11. Dynamics in Rotating Frames of Reference

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Abstract

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Motion in rotating frame of reference

Consider two frames of reference with identical origins. Frame $R$ is rotating with angular velocity $\vec{\omega}$ relative to the inertial frame $I$.

Coordinate axes: $e_i^{(I)} = \text{const}$, $\dot{e}_i^{(R)} = \vec{\omega} \times e_i^{(R)}$, $i = 1, 2, 3$.

Kinematics of a particle moving in frame $R$.

- Position: $\mathbf{r}_R = \mathbf{r}_I = \mathbf{r}$, where $\mathbf{r}_I = \sum_{i=1}^{3} x_i^{(I)} e_i^{(I)}$, $\mathbf{r}_R = \sum_{i=1}^{3} x_i^{(R)} e_i^{(R)}$.

- Velocity: $\frac{d\mathbf{r}}{dt} = \sum_{i=1}^{3} \dot{x}_i^{(I)} e_i^{(I)} = \sum_{i=1}^{3} \left[ \dot{x}_i^{(R)} e_i^{(R)} + x_i^{(R)} \dot{e}_i^{(R)} \right]$. 

  $\Rightarrow \sum_{i=1}^{3} \dot{x}_i^{(I)} e_i^{(I)} = \sum_{i=1}^{3} \left[ \dot{x}_i^{(R)} e_i^{(R)} + \vec{\omega} \times x_i^{(R)} e_i^{(R)} \right] \Rightarrow \mathbf{v}_I = \mathbf{v}_R + \vec{\omega} \times \mathbf{r}$.

- Acceleration: $\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}_I}{dt} = \left( \frac{d\mathbf{v}_R}{dt} \right)_I + \vec{\omega} \times \mathbf{r} + \vec{\omega} \times \left( \frac{d\mathbf{r}}{dt} \right)_I$.

  with $\left( \frac{d\mathbf{v}_R}{dt} \right)_I = \mathbf{a}_R + \vec{\omega} \times \mathbf{v}_R$, $\vec{\omega} \times \left( \frac{d\mathbf{r}}{dt} \right)_I = \vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r})$.

  $\Rightarrow \mathbf{a}_I = \mathbf{a}_R + \vec{\omega} \times \mathbf{r} + 2\vec{\omega} \times \mathbf{v}_R + \vec{\omega} \times (\vec{\omega} \times \mathbf{r})$.

Dynamics of a particle of mass $m$.

- Inertial frame: $m\mathbf{a}_I = \mathbf{F}_I$
- Rotating frame: $m\mathbf{a}_R = \mathbf{F}_I - m\vec{\omega} \times \mathbf{r} - 2m\vec{\omega} \times \mathbf{v}_R - m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$.

Real and fictitious forces:

- $\mathbf{F}_I$ (applied force).
- $-m\vec{\omega} \times \mathbf{r}$ (due to angular acceleration of frame $R$).
- $-2m\vec{\omega} \times \mathbf{v}_R$ (Coriolis force).
- $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$ (centrifugal force).

If the origin of frame $R$ undergoes a lateral motion in addition to the rotation, then a term $-m\left( R \frac{d^2\mathbf{R}}{dt^2} \right)$ must be added to the fictitious forces. Here $\mathbf{R}$ is the vector pointing from the origin of frame $I$ to the origin of frame $R$. 

Effect of Coriolis force on falling object

Consider a location at northern latitude $\lambda$ on the Earth’s surface. A particle of mass $m$ starts falling from rest at position $r_0 = (0, 0, h)$ in the local coordinate system with axes as shown in the figure. (a) Determine the position $r(t) = (x(t), y(t), z(t))$ during the fall. Perform the calculation to leading order in $\omega$, the Earth’s angular velocity of rotation. (b) If $h = 100\,\text{m}$, $g = 9.8\,\text{m/s}^2$ and $\lambda = 45^\circ$, what is the magnitude and direction of horizontal deflection from the vertical line of the point where the particle hits the ground.

Solution:
[mex62] Effects of Coriolis force on an object projected vertically up

A particle of mass \( m \) is projected vertically up from a point on the Earth’s surface at northern latitude \( \lambda \). (a) Find the deflection \((x_1, y_1)\) of the path from the vertical at \( z_1 = h \), where the particle reaches its maximum height. Use the local frame of reference with the origin at the launch site and the axes as shown. Express the result as a function of \( \lambda \) (angle of latitude), \( \omega \) (angular frequency of Earth’s rotation), \( g \) (acceleration due to gravity), and \( h \) (maximum height reached by particle). Keep only terms up to linear order in \( \omega \). (b) Find the deflection \((x_2, y_2)\) of the path at \( z_2 = 0 \), when the particle strikes the ground.

Solution:
Consider a location at northern latitude \( \lambda \) on the Earth’s surface. A pendulum (mass \( m \), length \( L \)) is free to swing in any direction. At time \( t = 0 \), the pendulum is set in motion from a small displacement \( x_0 > 0, \ y_0 = 0 \) with no initial velocity. (a) Show that the linearized equations of motion including the effect of the Coriolis force can be expressed in the form

\[
\ddot{q} + 2i\omega_z \dot{q} + \Omega^2 q = 0; \quad q \equiv x + iy, \quad \Omega = \sqrt{g/L}, \quad \omega_z = \omega \sin \lambda,
\]

where \( \omega \) is the angular frequency of the Earth’s rotation. This equation of motion describes a harmonic oscillator with imaginary damping. (b) Show that for the initial conditions stated above and for \( \omega_z \ll \Omega \) its solution is of the form \( q(t) = x_0 \cos \Omega te^{-i\omega_z t} \). (c) Show that the last factor in this solution describes a precession with angular frequency \( \omega_z \) of the plane in which the pendulum swings.

Solution:
Consider a location at northern latitude $\lambda$ on the Earth’s surface. A particle of mass $m$ starts falling from rest at position $\mathbf{r}_0 = (0, 0, h)$ in the local coordinate system with axes as shown in the figure. (a) Determine the position $\mathbf{r}(t) = (x(t), y(t), z(t))$ during the fall. Perform the calculation to second order order in $\omega$ using the first-order results of [mex61]. (b) Find the horizontal deflections $d_x, d_y$ from the vertical line of the point where the particle strikes the ground. Express the result as a function of $\lambda$ (angle of latitude), $\omega$ (angular frequency of Earth’s rotation), $g$ (acceleration due to gravity), and $h$ (height). (c) What are the values of $d_x, d_y$ if $h = 100\text{m}$, $g = 9.8\text{m/s}^2$ and $\lambda = 45^\circ$?

Solution:
[mex65] Lateral deflection of projectile due to Coriolis force

Consider a location at northern latitude $\lambda$ on the Earth’s surface. A particle is projected due east with initial speed $v_0$ and angle of inclination $\alpha$ above the horizontal. Find the lateral deflection $d_x$ due to the Coriolis force of the point where the particle strikes the ground. Perform the calculation to leading order in $\omega$, the angular frequency of the Earth’s rotation. Express $d_x$ as a function of $v_0, g, \alpha, \lambda, \omega$. Evaluate the range $R$ (to zeroth order in $\omega$) and the lateral deflection $d_x$ (to first order in $\omega$) for the case where a projectile is launched with speed $v_0 = 100\text{m/s}$ at angle $\alpha = 45^\circ$.

Solution:
Consider a location at northern latitude \( \lambda \) on the Earth's surface. A particle is projected due east with initial speed \( v_0 \) and angle of inclination \( \alpha \) above the horizontal. Show that the change in the range \( R = (2v_0^2/g) \sin \alpha \cos \alpha \) of the projectile due to the Coriolis force is

\[
\Delta R = \sqrt{\frac{2R^3}{g}} \omega \cos \lambda \left[ \cot^{1/2} \alpha - \frac{1}{3} \tan^{3/2} \alpha \right].
\]

Perform the calculation to leading order in \( \omega \), the angular frequency of the Earth's rotation.

**Solution:**

Diagram scheduling
What is vertical?

(a) Calculate the angular deviation $\epsilon$ of a plumb line from the direction to the Earth’s center at a point of latitude $\lambda$. (b) At what latitude does $\epsilon$ have its maximum value? (c) State $\epsilon_{\text{max}}$ in arc seconds. Use the value $g = 9.81 \text{m/s}^2$ for the acceleration due to gravity.

Solution:
[mex171] Lagrange equations in rotating frame

From [mex79] we know that the Lagrange equations are invariant under a point transformation. Here we use this property to transform the equation of motion of a particle in a potential $V(r)$ from an inertial frame to a frame rotating with constant angular velocity $\vec{\omega}$.

(a) Express the Lagrangian $\tilde{L}(r, \tilde{v}) = L(r, v) = \frac{1}{2}mv^2 - V(r)$ in terms of the rotating-frame coordinates.

(b) Derive the Lagrange equations $(d/dt)(\partial\tilde{L}/\partial\dot{x}_i) - (\partial\tilde{L}/\partial x_i) = 0, \ i = 1, 2, 3.$

(c) Bring the resulting Lagrange equations into the form

\[ m\ddot{\vec{a}} = -\frac{\partial V}{\partial \vec{r}} - 2m\vec{\omega} \times \vec{\dot{v}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}). \]

Solution:
Recipe for solving a Lagrangian mechanics problem with holonomic constraints between coordinates in the rotating frame.

- Formulate Lagrangian in inertial frame \((I)\) without imposing constraints.
- Transform coordinates to the rotating frame \((R)\).
- Impose holonomic constraints via independent generalized coordinates.
- Derive Lagrange equations in frame \(R\).

Example: particle of mass \(m\) moving on surface of rotating Earth in vertical plane parallel to meridian and subject to scalar potential \(V\).

- Lagrangian: \(L_I = \frac{1}{2}m(x_I^2 + y_I^2 + z_I^2) - V(x_I, y_I, z_I)\).
- Earth’s angular velocity: \(\vec{\omega} = (-\omega \cos \lambda, 0, \omega \sin \lambda)\).
- Transformation: \(v_I = v + \vec{\omega} \times r = \begin{pmatrix} \dot{x} - \omega y \sin \lambda \\ \dot{y} + \omega x \sin \lambda + \omega z \cos \lambda \\ \dot{z} - \omega y \cos \lambda \end{pmatrix}\).
- Constraint: \(y = 0\) \(\Rightarrow\) \(v_I = (\dot{x}_I, \dot{y}_I, \dot{z}_I) = \begin{pmatrix} \dot{x} \omega x \sin \lambda + \omega z \cos \lambda \\ \dot{y} \omega x \sin \lambda + \omega z \cos \lambda \\ \dot{z} \end{pmatrix}\).
- Substitute \(v_I\) into Lagrangian: \(L_I = L(x, z, \dot{x}, \dot{z})\).
- Lagrange equations: \(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0\), \(\frac{d}{dt} \frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0\).

Notes:

- The accelerated translational motion can be taken into account by a modified acceleration due to gravity: \(g = g_0 + \omega^2 r_\perp\) [mex170].
- In the local coordinate system, \(e_x\) is pointing south, \(e_y\) is pointing east, and \(e_z\) is pointing vertically up.
- It is common practice to drop subscripts \(R\) in the rotating frame to keep the notation simple.
[mex172] Parabolic slide on rotating Earth

A bead of mass $m$ slides without friction along a wire of parabolic shape, $z = Ay^2$, in a uniform gravitational field $g$ pointing in the negative $z$-direction. In generalization to [mex131], the effect of the Earth's rotation must be taken into account under the assumption that the slide is placed at latitude $\lambda$ with its (vertical) plane oriented perpendicular to the meridian.

(a) Construct the Lagrangian $L(y, \dot{y})$.
(b) Derive the Lagrange equation.

Solution: