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12. Rigid Body Dynamics I

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Abstract
Part twelve of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc12]

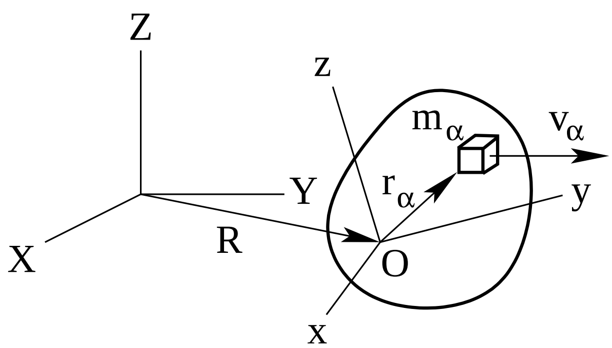
Dynamics of Rigid Bodies [mln24]

System with 6 degrees of freedom (3 translations and 3 rotations).

Equations of motion: $\dot{\mathbf{p}} = \mathbf{F}^{(e)}, \quad \dot{\mathbf{L}} = \mathbf{N}^{(e)}$ [mln2].

For the description of the rigid-body dynamics it is useful to introduce three coordinate systems:

- inertial coordinate system with axes (X, Y, Z),
- coordinate system with axes (x', y', z') parallel to (X, Y, Z) and origin O fixed to some point of the rigid body,
- coordinate system with axes (x, y, z) fixed to rigid body and with the same origin O as (x', y', z').



Motion of rigid body: $\mathbf{v}_\alpha = \dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha$.

- *Translational* motion of (x', y', z') relative to (X, Y, Z).
- *Rotational* motion of (x, y, z) relative to (x', y', z').

The optimal choice of the origin O is dictated by the circumstances:

- for freely rotating rigid bodies, the center of mass is the best choice;
- for rigid bodies rotating about at least one point fixed in the inertial system, one such fixed point is a good choice.

Analysis of rigid body motion:

- Solve the equations of motion in the coordinate system (x, y, z). They are called *Euler's equations* [mln27].
- Transform the solution to the coordinate system (x', y', z') via *Eulerian angles* [msl25], [msl26] and from there to the inertial system (X, Y, Z).

Rotational Kinetic Energy [mln25]

Consider a rigid body undergoing a purely rotational motion ($\dot{\mathbf{R}} = 0$).

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left(\vec{\omega} \times \mathbf{r}_\alpha\right)^2 = \frac{1}{2}\sum_\alpha m_\alpha \left[\omega^2 r_\alpha^2 - (\vec{\omega}\cdot\mathbf{r}_\alpha)^2\right].$$

$\vec{\omega} = (\omega_1, \omega_2, \omega_3)$: instantantaneous angular velocity of body frame relative to inertial frame (components in body frame).
$\mathbf{r}_\alpha = (r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3})$: position coordinates in body frame.

$$T = \frac{1}{2}\sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2\right)\left(\sum_k r_{\alpha k}^2\right) - \left(\sum_i \omega_i r_{\alpha i}\right)\left(\sum_j \omega_j r_{\alpha j}\right)\right].$$

Use $\omega_i = \sum_j \omega_j \delta_{ij}$.

$$T = \frac{1}{2}\sum_\alpha \sum_{ij} m_\alpha \left[\omega_i\omega_j\delta_{ij} r_\alpha^2 - \omega_i\omega_j r_{\alpha i} r_{\alpha j}\right] = \frac{1}{2}\sum_{ij} I_{ij}\,\omega_i\omega_j.$$

Inertia tensor: $I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j}\right].$

Inertia tensor of cont. mass distrib. $\rho(\mathbf{r})$: $I_{ij} = \int d^3r\,\rho(\mathbf{r})\left[\delta_{ij} r^2 - r_i r_j\right].$

Comments:

- Inertia tensor is symmetric: $I_{ij} = I_{ji}$.
- Matrix notation: $T = \dfrac{1}{2}(\omega_1, \omega_2, \omega_3)\cdot\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33}\end{pmatrix}\cdot\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}.$
- I_{ij} depends on choice of body frame.
- If $\omega = \omega_i$ then $T = \frac{1}{2} I_{ii}\omega_i^2$, where I_{ii} is called a moment of inertia.
- The use of the body frame guarantees that the inertia tensor is time-independent.

For $\dot{\mathbf{R}} \neq 0$ the kinetic energy generally has three parts, *translational, mixed*, and *rotational*, which are further discussed in [mex67], [mex173]:

$$T = \sum_\alpha \frac{1}{2} m_\alpha \left(\dot{\mathbf{R}} + \vec{\omega}\times\mathbf{r}_\alpha\right)^2 = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{R}}^2 + \sum_\alpha m_\alpha \dot{\mathbf{R}}\cdot\vec{\omega}\times\mathbf{r}_\alpha + \frac{1}{2}\sum_\alpha m_\alpha \left(\vec{\omega}\times\mathbf{r}_\alpha\right)^2.$$

[mex67] Translational and rotational kinetic energies

A rigid body is regarded as composed of particles with mass m_α whose relative positions have constant magnitude. Consider the inertial coordinate system (X, Y, Z) and the coordinate system (x, y, z) with axes fixed to the rigid body. The velocity of particle α as expressed in the inertial frame is $\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha$, where $\mathbf{R}$ is the position of the origin of the body frame measured in the inertial frame, $\vec{\omega}$ is the instantaneous angular velocity of the body frame relative to the inertial frame, and $\mathbf{r}_\alpha$ is the position of particle α in the body frame. (a) Calculate the kinetic energy T of the rigid body in the inertial system. (b) Show that if the origin of the body frame is chosen at the center of mass, then the kinetic energy can be written as the sum of two terms where one represents the *translational* kinetic energy and the other the *rotational* kinetic energy.

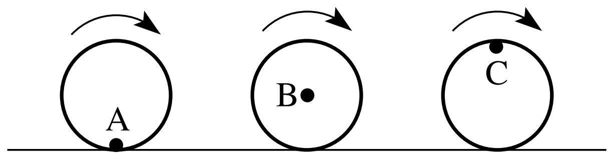
Solution:

[mex173] Kinetic energy of rolling cylinder

A solid cylinder of mass m and radius a is rolling with angular velocity ω on a level surface. Calculate the translational, mixed, and rotational parts of the kinetic energy, $T = T_t + T_m + T_r$, from the general formula

$$T = \sum_\alpha \frac{1}{2} m_\alpha \left(\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha\right)^2 = \frac{1}{2} \sum_\alpha m_\alpha \dot{\mathbf{R}}^2 + \sum_\alpha m_\alpha \dot{\mathbf{R}} \cdot \vec{\omega} \times \mathbf{r}_\alpha + \frac{1}{2} \sum_\alpha m_\alpha \left(\vec{\omega} \times \mathbf{r}_\alpha\right)^2$$

by using three different body frames with origins at points A, B, C, respectively.



Solution:

Principal Axes of Inertia [mln80]

The inertia tensor $\mathbf{I}$ is real and symmetric. Hence it has real eigenvalues.

The orthogonal transformation which diagonalizes the inertia tensor is a rotation of the body coordinate axes to the directions of the *principal axes*.

The eigenvalue problem,

$$\mathbf{I} \cdot \vec{\omega}_k = I_k \, \vec{\omega}_k, \quad k = 1, 2, 3,$$

amounts to a system of linear, homogeneous equations,

$$\begin{aligned} I_{11}\omega_{1k} + I_{12}\omega_{2k} + I_{13}\omega_{3k} &= I_1\omega_{1k} \\ I_{21}\omega_{1k} + I_{22}\omega_{2k} + I_{23}\omega_{3k} &= I_2\omega_{2k} \\ I_{31}\omega_{3k} + I_{32}\omega_{3k} + I_{33}\omega_{3k} &= I_3\omega_{3k}, \end{aligned}$$

where the principal moments of inertia $I_k, \ k = 1, 2, 3$ are the roots of the characteristic polynomial,

$$\begin{vmatrix} I_{11} - I_k & I_{12} & I_{13} \\ I_{21} & I_{22} - I_k & I_{23} \\ I_{31} & I_{32} & I_{33} - I_k \end{vmatrix} = 0.$$

The three eigenvectors $\vec{\omega}_k, \ k = 1, 2, 3$ have undetermined magnitude.
If normalized, we construct an orthonormal matrix from them:

$$\Omega = (\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}.$$

In matrix notation the transformation to principal axes looks as follows:

$$\mathbf{I} \cdot \Omega = \Omega \cdot \bar{\mathbf{I}} \qquad \Rightarrow \ \Omega^T \cdot \mathbf{I} \cdot \Omega = \Omega^T \cdot \Omega \cdot \bar{\mathbf{I}} = \bar{\mathbf{I}},$$

where

$$\mathbf{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}, \qquad \bar{\mathbf{I}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix},$$

$$\Omega^T \cdot \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[mex69] Parallel-axis theorem

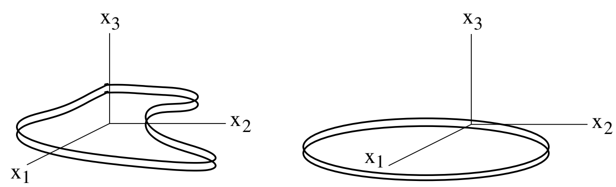
Consider a rigid body composed of particles with mass m_α whose relative positions have constant magnitude. The total mass of the rigid body is $\sum_\alpha m_\alpha = M$. Two body coordinate systems with parallel axes are labelled (r_1, r_2, r_3) and (R_1, R_2, R_3), respectively. The center of mass coordinate of the rigid body is $(0, 0, 0)$ in the former and (a_1, a_2, a_3) in the latter. Show that the inertia tensor of the rigid body in the displaced coordinate system, I_{ij}, is obtained from the inertia tensor in the center-of-mass system, $I_{ij}^{(cm)}$, by adding to it the inertia tensor of a point mass M at the position of the center of mass:

$$I_{ij} = I_{ij}^{(cm)} + M\left(a^2\delta_{ij} - a_i a_j\right).$$

Solution:

[mex73] Perpendicular-axis theorem

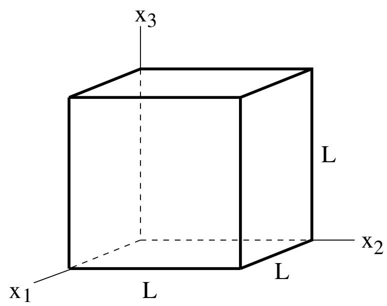
Consider a homogeneous sheet of material which has infinitesimal thickness dx_3 and some arbitray shape in the (x_1, x_2)-plane. (a) Prove the following relation between the moments of inertia for rotations about the coordinate axes: $I_{11} + I_{22} = I_{33}$. (b) Use the parallel-axis and perpendicular-axis theorems to calculate the principal moments of inertia I_{11}, I_{22}, I_{33} of a coin (mass M, radius R) for rotations about the axes of the following coordinate system: The origin is at the rim of the coin. The x_1-axis is radial toward the center of the coin, the x_2-axis is tangential to the coin, and the x_3-axis is perpendicular to the plane of the coin.



Solution:

[mex68] Inertia tensor of homogeneous cube

Consider a cube of mass M and side L and a coordinate system with origin at one corner of the cube and axes along the adjacent edges. (a) Calculate the inertia tensor I_{ij} for this body reference frame. (b) Determine the principal moments of inertia I_1, I_2, I_3. (c) Determine the directions of the principal axes of inertia relative to the coordinate system used originally.



Solution:

[mex252] Principal moments of solid cylinder

Calculate the principal moments of inertia I_1, I_2, I_3 of a solid cylinder as functions of its mass M, its radius R, and its height h for rotations about the center of mass..

Solution:

[mex253] Principal moments of a solid sphere.

Calculate the principal moments of inertia I_1, I_2, I_3 of a solid sphere as a function of its mass M and its radius R for rotations about the center of mass.

Solution:

[mex254] Principal moments of a solid ellipsoid

Calculate the principal moments I_1, I_2, I_3 of a solid ellipsoid as functions of its mass M and its semi-axes a, b, c for rotations about the center of mass.

Solution:

[mex255] Inertia tensor of four-atomic molecule

Consider a four-atomic molecule with the atoms at the corners of a pyramid. The base is an equilateral triangle of side a. The height is h. The three atoms at the base each have mass m_1 and the atom at the tip has mass m_2.
(a) Find the principal moments of inertia I_1, I_2, I_3 as functions of m_1, m_2, a, h for rotations about the center of mass.
(b) Identify the simplifications that occur when the pyramid is a regular tetrahedron of side a and all four masses are equal.

Solution:

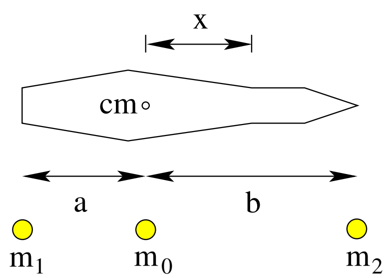
[mex71] Inertia tensor of a cone

Calculate the principal moments of inertia for a homogeneous cone of mass M, height h, and radius R at the base. Perform the calculation for rotations about an axis (a) through the apex of the cone, (b) through the center of mass. Express all results as functions of M, R, h.

Solution:

[mex143] Simulating a stick by three point masses

Consider a nonuniform rod with mass m and moment of inertia I_0 for rotations about an axis through the center of mass and perpendicular to the axis of the rod. The moment of inertia for rotations about a parallel axis displaced by x is then $I_x = I_0 + mx^2$.
Show that three point masses m_0, m_1, m_2 with $m_0 + m_1 + m_2 = m$ in the configuration shown can be chosen such that its moment of inertia for rotations about an axis through m_0 is I_0 and that for rotations about a parallel axis displaced by x is I_x just as is the case for the rod. Express the values of m_0, m_1, m_2 as functions of the specifications m, I_0, a, b of the rod.



Solution:

Angular Momentum [mln26]

$$\mathbf{L}_{tot} = \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}_\alpha) \times \left(\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha \right).$$

If the rigid body rotates freely, choose the origin of the body frame at the center of mass: $\sum_\alpha m_\alpha \mathbf{r}_\alpha = 0,\ \mathbf{R} = \mathbf{r}_{cm},\ \dot{\mathbf{R}} = \mathbf{v}_{cm}$.

$$\Rightarrow\ \mathbf{L}_{tot} = m\mathbf{r}_{cm} \times \mathbf{v}_{cm} + \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) = \mathbf{L}_{orb} + \mathbf{L}_{spin}.$$

If the rigid body rotates about a fixed point in the inertial frame, choose the origin of the body and inertial frames at the fixed point: $\mathbf{R} = 0,\ \dot{\mathbf{R}} = 0$.

$$\Rightarrow\ \mathbf{L}_{tot} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) = \mathbf{L}_{spin}.$$

Consider a rigid body undergoing a purely rotational motion ($\dot{\mathbf{R}} = 0$).

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) = \sum_\alpha m_\alpha \left[r_\alpha^2 \, \vec{\omega} - \mathbf{r}_\alpha (\mathbf{r}_\alpha \cdot \vec{\omega}) \right].$$

Use body frame components $\omega_1, \omega_2, \omega_3$ and $r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3}$:

$$L_i = \sum_\alpha m_\alpha \left[r_\alpha^2 \omega_i - r_{\alpha i} \sum_j r_{\alpha j} \omega_j \right] = \sum_\alpha m_\alpha \sum_j \left[\omega_j \delta_{ij} r_\alpha^2 - \omega_j r_{\alpha i} r_{\alpha j} \right].$$

$$\Rightarrow\ L_i = \sum_j \omega_j \sum_\alpha m_\alpha \left[\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j} \right] = \sum_j I_{ij} \, \omega_j.$$

Comments:

- The vectors $\vec{\omega}$ and $\mathbf{L}$ are, in general, not parallel.
- Matrix notation: $\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$
- Kinetic energy: $T = \frac{1}{2} \vec{\omega} \cdot \mathbf{L}$.
- If $\vec{\omega} = \text{const}$ then $\mathbf{L}$ varies $\ \Rightarrow\ \mathbf{N} \neq 0$ (torque).
- If $\mathbf{N} = 0$ then $\mathbf{L} = \text{const}\ \Rightarrow\ \vec{\omega}$ varies.
- If the body frame is along principal axes, then $I_{ij} = I_i \delta_{ij}$.
  $\Rightarrow\ L_i = I_i \omega_i,\ T = \frac{1}{2} \sum_i I_i \omega_i^2$.

Eulerian Angular Velocities [msl26]

The rotation of a rigid body is described by the vector $\vec{\omega}$ of angular velocity. In general, this vector changes magnitude and direction in both coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3).

The most natural formulation of the equations of motion for a rigid body is in the body frame (x_1, x_2, x_3). They are called Euler's equations.

However, the solution is incomplete unless we know how to express the vector $\vec{\omega}$ in the frame (x'_1, x'_2, x'_3), which is typically the frame of the observer.

Eulerian angular velocities:

$\dot{\phi}$ directed along z'-axis.
$\dot{\theta}$ directed along line of nodes.
$\dot{\psi}$ directed along z-axis.

Projections onto axes of (x_1, x_2, x_3):

$$\dot{\psi}_1 = 0,\ \dot{\psi}_2 = 0,\ \dot{\psi}_3 = \dot{\psi}.$$
$$\dot{\theta}_1 = \dot{\theta}\cos\psi,\ \dot{\theta}_2 = -\dot{\theta}\sin\psi,\ \dot{\theta}_3 = 0.$$
$$\dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi,\ \dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi,\ \dot{\phi}_3 = \dot{\phi}\cos\theta.$$

Projections onto axes of (x'_1, x'_2, x'_3):

$$\dot{\phi}'_1 = 0,\ \dot{\phi}'_2 = 0,\ \dot{\phi}'_3 = \dot{\phi}.$$
$$\dot{\theta}'_1 = \dot{\theta}\cos\phi,\ \dot{\theta}'_2 = \dot{\theta}\sin\phi,\ \dot{\theta}'_3 = 0.$$
$$\dot{\psi}'_1 = \dot{\psi}\sin\theta\sin\phi,\ \dot{\psi}'_2 = -\dot{\psi}\sin\theta\cos\phi,\ \dot{\psi}'_3 = \dot{\psi}\cos\theta.$$

Instantaneous angular velocity in the frame (x_1, x_2, x_3): $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$.

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi.$$
$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi.$$
$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}.$$

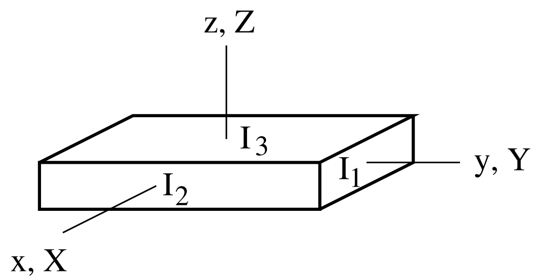
Instantaneous angular velocity in the frame (x'_1, x'_2, x'_3): $\vec{\omega}' = (\omega'_1, \omega'_2, \omega'_3)$.

$$\omega'_1 = \dot{\phi}'_1 + \dot{\theta}'_1 + \dot{\psi}'_1 = \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi.$$
$$\omega'_2 = \dot{\phi}'_2 + \dot{\theta}'_2 + \dot{\psi}'_2 = -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi.$$
$$\omega'_3 = \dot{\phi}'_3 + \dot{\theta}'_3 + \dot{\psi}'_3 = \dot{\psi}\cos\theta + \dot{\phi}.$$

Magnitude of angular velocity: $|\vec{\omega}|^2 = |\vec{\omega}'|^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta.$

[mex174] Rotating rectangular box

A rectangular box with principal moments of inertia $I_1 < I_2 < I_3$ spins with angular velocity $\dot{\alpha}$ about the y-axis of the body frame, which, in turn, rotates with angular velocity $\dot{\beta}$ about the Z-axis of the inertial frame. The origins of both coordinate systems are at the center of mass and the axes of the two systems coincide at time $t = 0$. Find the rotational kinetic energy.



Solution:

Euler's Equations [mln27]

Equation of motion in inertial frame: $\left(\frac{d\mathbf{L}}{dt}\right)_I = \mathbf{N}$.

Equation of motion in (rotating) body frame: $\left(\frac{d\mathbf{L}}{dt}\right)_R + \vec{\omega} \times \mathbf{L} = \mathbf{N}$.

Proof:

$$\mathbf{N} = \frac{d}{dt}\left[\sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha)\right] = \frac{d}{dt}\left[\sum_i L_i \mathbf{e}_i\right] = \frac{d}{dt}\left[\sum_{ij} I_{ij}\omega_j \mathbf{e}_i\right].$$

Use $\dot{\mathbf{e}}_i = \vec{\omega} \times \mathbf{e}_i$. $\Rightarrow$ $\mathbf{N} = \sum_{ij} I_{ij}\dot{\omega}_j \mathbf{e}_i + \vec{\omega} \times \sum_{ij} I_{ij}\omega_j \mathbf{e}_i$.

Choose body frame with principal coordinate axes: $L_i = I_i\omega_i, i = 1, 2, 3$.

Euler's equations:

$$\begin{aligned}
I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= N_1 \\
I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= N_2 \\
I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= N_3
\end{aligned}$$

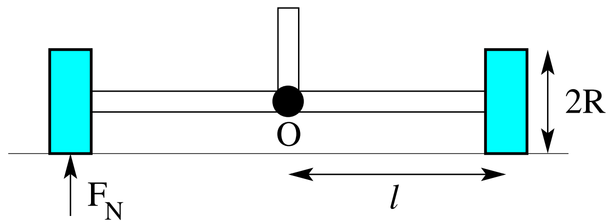
A purely rotating rigid body has 3 degrees of freedom.
The associated Lagrange equations are three 2$^{\text{nd}}$ order ODEs.

The solution via Euler's equations proceeds in two steps:

1. Euler's equations themselves are three 1$^{\text{st}}$ order ODEs for $\omega_1, \omega_2, \omega_3$.
2. The transformation to the inertial frame, $\omega_i = \omega_i(\phi, \theta, \psi; \dot{\phi}, \dot{\theta}, \dot{\psi})$ amounts to solving another three 1$^{\text{st}}$ order ODEs.

[mex175] Heavy wheels

The axes of two wheels (solid disks of mass m and radius R) are free to bend vertically about point O. A motor forces the axes to rotate with angular velocity ω about the vertical axis through O while the two wheels on the ground move in a circle of radius ℓ. Find the normal force F_N between each wheel and the ground by performing the calculation (a) in the inertial frame and (b) in the frame rotating with the vertical axis.



Solution: