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Like Terms: What's in a Name?

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Abstract:

This article discusses a lesson in which students' knowledge of operations with number and the base-10 place value system was connected with operations with algebraic polynomial expressions.

Keywords:

Algebra, Conceptual Development

Additional Keywords:

Mathematical Structure

On the white board:

$$\begin{array}{r} 4x^2+5x+3 \\ 3x^2+4x+6 \\ +\underline{\qquad\qquad} \end{array}$$

Teacher: Look x^2 is a double cheeseburger, and x is a cheeseburger. So how many double cheeseburgers do you have?

Students: Four

Teacher: How many double cheeseburgers do you add?

Students: Three

Teacher: How many double cheeseburgers do you have in all?

Students: Seven.

Teacher: Great! Get it?

In introductory algebra courses (and later) students seem to struggle with the concepts of like terms and combining like terms with algebraic expressions. These ideas appear to be unfamiliar to students and do not seem to relate to anything they have done in the past. In an attempt to link with something the students do know, teachers may resort to pseudo-explanations or contexts that are not mathematical in nature and which frequently are incorrect concretizations or contextualizations. These concretizations (e.g. different fruits, animals, and foods) incorrectly interpret part-part-whole structures of certain addition (and subtraction) problems and do not address the idea that an algebraic expression (or term) may contain variable quantities. In the above example, we can certainly join *single* cheeseburgers (part) and *double* cheeseburgers (part) in one collection of cheeseburgers (whole) for the given context, because the expressions x and x^2 are used to identify names of an object and not for identifying variable quantities.

While this approach of using letters to assign attributes of objects or the objects themselves may, at some point, get some students to combine like terms in limited circumstances, they will be misunderstanding what they are doing. It is not linked with any mathematical structure from their prior knowledge; rather a pseudo-mathematical explanation is given (Liping Ma, 1999). We are not advocating against using context. Proper employment of context has shown to be a fruitful bridge between arithmetic and algebra (Tabach and Friedlander, 2008). However, the context must help students develop structure sense, (Lichevski and Livneh, 1999; Lichevski and Herscovic, 1996). Developing structure sense is given an important place in both the process standards (NCTM, 2000) and the mathematical practices (CCSSI, 2010). In this article we wish to share the collaborative attempt of a student teacher and her supervisor to support instructing and learning the concept of combining like terms in algebra with the structural understandings of arithmetic. We realize that this is one of several choices. For a comprehensive treatise of the different perspectives that can be taken toward school algebra we recommend Chazan (2000).

Building on What We Know

The Common Core State Standards for Mathematics (CCSSI, 2010) explicitly point out in many places, especially in the K-8 standards, that the learning of new structures should be presented as an extension of prior (fundamental) structures. For example, the learning of fractions and rational numbers is built on the structure of whole numbers and integers, so that consistency with operations of numbers is preserved. Furthermore, the Standard for Mathematical Practice #7 makes looking for and making use of structure important at all grade levels within the Common Core. It is our contention in this article that secondary teachers of mathematics need to look for and make use of structures in the K-8 standards that can be extended to more generalized ideas in algebra. In this article we will present one example of how we, a student teacher (Author 2) and her university supervisor (Author 1), considered this idea in teaching an introductory lesson on polynomial expressions and operations with polynomial expressions.

In this article we will present the phases of the lesson that we co-constructed during the pre-lesson discussion, share how this lesson panned out, and reveal parts of the student teacher's reflection to represent her thinking and learning in hind-sight. It is our hope that the method described will be of use to teachers of algebra. The student teacher wrote a post-observation reflection of this lesson. In this reflection, she explains her initial concern expressed to her supervisor with regard to how students were learning about like terms in her algebra 1 class:

> I originally wanted to open my lesson on addition and subtraction of polynomials by referring back to solving systems of linear equations. I wanted to stress that when using the various solution methods one adds or subtracts like terms. During my lesson I was going to represent the different variables in the examples as different fruits. I knew this was not mathematically correct but this is what my cooperating teacher has always done in the past so I did not want to override her way of teaching addition and subtraction of polynomials. But it just did not feel right.

To address this concern we began by *finding a place of competence* in Author 2's students' past experiences. We determined that her students were comfortable and skilled with the base-10 place-value system through which they learned the fundamental structure of numeric polynomial expressions and how to add and subtract these (i.e. combine like terms). We can consider a multi-digit number, e.g. 957 as a numeric polynomial expression by expanding it as $9 \times 10^2 + 5 \times 10 + 7$. Author 2 articulated this thinking in her post-lesson reflection as follows:

> Based on my misgivings, I decided to have a discussion with my supervisor about what addition and subtraction of polynomials actually meant. We talked about how students have been adding and subtracting polynomials since second grade by thinking of numbers as polynomials, i.e. objects that have "many names." This idea of objects with more than one name became a big aha moment for me and I decided with my supervisor to completely change the launch phase of the lesson to establish this context for polynomials and operations with polynomials.

The context for this work was established by beginning with adding multi-digit numbers in an attempt to develop the idea that these can be written as numeric polynomial expressions providing an algebraic proto-structure for addition (and subtraction). We can write a multi-digit number in polynomial form by writing it in expanded scientific notation (see example for 957 above). We will refer to multi-digit numbers as numeric polynomials in this article. Each of the steps of the instructional process that follows were recorded on the cue cards we generated in preparation.

Step 1: Establishing competence

The student teacher began by putting the three-digit number 576 on the board. She asked the students what the value was of each of the digits. She then emphasized that they identified the value by naming it, that each digit has its own name. She then asked the students to show how they would add 957 to 576. The students said that they needed to line up the digits and then add column by column. They used the traditional addition algorithm to complete the task. She then asked students why they did not add the two sevens together. Students said that those were not in the same column and that you can only add digits in the same column. Next she shifted the language to adding digits that have the same name: ones to ones, tens to tens, and hundreds to hundreds. She represented this on the board as demonstrated in figure 1.

Figure 1. Establishing Competence.

		H	T	O	
		9	5	7	
+		5	7	6	
			1	3	O
		1	2		T
	1	4			H
	1	5	3	3	

She guided the students to see that 6 ones and 7 ones added to 13 ones; that 7 tens and 5 tens added to 12 tens; and that 5 hundreds and 9 hundreds added to 14 hundreds. Note that a) each partial sum moves up one place value as we go to each successive column, b) that it does not matter in what order we find the three partial sums, and c) that we add digits that have the same name. While this was a novel point of view for the students they found it easier than the traditional method.

Step 2: Multi-digit integers as polynomial expressions via expanded notation.

A multi-digit number does not yet look like a numeric polynomial expression. We found that some important misconceptions about polynomial expressions can be cleared up using the familiarity with the base-ten place-value number system. In the post-lesson reflection the student teacher stated:

> Many of the students were becoming involved with the lesson and started asking questions. We determined that polynomials are expressions with terms that have different names. Then one student asked me, “What about 444?” The student appeared to be thinking that since there are all fours in the expression, they cannot have different names. I then replied, “Well, in the number 444 we have 4 hundreds, 4 tens, and 4 ones. Do you see the difference? That was a great question!”

This nicely motivated the next step by writing the addends in expanded notation. The above addition was now presented as shown in figure 2.

Figure 2. Using Expanded Notation.

$$\begin{array}{rrcrcr} & 900 & + & 50 & + & 7 \\ + & 500 & + & 70 & + & 6 \\ \hline & 1400 & + & 120 & + & 13 \\ = & 1533 & & & & \end{array}$$

Several students were able to connect the work in step one with step two. They realized that 12 tens is equivalent to 120 and that 14 hundreds is equivalent to 1400. They also began to articulate during the lesson that step 2 looked more like their prior work on solving systems of two linear equations in two variables, where they learned to add and subtract like terms with the elimination method. While adding polynomial expressions is not the same as adding linear combinations of equations, the concept of like terms is essential in both structures. Students began to intuitively notice this. They began to realize that naming is an important idea that occurs both with numbers and variables.

Step 3: Linking number as polynomial expression with place value

In step three we attempted to connect the first two steps by explicitly naming each digit according to its place value. We must be careful not to interpret the names as a variable at this stage. Thus 9H represents nine units of hundred, 5T represents five units of ten, and 7O represents seven units of one. Note that we exchanged 10 ones for one ten and then 10 tens for one hundred in figure 3 after we did the addition according to *the algebraic structure of adding same named quantities.* Exchanging can also be thought of as renaming in a different unit. In algebra we do not do such renaming in a different unit, because the base is variable.

Figure 3. Connecting with Place Value

$$
\begin{array}{rrrrrr}
 & 9H & + & 5T & + & 7O \\
+ & 5H & + & 7T & + & 6O \\
\hline
 & 14H & + & 12T & + & 13O \\
= & 15H & + & 3T & + & 3O
\end{array}
$$

Several students started to notice that the H, T, and O seemed to function just like variables and some connected the distributive property to this process of addition. Connecting polynomials with place value clarifies the transition for students to visualize how the structure of a numeric polynomial is similar to that of an algebraic polynomial written with variables. The student teacher was careful to make sure that at this stage the letters H, T, and O represented names of units with fixed values. This thinking led to the next step.

Step 4: Connecting scientific notation with numeric polynomial expressions, place value, and the distributive property

We wanted to help students see the role of the distributive property more explicitly and then transition to helping them see that algebraic polynomial addition (and subtraction) is fundamentally the same in structure as numeric polynomial addition (and subtraction). To do this we decided to rewrite the expanded notation of step 2 in scientific notation. This was a crucial, but not a simple step in the development of the lesson. The student teacher described this process as follows in her post-lesson reflection:

> "Are you familiar with scientific notation?" was the transition question I asked my students when shifting from step three to step four. The students replied, "yes." I then told the class I could rewrite the numeric polynomials using scientific notation, because this uses 10 as the base and connects directly to the base-10 notation system. The room quickly filled with panic, but this vanished when I informed the class that we would use it to discover something important. I explained to the students that if 900 = 9(100) and 100 = (10^2), then 900 = 9(10^2). We concluded that the digit 9 in the hundreds place has a value of 9(10^2). Likewise the digit 5 in the tens place has a value of 5(10^1), and the digit 7 in the ones place has a value of 7(10^0). The students saw that the place of the digit was represented by the exponent of the base. Because of the consistency of adding terms in the same column, the students began seeing the pattern and relationship between a digit's value according to its place and like terms. After I set up the addends with scientific notation the class began to see that we used the distributive property to add quantities with the *same name*, that are in the *same place*, that are *like terms*: $9(10^2) + 5(10^2) = (9 + 5)(10^2) = 14(10^2)$. It was in this moment of the lesson that I started to really get it myself and I didn't need my notes anymore. My language had changed to mathematical language and so had my students'. No more cheeseburgers for me.

During this step of the lesson the student teacher and her students had changed their conceptions. It was at this point that she put the lesson notes on her desk. New thinking had become structurally clear and had become a strong foundation for moving forward.

Figure 4. Connecting with Scientific Notation and the Distributive Property.

$$\begin{array}{rrcrcr} & 9\text{x}10^2 & + & 5\text{x}10 & + & 7 \\ + & 5\text{x}10^2 & + & 7\text{x}10 & + & 6 \\ \hline & (9+5)\text{x}10^2 & + & (5+7)\text{x}10 & + & (7+6) \\ = & 14\text{x}10^2 & + & 12\text{x}10 & + & 13 \end{array}$$

While in step three we did include an exchange (renaming) of 10 tens for one hundred, we did not do this in step four. There we transitioned to the algebraic way of adding like terms using the distributive property and did not exchange anymore. Algebraically, base-10 is just one of many possible systems. In another base the coefficients 14, 12, and 13 from figure 4 will be decomposed differently. Therefore algebraically we cannot exchange because the base is variable. Here is where adding algebraic polynomials departs from adding numeric polynomials. This is why step four is such a crucial transition.

Step 5: Generalizing step 4 to addition with algebraic polynomial expressions

Using the scientific notation allowed us to abandon the specific case of the base-10 system and work toward the idea that in an algebraic polynomial the base of the system can be thought of as a variable. In Figure 5 you can see one of the many examples that were used, in this case a base-12 example.

Figure 5. Generalizing to operations with algebraic expressions.

$$\begin{array}{rrcrcr} & 3\text{x}12^2 & + & 7\text{x}12 & + & 6 \\ + & 6\text{x}12^2 & + & 5\text{x}12 & + & 7 \\ \hline & (3+6)\text{x}12^2 & + & (7+5)\text{x}12 & + & (6+7) \\ = & 9\text{x}12^2 & + & 12\text{x}12 & + & 13 \end{array}$$

It took a good number of examples for students to abstract the structure to that of an algebraic polynomial. In the days following this lesson the students looked at adding polynomials with a stronger structural perspective that was connected and deeply mathematical in nature. Note that the numeric polynomial sum above can be renamed to $10 \text{ x } 12^2 + 1 \text{ x } 12 + 1$. For algebraic polynomial expressions we cannot accomplish such renaming in general since the base is variable according to this perspective on such expressions. To rename a multiple of x to x^2 we will need x groups of x.

In this article we have only considered situations in which the terms are all added. We realize that combining like terms of expressions with mixed signs is not addressed here. That will need to be placed in the context of adding and subtracting integers. For example, we can expand 376 as $4 \text{ x } 10^2 – 2 \text{ x } 10 – 4$ and then add or subtract this from another numeric polynomial.

Conclusion

In this article we tried to accomplish two things. First we wanted to share our collaboration as student teacher and university supervisor during a student teaching experience. Second we wanted to share this experience in the context of linking important algebra concepts with fundamental structures from K-12 mathematics. The need for this collaboration was born from a concern by the student teacher regarding the teaching of combining like terms in adding and subtracting algebraic polynomials. She agreed to take a risk by teaching an approach that was novel to her with the safety of her supervisor's support. What appeared remarkable about this situation was that both the students' and the student teacher's mathematical language changed in the course of this and subsequent lessons, revealing a much deeper insight in the structural properties of algebraic polynomial addition and subtraction and its relationship with numeric polynomial addition and subtraction in a place value system. In addition the importance of the distributive property was amplified in this process. Translating polynomial to mean "many names" was not just a semantic device, but much more so a discovery that naming is our human way of distinguishing one place value from another and one term from another. We hope that you find equal promise in these ideas.

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