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11. Dynamics in Rotating Frames of Reference

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Abstract

Part eleven of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc11]

11. Dynamics in Rotating Frames of Reference

Motion in rotating frame of reference [mln22]

Consider two frames of reference with identical origins. Frame R is rotating with angular velocity $\vec{\omega}$ relative to the inertial frame I.

Coordinate axes: $\mathbf{e}_i^{(I)} = \text{const}, \quad \dot{\mathbf{e}}_i^{(R)} = \vec{\omega} \times \mathbf{e}_i^{(R)}, \quad i = 1, 2, 3.$

Kinematics of a particle moving in frame R.

- Position: $\mathbf{r}_R = \mathbf{r}_I \doteq \mathbf{r}, \quad$ where $\mathbf{r}_I = \sum_{i=1}^{3} x_i^{(I)} \mathbf{e}_i^{(I)}, \quad \mathbf{r}_R = \sum_{i=1}^{3} x_i^{(R)} \mathbf{e}_i^{(R)}.$

- Velocity: $\dfrac{d\mathbf{r}}{dt} = \sum_{i=1}^{3} \dot{x}_i^{(I)} \mathbf{e}_i^{(I)} = \sum_{i=1}^{3} \left[\dot{x}_i^{(R)} \mathbf{e}_i^{(R)} + x_i^{(R)} \dot{\mathbf{e}}_i^{(R)} \right].$

$$\Rightarrow \sum_{i=1}^{3} \dot{x}_i^{(I)} \mathbf{e}_i^{(I)} = \sum_{i=1}^{3} \left[\dot{x}_i^{(R)} \mathbf{e}_i^{(R)} + \vec{\omega} \times x_i^{(R)} \mathbf{e}_i^{(R)} \right] \qquad \Rightarrow \mathbf{v}_I = \mathbf{v}_R + \vec{\omega} \times \mathbf{r}.$$

- Acceleration: $\dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d\mathbf{v}_I}{dt} = \left(\dfrac{d\mathbf{v}_R}{dt} \right)_I + \dot{\vec{\omega}} \times \mathbf{r} + \vec{\omega} \times \left(\dfrac{d\mathbf{r}}{dt} \right)_I.$

with $\left(\dfrac{d\mathbf{v}_R}{dt} \right)_I = \mathbf{a}_R + \vec{\omega} \times \mathbf{v}_R, \quad \vec{\omega} \times \left(\dfrac{d\mathbf{r}}{dt} \right)_I = \vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$

$$\Rightarrow \mathbf{a}_I = \mathbf{a}_R + \dot{\vec{\omega}} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$$

Dynamics of a particle of mass m.

- Inertial frame: $m\mathbf{a}_I = \mathbf{F}_I$
- Rotating frame: $m\mathbf{a}_R = \mathbf{F}_I - m\dot{\vec{\omega}} \times \mathbf{r} - 2m\vec{\omega} \times \mathbf{v}_R - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$

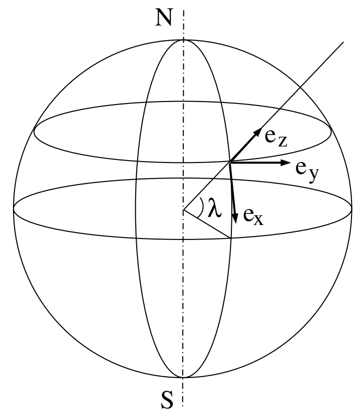
Real and fictitious forces:

- $\mathbf{F}_I$ (applied force).
- $-m\dot{\vec{\omega}} \times \mathbf{r}$ (due to angular acceleration of frame R).
- $-2m\vec{\omega} \times \mathbf{v}_R$ (Coriolis force).
- $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$ (centrifugal force).

If the origin of frame R undergoes a lateral motion in addition to the rotation, then a term $-m(d^2\mathbf{R}/dt^2)$ must be added to the fictitious forces. Here $\mathbf{R}$ is the vector pointing from the origin of frame I to the origin of frame R.

[mex61] Effect of Coriolis force on falling object

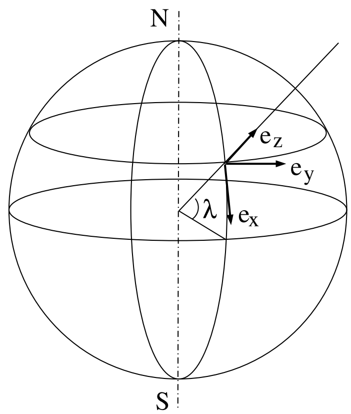
Consider a location at northern latitude λ on the Earth's surface. A particle of mass m starts falling from rest at position $\mathbf{r}_0 = (0, 0, h)$ in the local coordinate system with axes as shown in the figure. (a) Determine the position $\mathbf{r}(t) = (x(t), y(t), z(t))$ during the fall. Perform the calculation to leading order in ω, the Earth's angular velocity of rotation. (b) If $h = 100\text{m}$, $g = 9.8\text{m/s}^2$ and $\lambda = 45°$, what is the magnitude and direction of horizontal deflection from the vertical line of the point where the particle hits the ground.



Solution:

[mex62] Effects of Coriolis force on an object projected vertically up

A particle of mass m is projected vertically up from a point on the Earth's surface at northern latitude λ. (a) Find the deflection (x_1, y_1) of the path from the vertical at $z_1 = h$, where the particle reaches its maximum height. Use the local frame of reference with the origin at the launch site and the axes as shown. Express the result as a function of λ (angle of latitude), ω (angular frequency of Earth's rotation), g (acceleration due to gravity), and h (maximum height reached by particle). Keep only terms up to linear order in ω. (b) Find the deflection (x_2, y_2) of the path at $z_2 = 0$, when the particle strikes the ground.



Solution:

[mex64] Foucault pendulum

Consider a location at northern latitude λ on the Earth's surface. A pendulum (mass m, length L) is free to swing in any direction. At time $t = 0$, the pendulum is set in motion from a small displacement $x_0 > 0,\ y_0 = 0$ with no initial velocity. (a) Show that the linearized equations of motion including the effect of the Coriolis force can be expressed in the form

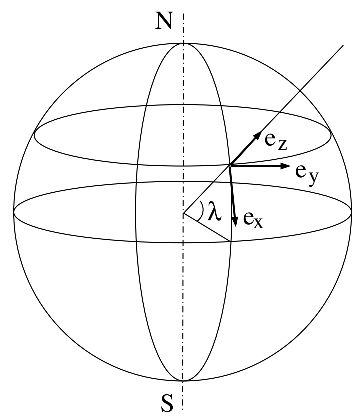
$$\ddot{q} + 2i\omega_z\dot{q} + \Omega^2 q = 0; \qquad q \equiv x + iy, \quad \Omega = \sqrt{g/L}, \quad \omega_z = \omega \sin\lambda,$$

where ω is the angular frequency of the Earth's rotation. This equation of motion describes a harmonic oscillator with imaginary damping. (b) Show that for the initial conditions stated above and for $\omega_z \ll \Omega$ its solution is of the form $q(t) = x_0 \cos\Omega t e^{-i\omega_z t}$. (c) Show that the last factor in this solution describes a precession with angular frequency ω_z of the plane in which the pendulum swings.

Solution:

[mex63] Effects of Coriolis and centrifugal forces on falling object

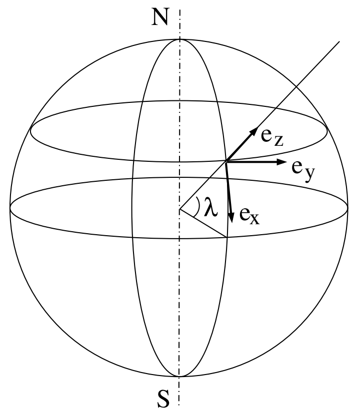
Consider a location at northern latitude λ on the Earth's surface. A particle of mass m starts falling from rest at position $\mathbf{r}_0 = (0, 0, h)$ in the local coordinate system with axes as shown in the figure. (a) Determine the position $\mathbf{r}(t) = (x(t), y(t), z(t))$ during the fall. Perform the calculation to second order order in ω using the first-order results of [mex61]. (b) Find the horizontal deflections d_x, d_y from the vertical line of the point where the particle strikes the ground. Express the result as a function of λ (angle of latitude), ω (angular frequency of Earth's rotation), g (acceleration due to gravity), and h (height). (c) What are the values of d_x, d_y if $h = 100\text{m}$, $g = 9.8\text{m/s}^2$ and $\lambda = 45°$?



Solution:

[mex65] Lateral deflection of projectile due to Coriolis force

Consider a location at northern latitude λ on the Earth's surface. A particle is projected due east with initial speed v_0 and angle of inclination α above the horizontal. Find the lateral deflection d_x due to the Coriolis force of the point where the particle strikes the ground. Perform the calculation to leading order in ω, the angular frequency of the Earth's rotation. Express d_x as a function of $v_0, g, \alpha, \lambda, \omega$. Evaluate the range R (to zeroth order in ω) and the lateral deflection d_x (to first order in ω) for the case where a projectile is launched with speed $v_0 = 100\text{m/s}$ at angle $\alpha = 45°$.



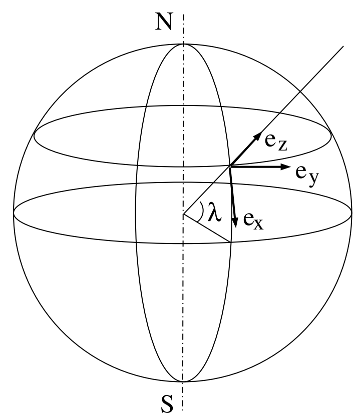
Solution:

[mex66] Effect of Coriolis force on range of projectile

Consider a location at northern latitude λ on the Earth's surface. A particle is projected due east with initial speed v_0 and angle of inclination α above the horizontal. Show that the change in the range $R = (2v_0^2/g)\sin\alpha\cos\alpha$ of the projectile due to the Coriolis force is

$$\Delta R = \sqrt{\frac{2R^3}{g}}\omega\cos\lambda\left[\cot^{1/2}\alpha - \frac{1}{3}\tan^{3/2}\alpha\right].$$

Perform the calculation to leading order in ω, the angular frequency of the Earth's rotation.



Solution:

[mex170] What is vertical?

(a) Calculate the the angular deviation ϵ of a plumb line from the direction to the Earth's center at a point of latitude λ. (b) At what latitude is does ϵ have its maximum value? (c) State ϵ_{max} in arc seconds. Use the value $g = 9.81\text{m/s}^2$ for the acceleration due to gravity.

Solution:

[mex171] Lagrange equations in rotating frame

From [mex79] we know that the Lagrange equations are invariant under a point transformation. Here we use this property to transform the equation of motion of a particle in a potential $V(\mathbf{r})$ from an inertial frame to a frame rotating with constant angular velocity $\vec{\omega}$.
(a) Express the Lagrangian $\tilde{L}(\mathbf{r}, \tilde{\mathbf{v}}) = L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 - V(\mathbf{r})$ in terms of the rotating-frame coordinates.
(b) Derive the Lagrange equations $(d/dt)(\partial\tilde{L}/\partial\dot{\tilde{x}}_i) - (\partial\tilde{L}/\partial\tilde{x}_i) = 0,\ i = 1, 2, 3.$
(c) Bring the resulting Lagrange equations into the form

$$m\tilde{\mathbf{a}} = -\frac{\partial V}{\partial \mathbf{r}} - 2m\vec{\omega} \times \tilde{\mathbf{v}} - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$$

Solution:

Holonomic constraints in rotating frame [mln23]

Recipe for solving a Lagrangian mechanics problem with holonomic constraints between coordinates in the rotating frame.

- Formulate Lagrangian in inertial frame (I) without imposing constraints.
- Transform coordinates to the rotating frame (R).
- Impose holonomic constraints via independent generalized coordinates.
- Derive Lagrange equations in frame R.

Example: particle of mass m moving on surface of rotating Earth in vertical plane parallel to meridian and subject to scalar potential V.

- Lagrangian: $L_I = \dfrac{1}{2}m(\dot{x_I}^2 + \dot{y_I}^2 + \dot{z_I}^2) - V(x_I, y_I, z_I)$.
- Earth's angular velocity: $\vec{\omega} = (-\omega\cos\lambda, 0, \omega\sin\lambda)$.
- Transformation: $\mathbf{v}_I = \mathbf{v} + \vec{\omega}\times\mathbf{r} = \begin{pmatrix} \dot{x} - \omega y\sin\lambda \\ \dot{y} + \omega x\sin\lambda + \omega z\cos\lambda \\ \dot{z} - \omega y\cos\lambda \end{pmatrix}$.
- Constraint: $y = 0 \quad\Rightarrow\quad \mathbf{v}_I = (\dot{x}_I, \dot{y}_I, \dot{z}_I) = \begin{pmatrix} \dot{x} \\ \omega x\sin\lambda + \omega z\cos\lambda \\ \dot{z} \end{pmatrix}$.
- Substitute $\mathbf{v}_I$ into Lagrangian: $L_I = L(x, z, \dot{x}, \dot{z})$.
- Lagrange equations: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} = 0, \quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{z}} - \dfrac{\partial L}{\partial z} = 0.$

Notes:

- The accelerated translational motion can be taken into account by a modified acceleration due to gravity: $\mathbf{g} = \mathbf{g}_0 + \omega^2\mathbf{r}_\perp$ [mex170].
- In the local coordinate system, $\mathbf{e}_x$ is pointing south, $\mathbf{e}_y$ is pointing east, and $\mathbf{e}_z$ is pointing vertically up.
- It is common practice to drop subscripts R in the rotating frame to keep the notation simple.

[mex172] Parabolic slide on rotating Earth

A bead of mass m slides without friction along a wire of parabolic shape, $z = Ay^2$, in a uniform gravitational field g pointing in the negative z-direction. In generalization to [mex131], the effect of the Earth's rotation rotation must be taken into account under the assumption that the slide is placed at latitude λ with its (vertical) plane oriented perpendicular to the meridian.
(a) Construct the Lagrangian $L(y, \dot{y})$.
(b) Derive the Lagrange equation.

Solution: