, which yield $\frac{2}{3}$ as a factor partner of -1, with $y = -\frac{2}{3}$. There exists one integer pair of factors with this relationship: 1 and -2. We will show this later on in this article. In the middle school we can encourage students to explore this relationship from a function perspective using tables, graphs, and equations. Next we will look at how we can further generalize this relationship.

Investigating this idea in the high school.

We would like high school students to consider the relationship in a broader, more general sense by investigating the question: For which pairs of factors (*a,b*) is their product (*ab*) equal to twice their sum [2(*a*+*b*)]? (See figure 2.)

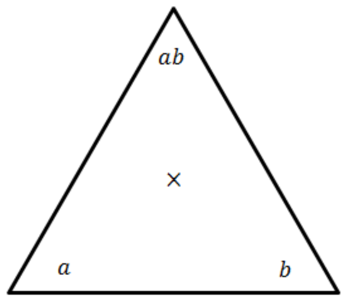

Figure 2. General fact triangle

From specific numerical examples, such as 6x3=2(6+3), students can conjecture that all possible factor pairs with this property have the following relationship, as we investigated in the elementary grades section above:

Twice the first factor ($2a$) is a multiple of the second factor (nb).

We can formulate this relationship symbolically as

(1) $2\times a = n\times b$, where $a, b, n \in \mathrm{N}$

The general relationship can be described as follows:

(2) $a\times b = 2(a+b)$

Solving for b in equation (1) yields:

(3) $b = \frac{2a}{n}$

Substitute (2) into (1) yields:

(4) $a\times b = (n+2)\times b$

From (4) we can show that

(5) $\boldsymbol{a = n + 2}$

Substituting (5) into (3) yields:

(6) $\boldsymbol{b = 2 + \frac{4}{n}}$

We will create a table (see table 3) with *n*, *a*, and *b*, using (5) and (6). We will be looking for all values of *n* such that *a* and *b* are natural numbers.

Table 3.

$\boldsymbol{n}$	$\boldsymbol{a = n + 2}$	$\boldsymbol{b = 2 + \frac{4}{n}}$	$\boldsymbol{a \times b}$
1	3	6	18
2	4	4	16
3	5	$3\frac{1}{3}$	$16\frac{2}{3}$
4	6	3	18
5	7	$2\frac{4}{5}$	$19\frac{3}{5}$

The values we can choose for *n*, such that *a* and *b* are natural numbers are constrained by (6). There we can see that *n* = 1, 2, 4. For all other values for *n* we will get rational values for *b* and this will make the product *ab,* a non-natural quantity. However, it should be noted that rule (2) does hold in these cases within the rational number set, as we investigated in the middle grades section above. Last, we consider the solutions for *n*=1 and *n*=4 to be identical. And thus we have shown that there are only two pairs of numbers whose product is equal to twice their sum for natural numbers. The children in grade 3 had found those with a little effort.

If we extend the domain to the Integers, we find additional solutions. Note that *n* cannot be equal to 0. We can now additionally substitute -1, -2, and -4 for *n* to yield Integer solutions (see Table 4 for results). For *n*=-2, we get a trivial solution.

Table 4.

n	$a = n + 2$	$b = 2 + \frac{4}{n}$	$a \times b = 2n + 8 + \frac{8}{n}$
-1	1	-2	-2
-2	0	0	0
-3	-1	$\frac{2}{3}$	$-\frac{2}{3}$
-4	-2	1	-2
-5	-3	$1\frac{1}{5}$	$-3\frac{3}{5}$

Investigating this idea in Calculus

If we allow this relationship over the Real number set, then we can generalize the product as follows: $ab = 2n + 8 + \frac{8}{2n}$. In Figure 3 we have plotted the function $f(n) = 2n + 8 + \frac{8}{2n}$, where $f(n)$ represents the possible products. We can see that for larger values of n, the function of products appears to be more like $f(n) = 2n + 8$ and for smaller values of n, the function appears to be more like $f(n) = \frac{8}{n}$. We can see that $f(n) = 2n + 8$ is the slanted asymptote of this function. Furthermore, by investigating the derivative of $f(n)$ we can see that for n=2, the function has a relative minimum and for n=-2 a relative maximum. This means that no products will be found between 0 and 16.

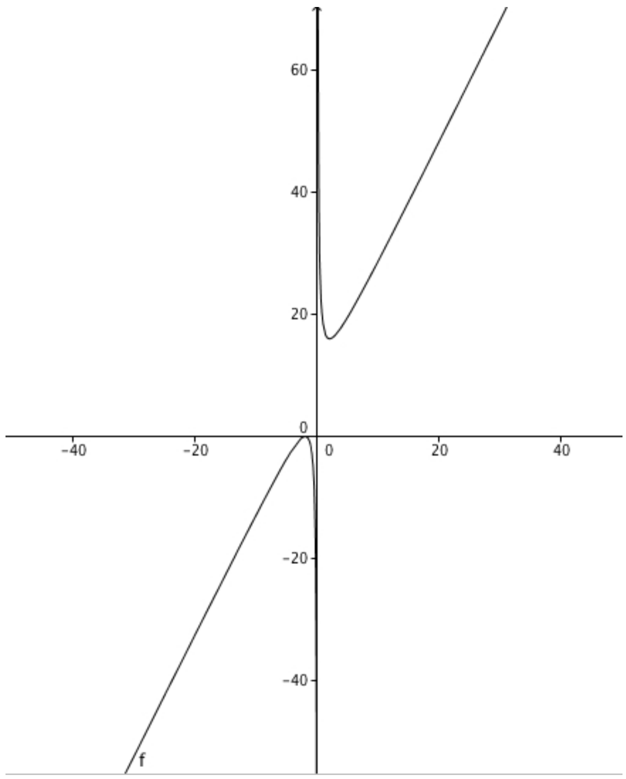

Figure 3. The function of products over R.

Once a thorough investigation of this property has been completed, high school teachers can provide additional challenges to their students. Consider posing the following question: For which pairs of numbers is their product equal to three times their sum? We will reason by analogy with the prior case. We first set up the following relationships for Figure 2.

$a \times b = 3(a + b)$, and $3 \times a = n \times b$.

From this follows that:

$a = n + 3$, and $b = 3 + \frac{9}{n}$

The number of solutions will depend on the term $\frac{9}{n}$. From this we can see that choices for n, such that a and b are natural numbers can only be 1, 3, and 9. We find that 4 x 12 = 3(4 +12) and 6 x 6 = 3(6+6). Additional Integer solutions are for n = -1,

-3, -9: 2 x -6 = -12, 0 x 0 = 0, and -6 x 2 = 12 (same as first solution by commutative property).

Investigating this idea beyond high school.

Looking back at this trajectory, we can generalize this situation even further. This will require applications of Number Theory. From the above high school level investigation students can conjecture that the terms in the expressions for *b* that constrained the choices for *n*, were the square of the multiple of a: $\frac{2^2}{n}$ and $\frac{3^2}{n}$. We will use this to generalize the prior two cases. Note that we have investigated two cases in which the multiple of *a* is a prime (2 and 3). Below we will investigate multiples that are non-prime as well. We can now consider the following question: For which pairs of factors (a,b) is their product a multiple of their sum [p(a+b)]?

From the prior example we derive the following two relationships:

(1) $ab = p(a + b) = pa + pb$, where p represents the multiple of the sum, where $a, b, p \in \mathbb{Z}$.

(2) $pa = nb$

Note that in this relationship the two multiples are connected. It states that a multiple of the first factor (pa) is equal to some other multiple of the second factor (nb).

Substituting (2) into (1) yields:

(3) $ab = nb + pb = (n + p)b$

From (3) it follows that

(4) $\boldsymbol{a = n + p.}$

From (2) we can solve for b

$b = \frac{pa}{n}$ and then substitute (4):

(5) $b = \frac{p(n+p)}{n} = \frac{pn+p^2}{n}$, thus $\boldsymbol{b = p + \frac{p^2}{n}}$

From this we can see that the constraining term for choice of multiples that satisfy (1) is $\frac{p^2}{n}$. From this we can note that *n* has to be a factor of p^2, such that $\frac{p^2}{n} \in \mathbb{Z}$.

If p is prime, we can only have six factors of p^2, namely ± 1, $\pm p$, and $\pm p^2$. We have summarized the results for *a*, *b*, and *ab* in Table 5 below.

Table 5

$\boldsymbol{n}$	$\boldsymbol{a = n + p}$	$\boldsymbol{b = p + \frac{p^2}{n}}$	$\boldsymbol{a \times b = np + 2p^2 + \frac{p^3}{n}}$
$-p^2$	$p - p^2$	$p - 1$	$-p^3 + 2p^2 - p$
$-p$	0	0	0
-1	$p - 1$	$p - p^2$	$-p^3 + 2p^2 - p$
1	$p + 1$	$p + p^2$	$p^3 + 2p^2 + p$
p	$2p$	$2p$	$4p^2$
p^2	$p + p^2$	$p + 1$	$p^3 + 2p^2 + p$

Note how the products for *n*=-1 and *n*=$-p^2$, as well as the products for *n*= 1 and *n*=p^2 are the same due to the commutative property. Therefore, we will always have four distinct pairs of numbers for which relationship (1) holds when *p* is prime. This, in part, answers our question from above. We now consider the remainder of the question for non-prime multiple relationships: Can we predict the amount of factor pairs that hold this relationship for any given multiple?

Again, we begin to consider the following relationships, first over N:

(1) $ab = p(a+b) = pa + pb$, where p represents the multiplier of the sum.

(2) $pa = nb$

We now consider the multiplier p to be non-prime. Our goal is to find the number of factors of p and subsequently p^2 so that we can determine exactly which values of n we can choose. This in turn will determine the number of pairs of numbers we can find for a given multiplier p.

First we determine the prime factorization of the multiplier p:

$$p = {p_1}^{m_1} \times {p_2}^{m_2} \times \ldots \times {p_n}^{m_n}$$

Then the prime factorization of p^2 is as follows:

$$p^2 = {p_1}^{2m_1} \times {p_2}^{2m_2} \times \ldots \times {p_n}^{2m_n}$$

From this we can determine the number of factors, f, of p^2 as follows:

$$f = (2m_1 + 1)(2m_2 + 1) \ldots (2m_n + 1)$$

Now that we know how many factors p^2 has, we can determine the number of pairs of factors that will satisfy (1) above. Note that the number of factors of p^2 is odd and that p is one of the factors and for the remaining factors (f-1), half are less than p and the other half greater than p. We can also see that each half yields the same pairs of numbers and therefore the total number of pairs for which relationship (1) holds is $\boldsymbol{\frac{f-1}{2} + 1}$.

Last we will consider the same question over Z. By symmetry we can see that the number of factors of p^2 over Z must be twice the number if factors of p^2 over N.

By extending the reasoning above the amount of pairs for which relationship (1) holds in Z is twice as much as in N: $2\left(\frac{f-1}{2}+1\right)=f+1$.

In exploring whether or not the number of factors pairs that satisfy the relationship described, the use of exponent properties and recalling prime factorization is key. Though these skills may be above the level of ability for elementary age students, high school students looking for applications of these topics would be able to investigate these relationships. The expression we determined as the number of pairs over N was denoted as $\frac{\boldsymbol{f-1}}{\boldsymbol{2}}+1$. Though this could have written equivalently as $\frac{\boldsymbol{f+1}}{\boldsymbol{2}}$, this does not allow for the symmetry of the solution to be observed. As stated above, half the factors of p^2 are greater than p and the other half are less than p. This shows symmetry in the list of factors around p, thus the $\frac{\boldsymbol{f-1}}{\boldsymbol{2}}+1$ is more useful in this case (See Figure 4.). When observed over Z, the situation for the negative integer factors is symmetrical with the positive integer factors. This is a good example in which "simplifying" the expressions hides the structure that we are after. It can teach us that "simplifying" is not always desirable or useful.

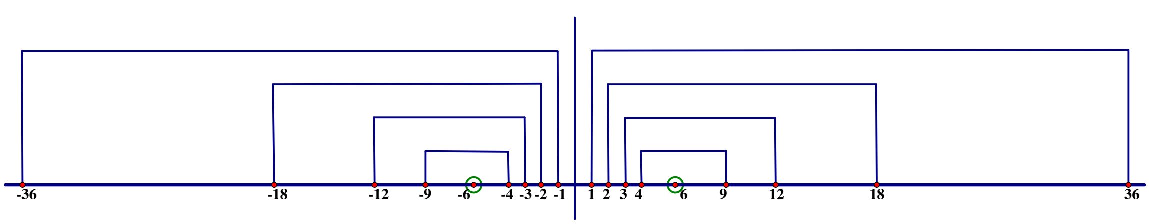

Figure 4. Symmetry of factors for p=6

Conclusion

The high-level math skills that teachers at all levels need to possess are apparent in exploring this problem as well. Having a deep understanding of the underlying mathematics, referred to as content knowledge in many instances, can give teachers the courage to ask students questions such as “is that always possible?”, “Why do you think so?”, and “Are there any other possible solutions?”. Being able to gauge the plausibility of students’ claims is an essential tool that all teacher must have. Ball, Thames, and Phelps (2008) discuss the numerous characteristics and abilities needed for mathematical teaching tasks. For instance, they attest that teachers “need not only understand *that* something is so; the teacher must further understand *why* it is so” (p. 391).

We took ourselves to task and attempted to find depth and structure in the discovery of a third grade student. This process involved much trial and error; lots of notes on little pieces of paper; many discussions trying to convince each other; and digging back into our own prior knowledge. We encourage teachers and teacher educators to go on these journeys together inspired by their students’ unique and unconstrained thoughts. It was fun and worthwhile for us.

We hope that teachers at elementary, middle, and high school levels are encouraged to use this thirds grader’s thinking in their classrooms with their students. We hope you and your students will experience as much pleasure as we did.

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Editor's Addendum:

Addendum to When Products are Sums

The more that the authors and I discussed the previous article **When Products are Sums**, the more we came to realize how mathematically rich the problem was. That it originated from a third grade student and can be adapted to various levels gives it wide appeal. The article itself was written to encourage student exploration, emphasizing tabular data and pattern recognition. Our discussions revealed a few insights that didn't quite fit into the framework of the original article. We decided that they were worth mentioning in this addendum. At the very least, this addendum will show just how rich a fairly simple question can be when re-examined. It also shows some mathematical "trickery" which are often utilized by mathematicians.

Looking at the original problem and solving $ab = 2(a + b)$ for b you get

$$ab = 2a + 2b$$

$$b = \frac{2a}{a-2} = \frac{(2a-4)+4}{a-2} = 2 + \frac{4}{a-2}$$

(Notice the trick of "un-cancelling" the 4's.)

At this point, the fact that $\frac{4}{a-2}$ must be a positive integer means that $a - 2 = 1, 2, 4$ so $a = 3, 4, 6$ with corresponding $b = 6, 4, 3$. This extends to negative integers as well with $a - 2 = -1, -2, -4$ and $a = 1, 0, -2$ with corresponding $b = -2, 0, 1$.

In general, suppose p is a fixed integer. The problem is to find integers a, b with

$$ab = p(a + b)$$

Again, solving for b and "un-cancelling", we get

$$b = \frac{pa}{a-p} = \frac{(pa - p^2) + p^2}{a-p} = p + \frac{p^2}{a-p}$$

This says that $a - p$ must be a divisor of p^2. For example, if p is prime then we only have six factors of p^2, namely ± 1, $\pm p$, and $\pm p^2$, so that

$$a = p \pm 1, p \pm p, p \pm p^2$$

Another approach involves exploring unit fractions. For example, consider that the original equation $ab = 2(a + b)$ can be rewritten as

$$2 = \frac{ab}{a+b}$$

$$\frac{1}{2} = \frac{a+b}{ab} = \frac{1}{b} + \frac{1}{a}$$

So the problem becomes one of writing $\frac{1}{2}$ as a sum of two unit fractions. The solutions $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$ and $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ are well known. The task is to show that these are the only positive solutions. If $a, b \geq 5$ then

$$\frac{1}{b} + \frac{1}{a} \leq \frac{1}{5} + \frac{1}{5} = \frac{2}{5} < \frac{1}{2}$$

so at least one of a or b must be 3 or 4, which we already have.

If one of the integers, say a, is negative, then $-a$ is positive, and we can rewrite the original equation as

$$\frac{1}{2} + \frac{1}{-a} = \frac{1}{b}$$

The only possible solution is when $a = -2$ and $b = 1$, which matches what we obtained before.

In general, students can rewrite $ab = p(a+b)$ as

$$\frac{1}{p} = \frac{1}{b} + \frac{1}{a}$$

and examine the ways to represent $\frac{1}{p}$ as a sum of two unit fractions.

The original problem also has a nice geometric interpretation. Positive solutions to the equation

$$ab = 2(a+b)$$

are the (integral) sides of a rectangle whose area is equal to its perimeter.

The point behind this addendum (and the original article) is that an innocent comment by a student can turn into a rich mathematical experience. We should encourage this as much as possible and be prepared to run with it when it happens. The excitement of learning for both the student and the teacher comes from finding out what can happen.