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11. Dynamics in Rotating Frames of Reference

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Abstract

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Motion in rotating frame of reference [mln22]

Consider two frames of reference with identical origins. Frame R is rotating with angular velocity $\vec{\omega}$ relative to the inertial frame I .

Coordinate axes: $\mathbf{e}_i^{(I)} = \text{const}$, $\dot{\mathbf{e}}_i^{(R)} = \vec{\omega} \times \mathbf{e}_i^{(R)}$, $i = 1, 2, 3$.

Kinematics of a particle moving in frame R .

- Position: $\mathbf{r}_R = \mathbf{r}_I \doteq \mathbf{r}$, where $\mathbf{r}_I = \sum_{i=1}^3 x_i^{(I)} \mathbf{e}_i^{(I)}$, $\mathbf{r}_R = \sum_{i=1}^3 x_i^{(R)} \mathbf{e}_i^{(R)}$.
- Velocity: $\frac{d\mathbf{r}}{dt} = \sum_{i=1}^3 \dot{x}_i^{(I)} \mathbf{e}_i^{(I)} = \sum_{i=1}^3 \left[\dot{x}_i^{(R)} \mathbf{e}_i^{(R)} + x_i^{(R)} \dot{\mathbf{e}}_i^{(R)} \right]$.
 $\Rightarrow \sum_{i=1}^3 \dot{x}_i^{(I)} \mathbf{e}_i^{(I)} = \sum_{i=1}^3 \left[\dot{x}_i^{(R)} \mathbf{e}_i^{(R)} + \vec{\omega} \times x_i^{(R)} \mathbf{e}_i^{(R)} \right] \Rightarrow \mathbf{v}_I = \mathbf{v}_R + \vec{\omega} \times \mathbf{r}$.
- Acceleration: $\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}_I}{dt} = \left(\frac{d\mathbf{v}_R}{dt} \right)_I + \dot{\vec{\omega}} \times \mathbf{r} + \vec{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_I$.
with $\left(\frac{d\mathbf{v}_R}{dt} \right)_I = \mathbf{a}_R + \vec{\omega} \times \mathbf{v}_R$, $\vec{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_I = \vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r})$.
 $\Rightarrow \mathbf{a}_I = \mathbf{a}_R + \dot{\vec{\omega}} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r})$.

Dynamics of a particle of mass m .

- Inertial frame: $m\mathbf{a}_I = \mathbf{F}_I$
- Rotating frame: $m\mathbf{a}_R = \mathbf{F}_I - m\dot{\vec{\omega}} \times \mathbf{r} - 2m\vec{\omega} \times \mathbf{v}_R - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$.

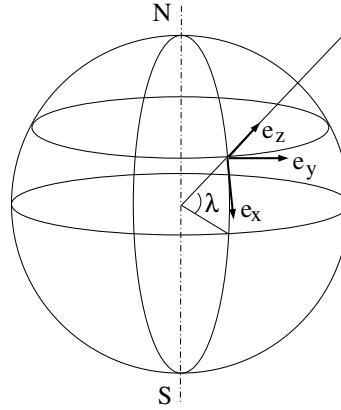
Real and fictitious forces:

- \mathbf{F}_I (applied force).
- $-m\dot{\vec{\omega}} \times \mathbf{r}$ (due to angular acceleration of frame R).
- $-2m\vec{\omega} \times \mathbf{v}_R$ (Coriolis force).
- $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$ (centrifugal force).

If the origin of frame R undergoes a lateral motion in addition to the rotation, then a term $-m(d^2\mathbf{R}/dt^2)$ must be added to the fictitious forces. Here \mathbf{R} is the vector pointing from the origin of frame I to the origin of frame R .

[mex61] Effect of Coriolis force on falling object

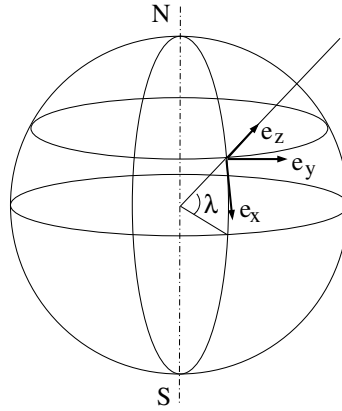
Consider a location at northern latitude λ on the Earth's surface. A particle of mass m starts falling from rest at position $\mathbf{r}_0 = (0, 0, h)$ in the local coordinate system with axes as shown in the figure. (a) Determine the position $\mathbf{r}(t) = (x(t), y(t), z(t))$ during the fall. Perform the calculation to leading order in ω , the Earth's angular velocity of rotation. (b) If $h = 100\text{m}$, $g = 9.8\text{m/s}^2$ and $\lambda = 45^\circ$, what is the magnitude and direction of horizontal deflection from the vertical line of the point where the particle hits the ground.



Solution:

[mex62] Effects of Coriolis force on an object projected vertically up

A particle of mass m is projected vertically up from a point on the Earth's surface at northern latitude λ . (a) Find the deflection (x_1, y_1) of the path from the vertical at $z_1 = h$, where the particle reaches its maximum height. Use the local frame of reference with the origin at the launch site and the axes as shown. Express the result as a function of λ (angle of latitude), ω (angular frequency of Earth's rotation), g (acceleration due to gravity), and h (maximum height reached by particle). Keep only terms up to linear order in ω . (b) Find the deflection (x_2, y_2) of the path at $z_2 = 0$, when the particle strikes the ground.



Solution:

[mex64] Foucault pendulum

Consider a location at northern latitude λ on the Earth's surface. A pendulum (mass m , length L) is free to swing in any direction. At time $t = 0$, the pendulum is set in motion from a small displacement $x_0 > 0$, $y_0 = 0$ with no initial velocity. (a) Show that the linearized equations of motion including the effect of the Coriolis force can be expressed in the form

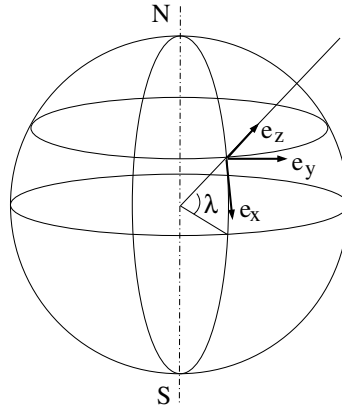
$$\ddot{q} + 2i\omega_z\dot{q} + \Omega^2q = 0; \quad q \equiv x + iy, \quad \Omega = \sqrt{g/L}, \quad \omega_z = \omega \sin \lambda,$$

where ω is the angular frequency of the Earth's rotation. This equation of motion describes a harmonic oscillator with imaginary damping. (b) Show that for the initial conditions stated above and for $\omega_z \ll \Omega$ its solution is of the form $q(t) = x_0 \cos \Omega t e^{-i\omega_z t}$. (c) Show that the last factor in this solution describes a precession with angular frequency ω_z of the plane in which the pendulum swings.

Solution:

[mex63] Effects of Coriolis and centrifugal forces on falling object

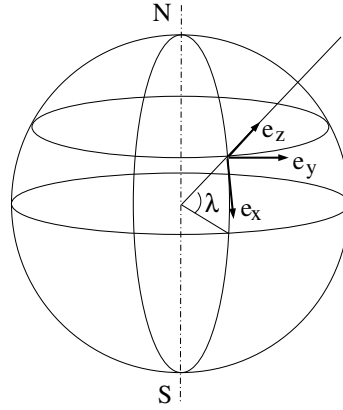
Consider a location at northern latitude λ on the Earth's surface. A particle of mass m starts falling from rest at position $\mathbf{r}_0 = (0, 0, h)$ in the local coordinate system with axes as shown in the figure. (a) Determine the position $\mathbf{r}(t) = (x(t), y(t), z(t))$ during the fall. Perform the calculation to second order order in ω using the first-order results of [mex61]. (b) Find the horizontal deflections d_x, d_y from the vertical line of the point where the particle strikes the ground. Express the result as a function of λ (angle of latitude), ω (angular frequency of Earth's rotation), g (acceleration due to gravity), and h (height). (c) What are the values of d_x, d_y if $h = 100\text{m}$, $g = 9.8\text{m/s}^2$ and $\lambda = 45^\circ$?



Solution:

[mex65] Lateral deflection of projectile due to Coriolis force

Consider a location at northern latitude λ on the Earth's surface. A particle is projected due east with initial speed v_0 and angle of inclination α above the horizontal. Find the lateral deflection d_x due to the Coriolis force of the point where the particle strikes the ground. Perform the calculation to leading order in ω , the angular frequency of the Earth's rotation. Express d_x as a function of $v_0, g, \alpha, \lambda, \omega$. Evaluate the range R (to zeroth order in ω) and the lateral deflection d_x (to first order in ω) for the case where a projectile is launched with speed $v_0 = 100\text{m/s}$ at angle $\alpha = 45^\circ$.



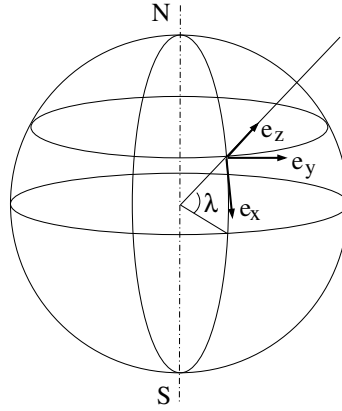
Solution:

[mex66] Effect of Coriolis force on range of projectile

Consider a location at northern latitude λ on the Earth's surface. A particle is projected due east with initial speed v_0 and angle of inclination α above the horizontal. Show that the change in the range $R = (2v_0^2/g) \sin \alpha \cos \alpha$ of the projectile due to the Coriolis force is

$$\Delta R = \sqrt{\frac{2R^3}{g}} \omega \cos \lambda \left[\cot^{1/2} \alpha - \frac{1}{3} \tan^{3/2} \alpha \right].$$

Perform the calculation to leading order in ω , the angular frequency of the Earth's rotation.



Solution:

[mex170] What is vertical?

(a) Calculate the the angular deviation ϵ of a plumb line from the direction to the Earth's center at a point of latitude λ . (b) At what latitude is does ϵ have its maximum value? (c) State ϵ_{max} in arc seconds. Use the value $g = 9.81\text{m/s}^2$ for the acceleration due to gravity.

Solution:

[mex171] Lagrange equations in rotating frame

From [mex79] we know that the Lagrange equations are invariant under a point transformation. Here we use this property to transform the equation of motion of a particle in a potential $V(\mathbf{r})$ from an inertial frame to a frame rotating with constant angular velocity $\vec{\omega}$.

(a) Express the Lagrangian $\tilde{L}(\mathbf{r}, \tilde{\mathbf{v}}) = L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 - V(\mathbf{r})$ in terms of the rotating-frame coordinates.

(b) Derive the Lagrange equations $(d/dt)(\partial\tilde{L}/\partial\dot{\tilde{x}}_i) - (\partial\tilde{L}/\partial\tilde{x}_i) = 0$, $i = 1, 2, 3$.

(c) Bring the resulting Lagrange equations into the form

$$m\tilde{\mathbf{a}} = -\frac{\partial V}{\partial \mathbf{r}} - 2m\vec{\omega} \times \tilde{\mathbf{v}} - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r}).$$

Solution:

Holonomic constraints in rotating frame [mln23]

Recipe for solving a Lagrangian mechanics problem with holonomic constraints between coordinates in the rotating frame.

- Formulate Lagrangian in inertial frame (I) without imposing constraints.
- Transform coordinates to the rotating frame (R).
- Impose holonomic constraints via independent generalized coordinates.
- Derive Lagrange equations in frame R .

Example: particle of mass m moving on surface of rotating Earth in vertical plane parallel to meridian and subject to scalar potential V .

- Lagrangian: $L_I = \frac{1}{2}m(\dot{x}_I^2 + \dot{y}_I^2 + \dot{z}_I^2) - V(x_I, y_I, z_I)$.
- Earth's angular velocity: $\vec{\omega} = (-\omega \cos \lambda, 0, \omega \sin \lambda)$.
- Transformation: $\mathbf{v}_I = \mathbf{v} + \vec{\omega} \times \mathbf{r} = \begin{pmatrix} \dot{x} - \omega y \sin \lambda \\ \dot{y} + \omega x \sin \lambda + \omega z \cos \lambda \\ \dot{z} - \omega y \cos \lambda \end{pmatrix}$.
- Constraint: $y = 0 \Rightarrow \mathbf{v}_I = (\dot{x}_I, \dot{y}_I, \dot{z}_I) = \begin{pmatrix} \dot{x} \\ \omega x \sin \lambda + \omega z \cos \lambda \\ \dot{z} \end{pmatrix}$.
- Substitute \mathbf{v}_I into Lagrangian: $L_I = L(x, z, \dot{x}, \dot{z})$.
- Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$.

Notes:

- The accelerated translational motion can be taken into account by a modified acceleration due to gravity: $\mathbf{g} = \mathbf{g}_0 + \omega^2 \mathbf{r}_\perp$ [mex170].
- In the local coordinate system, \mathbf{e}_x is pointing south, \mathbf{e}_y is pointing east, and \mathbf{e}_z is pointing vertically up.
- It is common practice to drop subscripts R in the rotating frame to keep the notation simple.

[mex172] Parabolic slide on rotating Earth

A bead of mass m slides without friction along a wire of parabolic shape, $z = Ay^2$, in a uniform gravitational field g pointing in the negative z -direction. In generalization to [mex131], the effect of the Earth's rotation must be taken into account under the assumption that the slide is placed at latitude λ with its (vertical) plane oriented perpendicular to the meridian.

- (a) Construct the Lagrangian $L(y, \dot{y})$.
- (b) Derive the Lagrange equation.

Solution: