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11. Electric potential of conductors, electric dipole, and point-charge configurations

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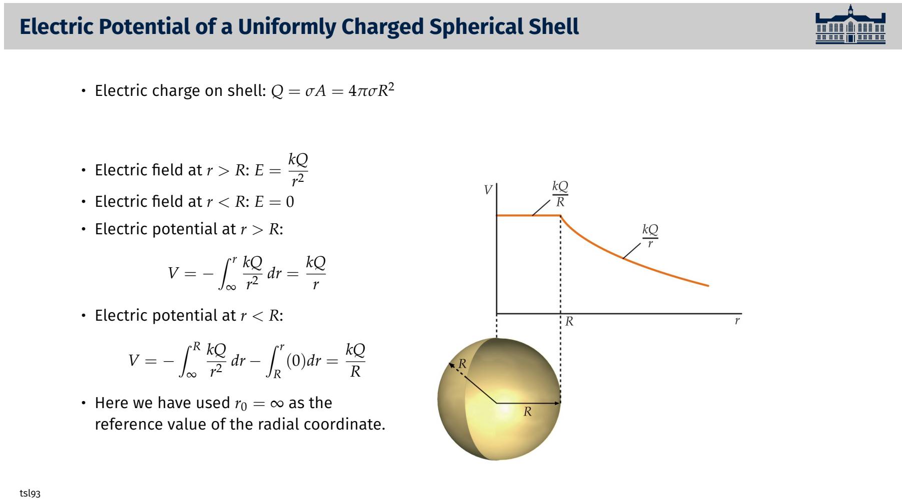
PHY204 Lecture 11 [rln11]



The first part of this lecture is about electric potential on and around electric conductors at equilibrium. We recall from lecture 6 that there is no electric field inside the conducting material. Hence the integral of $\vec{E} \cdot d\vec{s}$ along a path through conducting material vanishes. We conclude that the electric potential at all points on a conductor at equilibrium is the same.

From the previous lecture we know that the electric potential of a uniformly charged shell is that of a point charge on the outside and a constant on the inside as stated on the slide.

The electric potential inside and outside a solid conducting sphere is exactly the same. In both cases, the excess charge Q is all at radius R and there is no electric field at $r < R$.

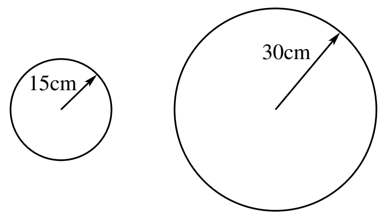
The expression for the electric potential uses a reference point out at infinity, $r_0 = \infty$. All that matters in applications are potential differences.

When we calculate potentials for two charged conductors, say V_1 and V_2, the physically relevant potential difference, $\Delta V = V_1 - V_2$, is only correct if the same reference point has been used in the calculation of V_1 and V_2. This is an important point that we must keep in mind.

Electric Potential of Conducting Spheres (2)

Consider a conducting sphere with radius $r = 15$cm and electric potential $V = 200$V relative to a point at infinity.

(a) Find the charge Q and the surface charge density σ on the sphere.
(b) Find the magnitude of the electric field E just outside the sphere.
(c) What happens to the values of Q, V, σ, E when the radius of the sphere is doubled?



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In this little exercise we consider, say, a balloon made of an elastic material that is an electric conductor. Its initial radius is given as well as its potential. Note the mention of the reference point.

(a) We use the result of the previous page to calculate the charge on the balloon.

$$V = \frac{kQ}{r} \quad \Rightarrow \; Q = \frac{Vr}{k} = \frac{(200\text{V})(0.15\text{m})}{9 \times 10^9 \text{Vm/C}} = 3.33 \times 10^{-9}\text{C}.$$

The spherical symmetry ensures that the charge is uniformly distributed:

$$\sigma = \frac{Q}{4\pi r^2} = \frac{3.33 \times 10^{-9}\text{C}}{4\pi(0.15\text{m})^2} = 1.18 \times 10^{-8}\text{C/m}^2.$$

(b) To determine the electric field just outside the conducting surface, we pull another result from lecture 6 that we derived using Gauss's law:

$$E = \frac{\sigma}{\epsilon_0} = \frac{1.18 \times 10^{-8}\text{C/m}^2}{8.85 \times 10^{-12}\text{C}^2\text{N}^{-1}\text{m}^{-2}} = 1.33 \times 10^3\text{N/C}.$$

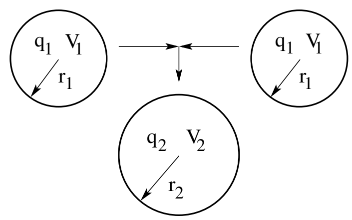
(c) When we inflate the balloon to twice its original radius, the surface area goes up by a factor of four. What happens to the quantities listed? The charge does not change. The potential, which is inversely proportional to the radius of the sphere decreases to half its value. The same charge now being spread over four times the original surface area reduces the surface charge density to one quarter of the original value. The electric field, which is proportional to the surface charge density, decreases by the same factor.

Electric Potential of Conducting Spheres (3)

A spherical raindrop of 1mm diameter carries a charge of 30pC.

(a) Find the electric potential of the drop relative to a point at infinity under the assumption that it is a conductor.

(b) If two such drops of the same charge and diameter combine to form a single spherical drop, what is its electric potential?



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What happens to the electric potential of tiny electrically charged water droplets, the constituents of mist and clouds, when they collide and merge into bigger droplets? The answer is of meteorological relevance.

Consider two such droplets, each of radius $r_1 = 5 \times 10^{-4}$m and carrying a charge $q_1 = 3.0 \times 10^{-11}$C. The electric potential of each droplet (a conducting sphere), is

$$V_1 = \frac{kq_1}{r_1} = 540\text{V}.$$

When the droplets collide and merge, the charge $q_2 = 2q_1$ is now spread over the surface of a sphere with twice the volume, implying that the radius is about 25% larger:

$$\frac{4\pi}{3} r_2^3 = 2\,\frac{4\pi}{3} r_1^3 \quad \Rightarrow \quad r_2 = 2^{1/3} r_1 \simeq 1.26\, r_1.$$

This has the consequence that the electric potential goes up significantly:

$$V_2 = \frac{kq_2}{r_2} = 857\text{V}.$$

Charged water droplets are a common occurrence in the atmosphere under humid conditions as a result of ionization processes caused by solar radiation. Large potential differences between clouds and the ground can build up. Strong electric fields associated with these potential differences can trigger events of fast electric discharge (lightning).

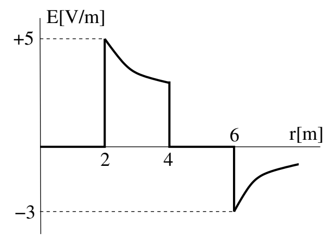
Electric Potential of Conducting Spheres (1)

A conducting sphere of radius $r_1 = 2\text{m}$ is surrounded by a concentric conducting spherical shell of radii $r_2 = 4\text{m}$ and $r_3 = 6\text{m}$. The graph shows the electric field $E(r)$.

(a) Find the charges q_1, q_2, q_3 on the three conducting surfaces.

(b) Find the values V_1, V_2, V_3 of the electric potential on the three conducting surfaces relative to a point at infinity.

(c) Sketch the potential $V(r)$.



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Here we revisit a configuration that we earlier analyzed in the context of Gauss's law applied to conductors with spherical symmetry: a conducting sphere surrounded by a conducting spherical shell (lecture 7). From what is given, we can reason as follows:

(a) The given electric field $E_1 = 5\text{V/m}$ determines the surface charge density on the sphere, which, in turn, determines the charge q_1:

$$\sigma_1 = \epsilon_0 E_1 = 4.41 \times 10^{-11}\text{C/m}^2 \quad \Rightarrow \quad q_1 = (4\pi r_1^2)\sigma_1 = 2.22 \times 10^{-9}\text{C}.$$

Gauss's law then dictates that the charge q_2 on the inner surface of the shell must be $q_2 = -q_1 = -2.22 \times 10^{-9}\text{C}$. We infer the charge q_3 on the outer surface of the shell from the given electric field $E_3 = -3\text{V/m}$:

$$\sigma_3 = \epsilon_0 E_3 = -2.66 \times 10^{-11}\text{C/m}^2 \quad \Rightarrow \quad q_3 = (4\pi r_3^2)\sigma_3 = -1.20 \times 10^{-8}\text{C}.$$

(b) Here we apply the method of calculating electric potential from electric field practiced in the previous lecture. We start at the reference point $r_0 = \infty$ and integrate first to r_3, then continue to r_2, and finally to r_1.

We have already carried out the first integral on page 1 of this lecture: $V_3 = kq_3/r_3 = -18.0\text{V}$. The electric potential does not change inside the conducting material of the shell. Hence we conclude that $V_2 = V_3 = -18.0\text{V}$.

The electric field between r_2 and r_1 is $E(r) = kq_1/r^2$, which we use to continue our integral:

$$V_1 = V_2 - kq_1 \int_{r_2}^{r_1} \frac{dr}{r^2} = V_2 + kq_1 \left[\frac{1}{r_1} - \frac{1}{r_2}\right] = 18.0\text{V} + 5.00\text{V} = -13.0\text{V}.$$

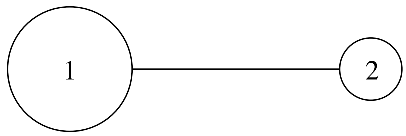
The potential does not change inside the conducting sphere.

Electric Potential of Conducting Spheres (4)

A positive charge is distributed over two conducting spheres 1 and 2 of unequal size and connected by a long thin wire. The system is at equilibrium.

Which sphere (1 or 2)...

(a) carries more charge on its surface?
(b) has the higher surface charge density?
(c) is at a higher electric potential?
(d) has the stronger electric field next to it?



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We finish (for now) our discussion of electric potential in the context of conductors at equilibrium with this quiz-like exercise.

The key relations for charged spherical conductors of radius R that we should have have at our fingertips are the following:

$$V = \frac{kQ}{R}, \quad E = \frac{kQ}{R^2}, \quad E = \frac{\sigma}{\epsilon_0}, \quad Q = (4\pi R^2)\sigma.$$

The first of five questions is the following: which of the four remaining questions should we tackle first? The answer is: question (c).

(c) The two spheres plus the wire are one conductor. Hence the electric potential is the same for both spheres: $V_1 = V_2$.

(a) The equality of the potentials implies that the bigger sphere carries more charge. We reason as follows:

$$V_1 = V_2 \quad \Rightarrow \frac{kQ_1}{R_1} = \frac{kQ_2}{R_2} \quad \Rightarrow \frac{Q_1}{Q_2} = \frac{R_1}{R_2} > 1.$$

(b) While the bigger sphere carries more charge, the smaller sphere has the higher surface charge density:

$$\frac{\sigma_1}{\sigma_2} = \frac{Q_1/4\pi R_1^2}{Q_2/4\pi R_2^2} = \frac{Q_1}{Q_2}\frac{R_2^2}{R_1^2} = \frac{R_2}{R_1} < 1.$$

(d) The electric field just outside the sphere is proportional to the surface charge density. Hence the smaller sphere has the stronger field.

Electric Potential Energy of Two Point Charges

Consider two different perspectives:

#1a Electric potential when q_1 is placed: $V(\vec{r}_2) \doteq V_2 = k\frac{q_1}{r_{12}}$

Electric potential energy when q_2 is placed into potential V_2: $U = q_2 V_2 = k\frac{q_1 q_2}{r_{12}}$

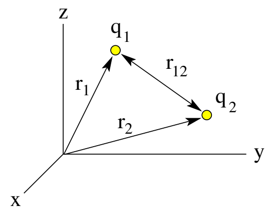
#1b Electric potential when q_2 is placed: $V(\vec{r}_1) \doteq V_1 = k\frac{q_2}{r_{12}}$

Electric potential energy when q_1 is placed into potential V_1: $U = q_1 V_1 = k\frac{q_1 q_2}{r_{12}}$.

#2 Electric potential energy of q_1 and q_2:

$$U = \frac{1}{2}\sum_{i=1}^{2} q_i V_i,$$

where $V_1 = k\frac{q_2}{r_{12}}$, $V_2 = k\frac{q_1}{r_{12}}$.



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What is the electric potential energy of a configuration of point charges in some region of space? We begin the analysis of this question with the case of two point charges q_1 and q_2 using the tools developed in lecture 8.

When we first place charge q_1 at position $\vec{r}_1$, it generates an electric potential around it. At position $\vec{r}_2$, where we next place charge q_2, this potential has the value V_2. We have learned that when we place charge q_2 into the potential V_2, it has potential energy $U_2 = q_2 V_2$.

Conversely, when we first place charge q_2 at position $\vec{r}_2$, it generates a potential around it. At position $\vec{r}_1$, where we next place charge q_1, this potential has the value V_1. Charge q_1 has potential energy $U_1 = q_1 V_1$.

We are not surprised to see that the result for the potential energy,

$$U_1 = U_2 \doteq U = k\,\frac{q_1 q_2}{r_{12}},$$

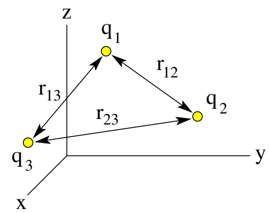
bears no trace of the sequence in which the charges are placed into the region of space, which justifies the declaration that it represents the potential energy of a configuration of two point charges at equilibrium, plain and simple.

In preparation of a generalization to more than two point charges, we rewrite this expression in a more symmetric form as stated near the bottom of the slide. In this rendition, each particle generates a potential and each particle has a potential energy. The over-counting of contributions is compensated by the factor $\frac{1}{2}$.

Electric Potential Energy of Three Point Charges

#1 Place q_1, then q_2, then q_3, and add all changes in potential energy:
$$U = 0 + k\frac{q_1q_2}{r_{12}} + k\left(\frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) = k\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right).$$

#2 Symmetric expression of potential energy U in terms of the potentials V_i experienced by point charges q_1:
$$U = \frac{1}{2}\sum_{i=1}^{3} q_iV_i = k\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right),$$
where
$$V_1 = k\left(\frac{q_2}{r_{12}} + \frac{q_3}{r_{13}}\right),$$
$$V_2 = k\left(\frac{q_1}{r_{12}} + \frac{q_3}{r_{23}}\right),$$
$$V_3 = k\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right).$$



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Here the generalized expression for the electric potential energy,

$$U = \frac{1}{2}\sum_{i=1}^{N} q_iV_i, \quad V_i = k\sum_{j\neq i}\frac{q_j}{r_{ij}},$$

of a configuration of N charged particles at equilibrium is illustrated for the case $N = 3$. The expression is arrived at in two different ways.

In item #1 we first place charge q_1, which generates a potential for charges q_2 and q_3. Its potential energy is zero because it is placed into zero potential.

Next we place charge q_2 into the potential generated by charge q_1. Its potential energy is nonzero. It also generates a potential for charge q_3.

Finally, we place charge q_3. Its potential energy has two terms, contributed by the potentials generated q_1 and q_2 placed previously. The total potential energy has $0 + 1 + 2 = 3$ terms as shown.

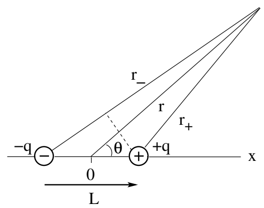
In item #2 we use the above expression. The V_i in that expression represents the potential experienced by charge q_i as if it is placed last. Therefore, each of the three V_i has two terms. They add up to $3\times 2 = 6$ terms, which combine into 3 pairs of identical terms. The factor $\frac{1}{2}$ takes care of the over-counting.

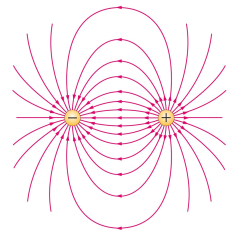
The final expression is the same for both ways of counting contributions.

The expression of electric potential energy developed here omits *self-energy* terms. We merely mention this as a caveat here. We are not yet ready to discuss self-energies of charged particles. When we compare the potential energy of point charges in different (static) configurations, self-energies have no impact.

Electric Dipole Potential

- Use spherical coordinates: $V = V(r,\theta)$ independent of azimuthal coordinate ϕ.
- Superposition principle: $V = V_+ + V_- = k\left(\frac{q}{r_+} + \frac{(-q)}{r_-}\right) = kq\,\frac{r_- - r_+}{r_- r_+}$
- Large distances $(r \gg L)$: $r_- - r_+ \simeq L\cos\theta,\ r_- r_+ \simeq r^2 \Rightarrow\ V(r,\theta) \simeq k\,\frac{qL\cos\theta}{r^2}$
- Electric dipole moment: $p = qL$ (magnitude)
- Electric dipole potential: $V(r,\theta) \simeq k\,\frac{p\cos\theta}{r^2}$





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We have introduced the electric dipole in lecture 3 with focus on the electric field generated by it. On the slide, the electric-dipole field is visualized by a set of field lines. Here we continue the discussion of the electric dipole with focus on the electric potential generated around it.

It is simple enough the calculate the electric potential of two opposite point charges $\pm q$ positioned a distance L from each other.

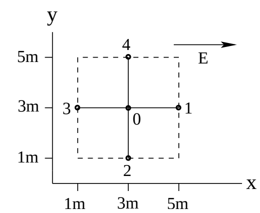
Most dipoles of interest in physics, chemistry, engineering, and biology are of molecular origin with L in the nanometer range. Therefore, in most applications we are interested in the potential at distances much larger than the size of the molecule, $r \gg L$.

A simplified asymptotic expression $V(r,\theta)$ for the electric potential of an electric dipole $\vec{p} = q\vec{L}$ is worked out on the slide. Note that θ is the angle between the vector $\vec{L}$, which points from the negative to the positive charge, and the vector $\vec{r}$ which points from the dipole (source point) to the field point.

Molecular dipoles have complex charge distributions. Calculating the dipole moment $\vec{p}$ from it is more complicated. Tabulated data exist for its magnitude and its direction relative to the atomic positions on the molecule. The only thing that goes into the expression for the electric potential is the vector $\vec{p}$.

Consider a region of space with a uniform electric field $\mathbf{E} = 0.5\text{V/m}\,\hat{\mathbf{i}}$. Ignore gravity.

(a) If the electric potential vanishes at point 0, what are the electric potentials at points 1 and 2?
(b) If an electron ($m = 9.11 \times 10^{-31}$kg, $q = -1.60 \times 10^{-19}$C) is released from rest at point 0, toward which point will it start moving?
(c) What will be the speed of the electron when it gets there?



Solution:

(a) $V_1 = -(0.5\text{V/m})(2\text{m}) = -1\text{V}, \quad V_2 = 0.$
(b) $\mathbf{F} = q\mathbf{E} = -|qE|\hat{\mathbf{i}}$ (toward point 3).
(c) $\Delta V = (V_3 - V_0) = 1\text{V}, \quad \Delta U = q\Delta V = -1.60 \times 10^{-19}\text{J},$
$K = -\Delta U = 1.60 \times 10^{-19}\text{J}, \quad v = \sqrt{\frac{2K}{m}} = 5.93 \times 10^5\text{m/s}.$
Alternatively:
$F = qE = 8.00 \times 10^{-20}\text{N}, \quad a = \frac{F}{m} = 8.78 \times 10^{10}\text{m/s}^2,$
$|\Delta x| = 2\text{m}, \quad v = \sqrt{2a|\Delta x|} = 5.93 \times 10^5\text{m/s}.$

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We conclude this lecture with two simple applications of electric potential previously used as exam problems.

We begin with a region of uniform electric field. The mention of this specification rings a few bells that tell us in which compartment of our toolbox to look for the necessary resources.

Uniform field means constant force on charged particles, which, in turn, means motion with constant acceleration.

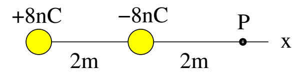
We have figured out the relation between a uniform field and the associated potential early on in lecture 8.

Part (c) we can solve by either using energy conservation or motion with constant acceleration.

Unit Exam I: Problem #1 (Spring '09)

Consider two point charges positioned on the x-axis as shown.

(a) Find magnitude and direction of the electric field at point P.
(b) Find the electric potential at point P.
(c) Find the electric potential energy of an electron (mass $m = 9.1 \times 10^{-31}$kg, charge $q = -1.6 \times 10^{-19}$C) when placed at point P.
(d) Find magnitude and direction of the acceleration the electron experiences when released at point P.



Solution:

(a) $E_x = +k\dfrac{8\text{nC}}{(4\text{m})^2} + k\dfrac{(-8\text{nC})}{(2\text{m})^2} = 4.5\text{N/C} - 18\text{N/C} = -13.5\text{N/C}$ (directed left).

(b) $V = +k\dfrac{8\text{nC}}{4\text{m}} + k\dfrac{(-8\text{nC})}{2\text{m}} = 18\text{V} - 36\text{V} = -18\text{V}$.

(c) $U = qV = (-18\text{V})(-1.6 \times 10^{-19}\text{C}) = 2.9 \times 10^{-18}\text{J}$.

(d) $a_x = \dfrac{qE_x}{m} = \dfrac{(-1.6 \times 10^{-19}\text{C})(-13.5\text{N/C})}{9.1 \times 10^{-31}\text{kg}} = 2.4 \times 10^{12}\text{ms}^{-2}$ (directed right).

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The main difference from the previous situation is that here we are dealing with an electric field that is not uniform. Therefore, we go to different compartments in our mental toolbox.

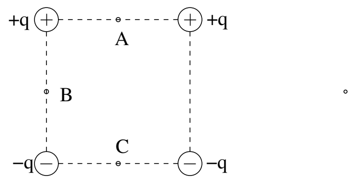
Parts (a) through (c) could not be more elementary. Explanations are hardly needed.

The acceleration is no longer constant when something moves. Nevertheless, the relations between acceleration, force, and electric field remain the same. Those are all we need in part (d)

Electric Potential and Potential Energy: Application (9)

Consider four point charges of equal magnitude positioned at the corners of a square as shown. Answer the following questions for points A, B, C.

(1) Which point is at the highest electric potential?
(2) Which point is at the lowest electric potential?
(3) At which point is the electric field the strongest?
(4) At which point is the electric field the weakest?



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This is the quiz for lecture 11.

In one of the four questions, the answer is a tie between two points.