12. Rigid Body Dynamics I

Gerhard Müller  
*University of Rhode Island, gmuller@uri.edu*

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Abstract

Part twelve of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

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12. Rigid Body Dynamics I

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System with 6 degrees of freedom (3 translations and 3 rotations).

Equations of motion: \( \ddot{p} = F^{(e)} \), \( \dot{L} = N^{(e)} \) [mln2].

For the description of the rigid-body dynamics it is useful to introduce three coordinate systems:

- inertial coordinate system with axes \((X, Y, Z)\),
- coordinate system with axes \((x', y', z')\) parallel to \((X, Y, Z)\) and origin \(O\) fixed to some point of the rigid body,
- coordinate system with axes \((x, y, z)\) fixed to rigid body and with the same origin \(O\) as \((x', y', z')\).

Motion of rigid body: \( \mathbf{v}_\alpha = \dot{\mathbf{R}} + \mathbf{\omega} \times \mathbf{r}_\alpha \).

- Translational motion of \((x', y', z')\) relative to \((X, Y, Z)\).
- Rotational motion of \((x, y, z)\) relative to \((x', y', z')\).

The optimal choice of the origin \(O\) is dictated by the circumstances:

- for freely rotating rigid bodies, the center of mass is the best choice;
- for rigid bodies rotating about at least one point fixed in the inertial system, one such fixed point is a good choice.

Analysis of rigid body motion:

- Solve the equations of motion in the coordinate system \((x, y, z)\). They are called Euler’s equations [mln27].
- Transform the solution to the coordinate system \((x', y', z')\) via Eulerian angles [msl25], [msl26] and from there to the inertial system \((X, Y, Z)\).
Rotational Kinetic Energy

Consider a rigid body undergoing a purely rotational motion \( \dot{\mathbf{R}} = 0 \).

\[
T = \frac{1}{2} \sum_{\alpha} m_{\alpha} (\vec{\omega} \times \mathbf{r}_{\alpha})^2 = \frac{1}{2} \sum_{\alpha} m_{\alpha} \left[ \omega^2 r_{\alpha}^2 - (\vec{\omega} \cdot \mathbf{r}_{\alpha})^2 \right].
\]

\( \vec{\omega} = (\omega_1, \omega_2, \omega_3) \): instantaneous angular velocity of body frame relative to inertial frame (components in body frame).

\( \mathbf{r}_{\alpha} = (r_{\alpha1}, r_{\alpha2}, r_{\alpha3}) \): position coordinates in body frame.

\[
T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \left[ \left( \sum_{i} \omega_i^2 \right) \left( \sum_{k} r_{\alpha k}^2 \right) - \left( \sum_{i} \omega_i r_{\alpha i} \right) \left( \sum_{j} \omega_j r_{\alpha j} \right) \right].
\]

Use \( \omega_i = \sum_j \omega_j \delta_{ij} \).

\[
T = \frac{1}{2} \sum_{\alpha} \sum_{ij} m_{\alpha} \left[ \omega_i \omega_j r_{\alpha i}^2 - \omega_i \omega_j r_{\alpha i} r_{\alpha j} \right] = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j.
\]

Inertia tensor: \( I_{ij} = \sum_{\alpha} m_{\alpha} \left[ \delta_{ij} r_{\alpha i}^2 - r_{\alpha i} r_{\alpha j} \right] \).

Inertia tensor of cont. mass distrib. \( \rho(\mathbf{r}) \): \( I_{ij} = \int d^3r \, \rho(\mathbf{r}) \left[ \delta_{ij} r^2 - r_i r_j \right] \).

Comments:

- Inertia tensor is symmetric: \( I_{ij} = I_{ji} \).
- Matrix notation: \( T = \frac{1}{2} \left( \omega_1, \omega_2, \omega_3 \right) \cdot \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \).
- \( I_{ij} \) depends on choice of body frame.
- If \( \omega = \omega_i \) then \( T = \frac{1}{2} I_{ii} \omega_i^2 \), where \( I_{ii} \) is called a moment of inertia.
- The use of the body frame guarantees that the inertia tensor is time-independent.

For \( \dot{\mathbf{R}} \neq 0 \) the kinetic energy generally has three parts, translational, mixed, and rotational, which are further discussed in \( \text{mex67}, \text{mex173} \):

\[
T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \left( \dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_{\alpha} \right)^2 = \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{\mathbf{R}}^2 + \sum_{\alpha} m_{\alpha} \dot{\mathbf{R}} \cdot \vec{\omega} \times \mathbf{r}_{\alpha} + \frac{1}{2} \sum_{\alpha} m_{\alpha} (\vec{\omega} \times \mathbf{r}_{\alpha})^2.
\]
Translational and rotational kinetic energies

A rigid body is regarded as composed of particles with mass $m_\alpha$ whose relative positions have constant magnitude. Consider the inertial coordinate system $(X, Y, Z)$ and the coordinate system $(x, y, z)$ with axes fixed to the rigid body. The velocity of particle $\alpha$ as expressed in the inertial frame is $\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha$, where $\mathbf{R}$ is the position of the origin of the body frame measured in the inertial frame, $\vec{\omega}$ is the instantaneous angular velocity of the body frame relative to the inertial frame, and $\mathbf{r}_\alpha$ is the position of particle $\alpha$ in the body frame. (a) Calculate the kinetic energy $T$ of the rigid body in the inertial system. (b) Show that if the origin of the body frame is chosen at the center of mass, then the kinetic energy can be written as the sum of two terms where one represents the translational kinetic energy and the other the rotational kinetic energy.

Solution:
A solid cylinder of mass $m$ and radius $a$ is rolling with angular velocity $\omega$ on a level surface. Calculate the translational, mixed, and rotational parts of the kinetic energy, $T = T_t + T_m + T_r$, from the general formula

$$T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \left( \mathbf{\dot{R}} + \mathbf{\omega} \times \mathbf{r}_{\alpha} \right)^2 = \frac{1}{2} \sum_{\alpha} m_{\alpha} \mathbf{\dot{R}}^2 + \sum_{\alpha} m_{\alpha} \mathbf{R} \cdot \mathbf{\dot{\omega}} \times \mathbf{r}_{\alpha} + \frac{1}{2} \sum_{\alpha} m_{\alpha} (\mathbf{\dot{\omega}} \times \mathbf{r}_{\alpha})^2$$

by using three different body frames with origins at points $A, B, C$, respectively.

Solution:
Principal Axes of Inertia

The inertia tensor $I$ is real and symmetric. Hence it has real eigenvalues. The orthogonal transformation which diagonalizes the inertia tensor is a rotation of the body coordinate axes to the directions of the principal axes.

The eigenvalue problem,

$$ I \cdot \vec{\omega}_k = I_k \vec{\omega}_k, \quad k = 1, 2, 3, $$

amounts to a system of linear, homogeneous equations,

$$ I_{11}\omega_{1k} + I_{12}\omega_{2k} + I_{13}\omega_{3k} = I_1\omega_{1k}, $$
$$ I_{21}\omega_{1k} + I_{22}\omega_{2k} + I_{23}\omega_{3k} = I_2\omega_{2k}, $$
$$ I_{31}\omega_{3k} + I_{32}\omega_{3k} + I_{33}\omega_{3k} = I_3\omega_{3k}, $$

where the principal moments of inertia $I_k$, $k = 1, 2, 3$ are the roots of the characteristic polynomial,

$$ \begin{vmatrix} I_{11} - I_k & I_{12} & I_{13} \\ I_{21} & I_{22} - I_k & I_{23} \\ I_{31} & I_{32} & I_{33} - I_k \end{vmatrix} = 0. $$

The three eigenvectors $\vec{\omega}_k$, $k = 1, 2, 3$ have undetermined magnitude. If normalized, we construct an orthonormal matrix from them:

$$ \Omega = (\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}. $$

In matrix notation the transformation to principal axes looks as follows:

$$ I \cdot \Omega = \Omega \cdot \bar{I} \quad \Rightarrow \quad \Omega^T \cdot I \cdot \Omega = \Omega^T \cdot \Omega \cdot \bar{I} = \bar{I}, $$

where

$$ I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}, \quad \bar{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, $$

$$ \Omega^T \cdot \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$
Parallel-axis theorem

Consider a rigid body composed of particles with mass $m_\alpha$ whose relative positions have constant magnitude. The total mass of the rigid body is $\sum_\alpha m_\alpha = M$. Two body coordinate systems with parallel axes are labelled $(r_1, r_2, r_3)$ and $(R_1, R_2, R_3)$, respectively. The center of mass coordinate of the rigid body is $(0, 0, 0)$ in the former and $(a_1, a_2, a_3)$ in the latter. Show that the inertia tensor of the rigid body in the displaced coordinate system, $I_{ij}$, is obtained from the inertia tensor in the center-of-mass system, $I_{ij}^{(cm)}$, by adding to it the inertia tensor of a point mass $M$ at the position of the center of mass:

$$I_{ij} = I_{ij}^{(cm)} + M \left( a_i^2 \delta_{ij} - a_i a_j \right).$$

Solution:
Perpendicular-axis theorem

Consider a homogeneous sheet of material which has infinitesimal thickness $dx_3$ and some arbitrary shape in the $(x_1, x_2)$-plane. (a) Prove the following relation between the moments of inertia for rotations about the coordinate axes: $I_{11} + I_{22} = I_{33}$. (b) Use the parallel-axis and perpendicular-axis theorems to calculate the principal moments of inertia $I_{11}, I_{22}, I_{33}$ of a coin (mass $M$, radius $R$) for rotations about the axes of the following coordinate system: The origin is at the rim of the coin. The $x_1$-axis is radial toward the center of the coin, the $x_2$-axis is tangential to the coin, and the $x_3$-axis is perpendicular to the plane of the coin.

Solution:
Consider a cube of mass $M$ and side $L$ and a coordinate system with origin at one corner of the cube and axes along the adjacent edges. (a) Calculate the inertia tensor $I_{ij}$ for this body reference frame. (b) Determine the principal moments of inertia $I_1, I_2, I_3$. (c) Determine the directions of the principal axes of inertia relative to the coordinate system used originally.

**Solution:**
Principal moments of solid cylinder

Calculate the principal moments of inertia $I_1, I_2, I_3$ of a solid cylinder as functions of its mass $M$, its radius $R$, and its height $h$ for rotations about the center of mass.

Solution:
[mex253] Principal moments of a solid sphere.

Calculate the principal moments of inertia $I_1$, $I_2$, $I_3$ of a solid sphere as a function of its mass $M$ and its radius $R$ for rotations about the center of mass.

Solution:
[mex254] Principal moments of a solid ellipsoid

Calculate the principal moments $I_1, I_2, I_3$ of a solid ellipsoid as functions of its mass $M$ and its semi-axes $a, b, c$ for rotations about the center of mass.

Solution:
Consider a four-atomic molecule with the atoms at the corners of a pyramid. The base is an equilateral triangle of side $a$. The height is $h$. The three atoms at the base each have mass $m_1$ and the atom at the tip has mass $m_2$.

(a) Find the principal moments of inertia $I_1, I_2, I_3$ as functions of $m_1, m_2, a, h$ for rotations about the center of mass.

(b) Identify the simplifications that occur when the pyramid is a regular tetrahedron of side $a$ and all four masses are equal.

Solution:
Inertia tensor of a cone

Calculate the principal moments of inertia for a homogeneous cone of mass $M$, height $h$, and radius $R$ at the base. Perform the calculation for rotations about an axis (a) through the apex of the cone, (b) through the center of mass. Express all results as functions of $M, R, h$.

Solution:
Simulating a stick by three point masses

Consider a nonuniform rod with mass $m$ and moment of inertia $I_0$ for rotations about an axis through the center of mass and perpendicular to the axis of the rod. The moment of inertia for rotations about a parallel axis displaced by $x$ is then $I_x = I_0 + mx^2$.

Show that three point masses $m_0, m_1, m_2$ with $m_0 + m_1 + m_2 = m$ in the configuration shown can be chosen such that its moment of inertia for rotations about an axis through $m_0$ is $I_0$ and that for rotations about a parallel axis displaced by $x$ is $I_x$ just as is the case for the rod. Express the values of $m_0, m_1, m_2$ as functions of the specifications $m, I_0, a, b$ of the rod.

Solution:
Angular Momentum

\[ L_{\text{tot}} = \sum_{\alpha} m_{\alpha} (R + r_{\alpha}) \times \left( \dot{R} + \vec{\omega} \times r_{\alpha} \right). \]

If the rigid body rotates freely, choose the origin of the body frame at the center of mass: \( \sum_{\alpha} m_{\alpha} r_{\alpha} = 0, \quad R = r_{\text{cm}}, \quad \dot{R} = v_{\text{cm}}. \)

\[ L_{\text{tot}} = m r_{\text{cm}} \times v_{\text{cm}} + \sum_{\alpha} m_{\alpha} r_{\alpha} \times (\vec{\omega} \times r_{\alpha}) = L_{\text{orb}} + L_{\text{spin}}. \]

If the rigid body rotates about a fixed point in the inertial frame, choose the origin of the body and inertial frames at the fixed point: \( R = 0, \quad \dot{R} = 0. \)

\[ L_{\text{tot}} = \sum_{\alpha} m_{\alpha} r_{\alpha} \times (\vec{\omega} \times r_{\alpha}) = L_{\text{spin}}. \]

Consider a rigid body undergoing a purely rotational motion (\( \dot{R} = 0 \)).

\[ \mathbf{L} = \sum_{\alpha} m_{\alpha} r_{\alpha} \times (\vec{\omega} \times r_{\alpha}) = \sum_{\alpha} m_{\alpha} \left[ r_{\alpha}^2 \vec{\omega} - r_{\alpha} (r_{\alpha} \cdot \vec{\omega}) \right]. \]

Use body frame components \( \omega_1, \omega_2, \omega_3 \) and \( r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3} \):

\[ L_i = \sum_{\alpha} m_{\alpha} \left[ r_{\alpha i}^2 \omega_i - r_{\alpha i} \sum_j r_{\alpha j} \omega_j \right] = \sum_{\alpha} m_{\alpha} \sum_j \left[ \omega_j \delta_{ij} r_{\alpha j}^2 - \omega_j r_{\alpha i} r_{\alpha j} \right]. \]

\[ L_i = \sum_j \omega_j \sum_{\alpha} m_{\alpha} \left[ \delta_{ij} r_{\alpha j}^2 - r_{\alpha i} r_{\alpha j} \right] = \sum_j I_{ij} \omega_j. \]

Comments:

- The vectors \( \vec{\omega} \) and \( \mathbf{L} \) are, in general, not parallel.
- Matrix notation:

\[
\begin{pmatrix}
L_1 \\
L_2 \\
L_3
\end{pmatrix} =
\begin{pmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{pmatrix}
\cdot
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}.
\]
- Kinetic energy: \( T = \frac{1}{2} \vec{\omega} \cdot \mathbf{L}. \)
- If \( \vec{\omega} = \text{const} \) then \( \mathbf{L} \) varies \( \Rightarrow \mathbf{N} \neq 0 \) (torque).
- If \( \mathbf{N} = 0 \) then \( \mathbf{L} = \text{const} \) \( \Rightarrow \vec{\omega} \) varies.
- If the body frame is along principal axes, then \( I_{ij} = I_i \delta_{ij}. \)

\[ L_i = I_i \omega_i, \quad T = \frac{1}{2} \sum_i I_i \omega_i^2. \]
Eulerian Angular Velocities

The rotation of a rigid body is described by the vector $\vec{\omega}$ of angular velocity. In general, this vector changes magnitude and direction in both coordinate systems $(x_1, x_2, x_3)$ and $(x'_1, x'_2, x'_3)$.

The most natural formulation of the equations of motion for a rigid body is in the body frame $(x_1, x_2, x_3)$. They are called Euler’s equations.

However, the solution is incomplete unless we know how to express the vector $\vec{\omega}$ in the frame $(x'_1, x'_2, x'_3)$, which is typically the frame of the observer.

Eulerian angular velocities:

- $\dot{\phi}$ directed along $z'$-axis.
- $\dot{\theta}$ directed along line of nodes.
- $\dot{\psi}$ directed along $z$-axis.

Projections onto axes of $(x_1, x_2, x_3)$:

- $\dot{\psi}_1 = 0, \dot{\psi}_2 = 0, \dot{\psi}_3 = \dot{\psi}$.
- $\dot{\theta}_1 = \dot{\theta} \cos \psi, \dot{\theta}_2 = -\dot{\theta} \sin \psi, \dot{\theta}_3 = 0$.
- $\dot{\phi}_1 = \dot{\phi} \sin \theta \sin \psi, \dot{\phi}_2 = \dot{\phi} \sin \theta \cos \psi, \dot{\phi}_3 = \dot{\phi} \cos \theta$.

Projections onto axes of $(x'_1, x'_2, x'_3)$:

- $\dot{\phi}'_1 = 0, \dot{\phi}'_2 = 0, \dot{\phi}'_3 = \dot{\phi}$.
- $\dot{\theta}'_1 = \dot{\theta} \cos \phi, \dot{\theta}'_2 = \dot{\theta} \sin \phi, \dot{\theta}'_3 = 0$.
- $\dot{\psi}'_1 = \dot{\psi} \sin \theta \sin \phi, \dot{\psi}'_2 = -\dot{\psi} \sin \theta \cos \phi, \dot{\psi}'_3 = \dot{\psi} \cos \theta$.

Instantaneous angular velocity in the frame $(x_1, x_2, x_3)$: $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$.

- $\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$.
- $\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$.
- $\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi} \cos \theta + \dot{\psi}$.

Instantaneous angular velocity in the frame $(x'_1, x'_2, x'_3)$: $\vec{\omega}' = (\omega'_1, \omega'_2, \omega'_3)$.

- $\omega'_1 = \dot{\phi}'_1 + \dot{\theta}'_1 + \dot{\psi}'_1 = \dot{\phi} \sin \theta \sin \phi + \dot{\theta} \cos \phi$.
- $\omega'_2 = \dot{\phi}'_2 + \dot{\theta}'_2 + \dot{\psi}'_2 = -\dot{\psi} \sin \theta \cos \phi + \dot{\theta} \sin \phi$.
- $\omega'_3 = \dot{\phi}'_3 + \dot{\theta}'_3 + \dot{\psi}'_3 = \dot{\psi} \cos \theta + \dot{\phi}$.

Magnitude of angular velocity: $|\vec{\omega}|^2 = |\vec{\omega}'|^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi} \cos \theta$. 

Rotating rectangular box

A rectangular box with principal moments of inertia $I_1 < I_2 < I_3$ spins with angular velocity $\dot{\alpha}$ about the $y$-axis of the body frame, which, in turn, rotates with angular velocity $\dot{\beta}$ about the $Z$-axis of the inertial frame. The origins of both coordinate systems are at the center of mass and the axes of the two systems coincide at time $t = 0$. Find the rotational kinetic energy.

Solution:
Euler’s Equations

Equation of motion in inertial frame: \( \left( \frac{dL}{dt} \right)_I = N \).

Equation of motion in (rotating) body frame: \( \left( \frac{dL}{dt} \right)_B + \vec{\omega} \times L = N \).

Proof:

\[
N = \frac{d}{dt} \left[ \sum m_a r_a \times (\vec{\omega} \times r_a) \right] = \frac{d}{dt} \left[ \sum L_i e_i \right] = \frac{d}{dt} \left[ \sum I_{ij} \omega_j e_i \right].
\]

Use \( \dot{e}_i = \vec{\omega} \times e_i \). \( \Rightarrow \) \( N = \sum_{ij} I_{ij} \dot{\omega}_j e_i + \vec{\omega} \times \sum_{ij} I_{ij} \omega_j e_i. \)

Choose body frame with principal coordinate axes: \( L_i = I_i \omega_i, i = 1, 2, 3. \)

Euler’s equations:

\[
\begin{align*}
I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) &= N_1 \\
I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) &= N_2 \\
I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) &= N_3
\end{align*}
\]

A purely rotating rigid body has 3 degrees of freedom. The associated Lagrange equations are three 2\textsuperscript{nd} order ODEs.

The solution via Euler’s equations proceeds in two steps:

1. Euler’s equations themselves are three 1\textsuperscript{st} order ODEs for \( \omega_1, \omega_2, \omega_3. \)
2. The transformation to the inertial frame, \( \omega_i = \omega_i(\phi, \theta, \psi; \dot{\phi}, \dot{\theta}, \dot{\psi}) \) amounts to solving another three 1\textsuperscript{st} order ODEs.
[mex175] Heavy wheels

The axes of two wheels (solid disks of mass $m$ and radius $R$) are free to bend vertically about point $O$. A motor forces the axes to rotate with angular velocity $\omega$ about the vertical axis through $O$ while the two wheels on the ground move in a circle of radius $\ell$. Find the normal force $F_N$ between each wheel and the ground by performing the calculation (a) in the inertial frame and (b) in the frame rotating with the vertical axis.

Solution: