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10. Scattering from Central Force Potential

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Abstract
Part ten of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

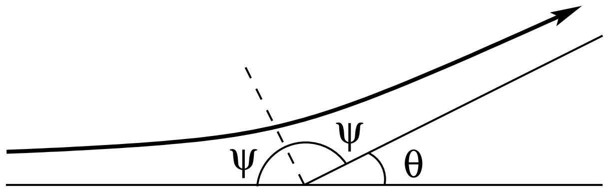
Contents of this Document [mtc10]

- Decay of particle: directions in lab frame I [mex238]
- Decay of particle: directions in lab frame II [mex239]

Determination of scattering angle [mln20]

Orbital integral: $\vartheta = \int_\infty^r dr \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}}.$

Periapsis: $\vartheta(r_{min}) \doteq \psi \quad \Rightarrow\ 2\psi + \theta = \pi.$



$$\Rightarrow\ \psi = \int_{r_{min}}^\infty \frac{dr/r^2}{\sqrt{\frac{2m}{\ell^2}[E - V(r)] - \frac{1}{r^2}}} = \int_{r_{min}}^\infty \frac{sdr/r^2}{\sqrt{1 - \frac{V(r)}{E} - \frac{s^2}{r^2}}}.$$

Substitute $u = 1/r$:

$$\Rightarrow\ \psi = \int_0^{u_{max}} \frac{sdu}{\sqrt{1 - \frac{V(1/u)}{E} - s^2u^2}}.$$

Use energy conservation to determine u_{max}:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) = \frac{\ell^2}{2mr_{min}^2} + V(r_{min}) = Es^2u_{max}^2 + V(1/u_{max}).$$

$$\Rightarrow\ s^2u_{max}^2 + V(1/u_{max})/E = 1 \quad \Rightarrow\ u_{max} = u_{max}(s, E).$$

Scattering angle: $\theta(s, E) = \pi - 2\int_0^{u_{max}} \frac{sdu}{\sqrt{1 - V(1/u)/E - s^2u^2}}.$

Total scattering cross section:

$$\sigma_T = \int \sigma(\theta)d\Omega = 2\pi \int_0^\pi \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right| \sin\theta\, d\theta = 2\pi \int_0^{s_{max}} s\, ds.$$

Note: In quantum mechanics σ_T can be finite even if s_{max} is infinite.

[mex58] Total cross section for shower of meteorites

A uniform beam of small particles with mass m_1 and velocity v_0 is directed toward a planet of mass m_2 and radius R. Calculate the total cross section σ_T for the particles to be absorbed by the planet.

Solution:

[mex56] Rutherford scattering formula

Derive the scattering cross section

$$\sigma(\theta) = \left(\frac{\kappa}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}, \quad \kappa = \frac{ZZ'e^2}{4\pi\epsilon_0}$$

for elastic scattering of particles with electric charge Ze and energy E from stationary atomic nuclei with charge $Z'e$. Note that $\sigma(\theta)$ does not depend on whether the beam is positively or negatively charged.

Solution:

[mex55] Scattering from hard spheres

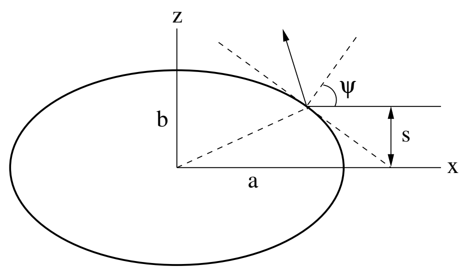
(a) Calculate the scattering cross section $\sigma(\theta)$ for elastic scattering from hard spheres of radius a.
(b) Calculate the total scattering cross section $\sigma_T = \int d^2\Omega\, \sigma(\theta)$.

Solution:

[mex60] Elastic scattering from a hard ellipsoid

Show that the cross section for elastic scattering from a hard ellipsoid described by the equation $x^2/a^2 + (y^2 + z^2)/b^2 = 1$ with the incident beam along the x-axis is

$$\sigma(\theta) = \frac{1}{4} b^2 \frac{a^2 b^2}{[a^2 \sin^2(\theta/2) + b^2 \cos^2(\theta/2)]^2}.$$



Solution:

[mex59] Scattering cross section for inverse square potential

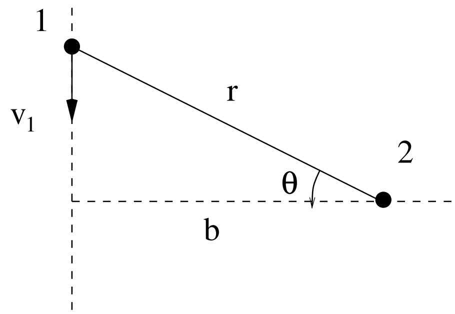
Show that the cross section for scattering from the stationary potential $V(r) = \kappa/r^2$ with $\kappa > 0$ is

$$\sigma(\theta) = \frac{\kappa\pi^2}{E}\frac{\pi-\theta}{\theta^2(2\pi-\theta)^2\sin\theta}.$$

Solution:

[mex10] Particle experiencing soft Coulomb kick

A particle with charge Q_1 and mass m_1 moves at very high velocity v_1 along a (nearly) straight line that passes at a distance b from a particle with charge Q_2 and mass m_2, which is initially at rest. The assumptions are that the two particles interact via a Coulomb central force and that the second particle does not change its position significantly during the encounter.
(a) Find the direction in which the second particle will move after the encounter.
(b) Find the energy ΔE transferrred from the first to the second particle during the encounter.



Solution:

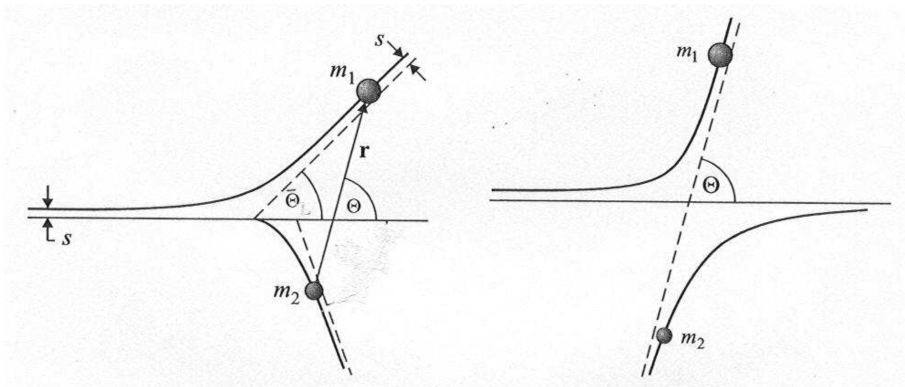
Scattering angle in the laboratory frame [msl3]

The scattering experiment is performed in the **laboratory frame**.

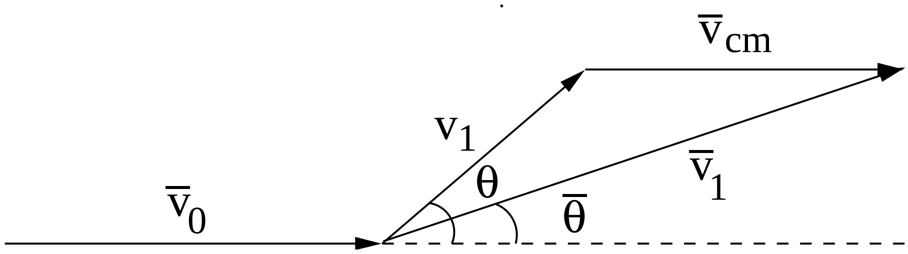
- observed scattering angle: $\bar{\theta}$,
- observed scattering cross section: $\bar{\sigma}(\bar{\theta})$,
- projectile of mass m_1 and target of mass m_2.

The theoretical analysis is performed in the **center-of-mass frame**:

- problem reduced to one degree of freedom,
- total mass $M = m_1 + m_2$,
- reduced mass $m = m_1 m_2/(m_1 + m_2)$,
- calculated scattering angle: θ,
- calculated scattering cross section: $\sigma(\theta)$.



Task #1: establish the relation between θ and $\bar{\theta}$.



$$m_1 \bar{\mathbf{v}}_0 = (m_1 + m_2)\bar{\mathbf{v}}_{cm} \quad \Rightarrow \bar{\mathbf{v}}_{cm} = \frac{m_1}{m_1 + m_2}\,\bar{\mathbf{v}}_0 = \frac{m}{m_2}\,\bar{\mathbf{v}}_0.$$

$$\bar{v}_1 \sin\bar{\theta} = v_1 \sin\theta, \quad \bar{v}_1 \cos\bar{\theta} = v_1 \cos\theta + \bar{v}_{cm}.$$

Relative velocity after collision: $\mathbf{v} = \bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_1 = \mathbf{v}_2 - \mathbf{v}_1$ (frame-independent).

Linear momentum in center-of-mass frame: $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$.

$$\Rightarrow \ \mathbf{v}_1 = -\frac{m_2}{m_1+m_2}\mathbf{v}, \quad \mathbf{v}_2 = \frac{m_1}{m_1+m_2}\mathbf{v} \quad \Rightarrow \ m_1 v_1 = mv.$$

$$\Rightarrow \ \tan\bar{\theta} = \frac{v_1\sin\theta}{v_1\cos\theta + \bar{v}_{cm}} = \frac{\sin\theta}{\cos\theta+\rho}, \quad \rho = \frac{m}{m_2}\frac{\bar{v}_0}{v_1} = \frac{m_1}{m_2}\frac{\bar{v}_0}{v}.$$

$$\Rightarrow \ \cos\theta = -\rho(1-\cos^2\bar{\theta}) + \cos\bar{\theta}\sqrt{1-\rho^2(1-\cos^2\bar{\theta})}.$$

Elastic scattering: $T = \frac{1}{2}m\bar{v}_0^2 = \frac{1}{2}mv^2$ (in center-of-mass frame)

$$\Rightarrow \ \bar{v}_0 = v \quad \Rightarrow \ \rho = m_1/m_2.$$

Task #2: establish the relation between σ and $\bar{\sigma}$.

Number of particles scattered into infinitesimal solid angle:
$2\pi I\sigma(\theta)\sin\theta|d\theta| = 2\pi I\bar{\sigma}(\bar{\theta})\sin\bar{\theta}|d\bar{\theta}|$.

$$\Rightarrow \ \bar{\sigma}(\bar{\theta}) = \sigma(\theta)\,\frac{\sin\theta}{\sin\bar{\theta}}\left|\frac{d\theta}{d\bar{\theta}}\right| = \sigma(\theta)\,\left|\frac{d\cos\theta}{d\cos\bar{\theta}}\right|.$$

$$\Rightarrow \ \bar{\sigma}(\bar{\theta}) = \sigma(\theta)\left[2\rho\cos\bar{\theta} + \frac{1+\rho^2\cos(2\bar{\theta})}{\sqrt{1-\rho^2\sin^2\bar{\theta}}}\right].$$

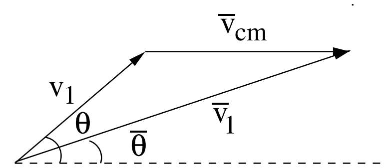
Special case: elastic scattering between particles of equal mass:

$$m_1 = m_2 \quad \Rightarrow \ \cos\theta = \cos(2\bar{\theta}) \quad \Rightarrow \ \bar{\theta} = \frac{\theta}{2}, \quad \bar{\sigma}(\bar{\theta}) = 4\cos\frac{\theta}{2}\,\sigma(\theta).$$

[mex57] Loss of kinetic energy in elastic collision

Consider a particle of mass m_1 and incident velocity $\bar{\mathbf{v}}_0$ undergoing an elastic collision via central force with a target of mass m_2 that is initially at rest. The particle emerges with velocity $\bar{\mathbf{v}}_1$ from the collision as viewed in the laboratory frame. The figure shows this velocity in relation to the center-of-mass velocity $\bar{\mathbf{v}}_{cm}$ and the final velocity $\mathbf{v}_1$ of the particle in the center-of-mass frame. Also shown are the scattering angles θ (center-of mass frame) and $\bar{\theta}$ (laboratory frame). Show that the ratio of the final and initial kinetic energies in the laboratory frame is

$$\frac{T_1}{T_0} = \frac{1 + 2\rho\cos\theta + \rho^2}{(1+\rho)^2}, \qquad \rho = \frac{m_1}{m_2}.$$



Solution:

[mex240] Elastic collision: angle between scattered particles

A particle of mass m_1 and incident velocity $\bar{\mathbf{v}}_0$ undergoes an elastic collision via central force with a particle of mass m_2 that is initially at rest. Given the scattering angles θ_1, $\theta_2 = \pi - \theta_1$ in the center-of-mass frame, find the sum $\bar{\theta}_1 + \bar{\theta}_2$ of the scattering angles in the laboratory frame as a function of θ_1 and m_1/m_2. Show that if $m_1 = m_2$ then we have $\bar{\theta}_1 + \bar{\theta}_2 = \pi/2$ for $0 < \theta_1 < \pi$.

Solution:

[mex241] Elastic collision: velocities of scattered particles

A particle of mass m_1 and incident velocity $\bar{\mathbf{v}}_0$ undergoes an elastic collision via central force with a particle of mass m_2 that is initially at rest. Show that the velocities of the scattered particles depend on the scattering angles in the laboratory frmae as follows:

$$\frac{\bar{v}_2}{\bar{v}_0} = 2\frac{m}{m_2}\cos\bar{\theta}_2, \quad m \doteq \frac{m_1 m_2}{m_1 + m_2},$$

$$\frac{\bar{v}_1}{\bar{v}_0} = \frac{m}{m_2}\left(\cos\bar{\theta}_1 \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\bar{\theta}_1}\right),$$

where two solutions ($\pm$) exist for $m_1 > m_2$ and one solution ($+$) for $m_1 < m_2$.

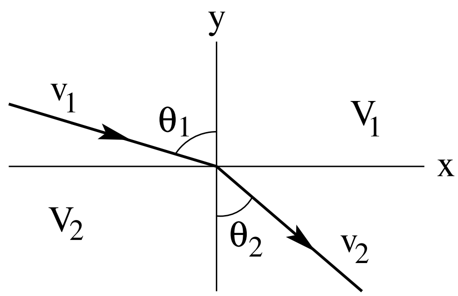
Solution:

[mex167] Mechanical refraction

A particle of mass m moving in the xy-plane is subject to a potential energy which assumes the constant value V_1 at $y \geq 0$ and the constant value V_2 at $y < 0$. Let us assume that $V_2 < V_1$. Use conversation laws to show that if the particle approaches the x-axis with speed v_1 at an angle θ_1 as shown, it will proceed with a different speed v_2 at a different angle θ_2 after crossing the line where the potential energy changes abruptly. Show in particular that the relation

$$n \equiv \frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 + \frac{2}{mv_1^2}(V_1 - V_2)},$$

between the two angles holds, where n plays the role of index of refraction.

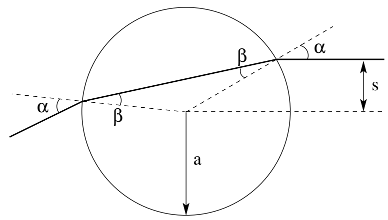


Solution:

[mex168] Scattering from a spherical potential well

Consider a spherical potential well of depth U and radius a, decribed by the potential energy $V(r) = -U\Theta(a-r)$. According to [mex167], the path of an incident particle with energy E encountering this potential will then be that of a ray of light refracted from a sphere with refractive index $n = \sqrt{1+U/E}$. (a) Calculate the maximum scattering angle as a function of n. (b) Show that the scattering cross section has the form

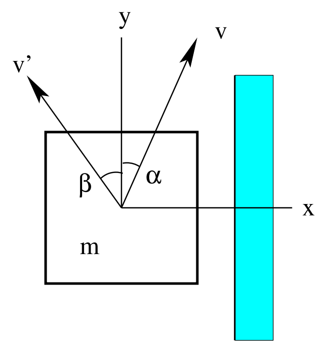
$$\sigma(\theta) = \frac{a^2n^2}{4\cos(\theta/2)}\left|\frac{[n\cos(\theta/2)-1][n-\cos(\theta/2)]}{[n^2+1-2n\cos(\theta/2)]^2}\right|.$$



Solution:

[mex219] Grazing collison between flat surfaces

Consider a cube of mass m in translational motion with velocity $\mathbf{v}$ on a frictionless airtrack. The cube is approaching a wall at a grazing angle α with one of its sides parallel to the wall. The coefficient of kinetic friction between the cube and the wall is μ. Determine the angle β describing the direction of the velocity $\mathbf{v}'$ the cube has after the collison. Assume that the recoil motion of the wall is negligible.



Solution:

[mex242] Absorption cross section of power-law potential

A uniform beam of particles of mass m and velocity v_0 is directed toward an attractive power-law potential $V(r) = -\kappa/r^\alpha$ with $\alpha > 2$. Depending on the energy E and the angular momentum ℓ the orbit of the particle leads to the center of force or it passes by at a nonvanishing minimum distance. Assume that all particles that arrive at the center of force are absorbed whereas all other particles are scattered elastically. Calculate the total cross section σ_T for particle absorption as a function of α, E, κ.

Solution:

Small-Angle Scattering [mln105]

Scattering angle from transverse momentum: $\sin\theta = \frac{p_y}{p} \quad \Rightarrow \quad \theta = \frac{p_y}{mv_0} + \ldots$

Impact parameter: s.

Impulse and transverse momentum: $p_y = \int_{-\infty}^{+\infty} dt\, F_y$.

Transverse force: $F_y = -\frac{\partial V}{\partial y} = -\frac{dV}{dr}\frac{\partial r}{\partial y} = -\frac{dV}{dr}\frac{y}{r}, \quad r = \sqrt{x^2+y^2+z^2}.$

Amount of transverse motion during collision assumed negligible: $F_y = -\frac{dV}{dr}\frac{s}{r}$.

Change in speed of particle during collision assumed negligible:

$$dt = \frac{dx}{v_0} \quad \Rightarrow \quad p_y = -\frac{s}{v_0}\int_{-\infty}^{+\infty} \frac{dV}{dr}\frac{dx}{r}.$$

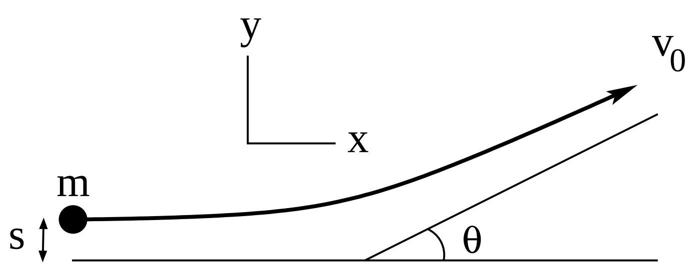
Eliminate dx: $x = \sqrt{r^2 - s^2} \quad \Rightarrow \quad \frac{dx}{dr} = \frac{r}{\sqrt{r^2-s^2}}$.

Transverse momentum: $p_y = -\frac{2s}{v_0}\int_s^{+\infty} \frac{dV}{dr}\frac{dr}{\sqrt{r^2-s^2}}$.

Scattering angle: $\theta(s) = -\frac{s}{E}\int_s^{+\infty} \frac{dV}{dr}\frac{dr}{\sqrt{r^2-s^2}}, \qquad E = \frac{1}{2}mv_0^2.$

Scattering cross section: $\sigma(\theta) = \frac{s(\theta)}{\theta}\left|\frac{ds}{d\theta}\right|$ with $s(\theta)$ from inversion of $\theta(s)$.

Application to power-law potential: [mex246]



[mex246] Small-angle scattering from power-law potential

Consider small-angle scattering from a repulsive power-law potential $V(r) = \kappa/r^\alpha$ using the relations derived in [mln105].
(a) Find the scattering cross section $\theta(s)$.
(b) Find the scattering cross section $\sigma(\theta)$.
(c) Show that the small-angle results of $\sigma(\theta)$ for $\alpha = 1$ and $\alpha = 2$ are consistent with the general results from [mex56] and [mex59], respectively.

Solution:

Classical inverse scattering [mln104]

Goal: reconstruction of potential $V(r)$ from cross section $\sigma(\theta)$.

Assumptions: $dV/dr < 0$ (repulsive force), $V(0) > E$, $V(\infty) = 0$.

Consequence: $\theta(s_1) > \theta(s_2)$ if $s_1 < s_2$.

Calculate $s(\theta)$ from $\sigma(\theta)$: $2\pi \int_\theta^\pi d\theta \sin\theta\, \sigma(\theta) = 2\pi \int_0^s ds' s' = \pi s^2$.

Orbital integral from [mln20] with $u \doteq 1/r$:

$$\frac{\pi - \theta(s)}{2} = \int_0^{u_m} \frac{s du}{\sqrt{1 - \frac{V(1/u)}{E} - s^2 u^2}}, \qquad s^2 u_m^2 + V(1/u_m)/E = 1.$$

Transformation with $x \doteq 1/s^2$, $\tilde{\theta}(x) = \theta(s)$, and $w(u) \doteq \sqrt{1 - V(1/u)/E}$:

$$\frac{\pi - \tilde{\theta}(x)}{2} = \int_0^{u_m} \frac{du}{\sqrt{x[w(u)]^2 - u^2}}, \qquad u_m^2 = x[w(u_m)]^2 \quad \Rightarrow \; u_m(x).$$

Transformation: $\displaystyle \frac{1}{2}\int_0^\alpha dx \frac{\pi - \tilde{\theta}(x)}{\sqrt{\alpha - x}} = \int_0^\alpha dx \int_0^{u_m} \frac{du}{\sqrt{\big(x[w(u)]^2 - u^2\big)\big(\alpha - x\big)}}.$

$$\Rightarrow \; \pi\sqrt{\alpha} - \int_0^\alpha dx\, \tilde{\theta}'(x)\sqrt{\alpha - x} = \pi \int_0^{u_m(\alpha)} \frac{du}{w(u)}. \qquad \text{[mex243]}$$

Set $\alpha = u^2/w^2$, implying $u_m(\alpha) \to u$, take d/du, and multiply by du:

$$\frac{\pi}{w} dw = -d\left(\frac{u}{w}\right) \int_0^{u^2/w^2} dx \frac{\tilde{\theta}'(x)}{\sqrt{u^2/w^2 - x}}. \qquad \text{[mex244]}$$

Integrate differentials dw and $d(u/w)$ with consistent boundary values:

$$\Rightarrow \; w(u) = \exp\left(\frac{1}{\pi}\int_{w/u}^\infty ds \frac{\theta(s)}{\sqrt{s^2 - [w(u)]^2/u^2}}\right). \qquad \text{[mex245]}$$

The solution $w(u)$ of this integral equation for given $\theta(s)$ thus determines $V(r)$ from $\sigma(\theta)$.

[Landau and Lifshitz 1976]

[mex243] Classical inverse scattering problem I

The reconstruction of the (central force) scattering potential $V(r)$ from the observed scattering cross section $\sigma(\theta)$ as outlined in [mln104] involves the transformation of the orbital integral

$$\frac{\pi - \theta(s)}{2} = \int_0^{u_m} \frac{s du}{\sqrt{1 - \frac{V(1/u)}{E} - s^2 u^2}}, \qquad s^2 u_m^2 + V(1/u_m)/E = 1,$$

where $u \doteq 1/r$ and $\theta(s)$ is the scattering angle as a function of the impact parameter, into the relation

$$\pi\sqrt{\alpha} - \int_0^{\alpha} dx\, \tilde{\theta}'(x)\sqrt{\alpha - x} = \pi \int_0^{u_m(\alpha)} \frac{du}{w(u)}, \qquad u_m^2 = \alpha[w(u_m)]^2,$$

for the unknown function $w(u) \doteq \sqrt{1 - V(1/u)/E}$, where $x \doteq 1/s^2$ and $\tilde{\theta}(x) = \theta(s)$. Carry out the initial steps as indicated in [mln104]. Integrate by parts on the left and interchange the order of integrations on the right.

Solution:

[mex244] Classical inverse scattering problem II

The reconstruction of the (central force) scattering potential $V(r)$ from the observed scattering cross section $\sigma(\theta)$ as outlined in [mln104] involves the conversion of the relation

$$\pi\sqrt{\alpha} - \int_0^\alpha dx\,\tilde{\theta}'(x)\sqrt{\alpha - x} = \pi\int_0^{u_m(\alpha)} \frac{du}{w(u)}, \qquad u_m^2 = \alpha[w(u_m)]^2,$$

as derived in [mex243] into the differential relation

$$\pi\frac{dw}{w} = -d\left(\frac{u}{w}\right)\int_0^{u^2/w^2} dx \frac{\tilde{\theta}'(x)}{\sqrt{u^2/w^2 - x}}$$

by setting $\alpha = u^2/w^2$, taking the derivative with respect to $u$on both sides and multiplying back by du. Show that the boundary value $u_m(\alpha)$ becomes the unrestricted u in the process.

Solution:

[mex245] Classical inverse scattering problem III

The final step in the reconstruction of the (central force) scattering potential $V(r)$ from the observed scattering cross section $\sigma(\theta)$ as outlined in [mln104] involves the integration of the differential relation

$$\frac{\pi}{w}dw = -d\left(\frac{u}{w}\right)\int_0^{u^2/w^2} dx \frac{\tilde{\theta}'(x)}{\sqrt{u^2/w^2 - x}}$$

into the integral equation

$$\Rightarrow \; w(u) = \exp\left(\frac{1}{\pi}\int_{w/u}^{\infty} ds \frac{\theta(s)}{\sqrt{s^2 - [w(u)]^2/u^2}}\right)$$

for the quantity $w(u) \doteq \sqrt{1 - V(1/u)/E}$ with $u = 1/r$. Carry out this step by interchanging the order of integration on the right.

Solution:

Decay of Particle I [mln102]

Particle at rest decays into two particles:

Decay energy: $\epsilon = E_{\text{int}}^{(0)} - E_{\text{int}}^{(1)} - E_{\text{int}}^{(2)}$ (change in internal energy).

Masses of decay products: m_1, m_2.

Momentum conservation: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \quad p_1 = p_2 \doteq p.$

Energy conservation: $E_{\text{int}}^{(0)} = E_{\text{int}}^{(1)} + \dfrac{p^2}{2m_1} + E_{\text{int}}^{(2)} + \dfrac{p^2}{2m_2}.$

$$\Rightarrow \ \epsilon = \frac{p^2}{2m} = T_1 + T_2.$$

Reduced mass $m = \dfrac{m_1 m_2}{m_1 + m_2}.$

Kinetic energies: $T_1 = \dfrac{p^2}{2m_1} = \dfrac{\epsilon m_2}{m_1 + m_2}, \quad T_2 = \dfrac{p^2}{2m_2} = \dfrac{\epsilon m_1}{m_1 + m_2}.$

- Decay products move in opposite directions.
- All directions of $\mathbf{p_1}$ equally likely.
- Kinetic energies T_1, T_2 determined by conservation laws alone.

Particle at rest decays into three particles:

Decay energy: $\epsilon = E_{\text{int}}^{(0)} - E_{\text{int}}^{(1)} - E_{\text{int}}^{(2)} - E_{\text{int}}^{(3)}.$

Masses of decay products: m_1, m_2, m_3.

Momentum conservation: $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0.$

Energy conservation: $E_{\text{int}}^{(0)} = E_{\text{int}}^{(1)} + \dfrac{p_1^2}{2m_1} + E_{\text{int}}^{(2)} + \dfrac{p_2^2}{2m_2} + E_{\text{int}}^{(3)} + \dfrac{p_3^2}{2m_3}.$

- Relative directions between decay products constrained but not determined by conservation laws.
- Kinetic energies T_1, T_2, T_3 constrained but not determined by conservation laws.
- Maximum kinetic energy T_i limited by ϵ, m_i. →[mex237]

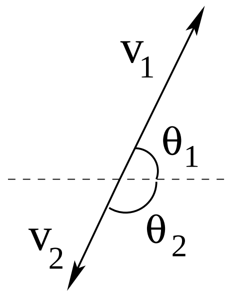
Decay of Particle II [mln103]

Particle in motion decays into two particles.

View from center-of-mass frame:

Momenta: $m_1\mathbf{v}_1 = -m_2\mathbf{v}_2$.

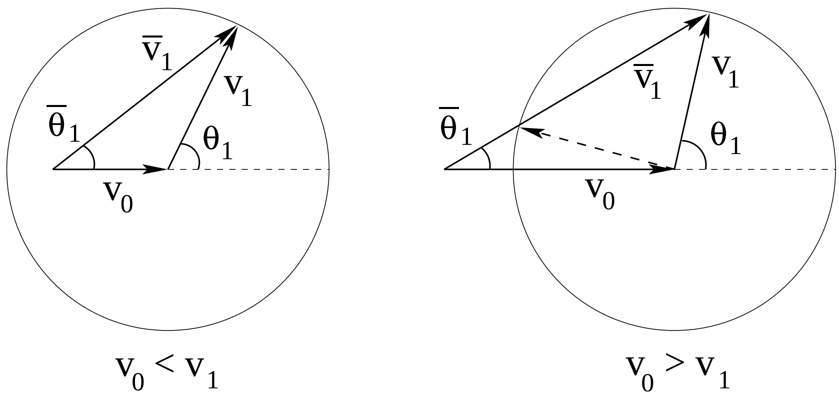
Directions of decay products: $\theta_1 + \theta_2 = \pi$.



View from laboratory frame:

Momentum of particle before decay: $(m_1 + m_2)\mathbf{v}_0$.

Directions of decay products: $\bar{\theta}_1, \bar{\theta}_2$.



Task #1: Find the relation between θ_1 and $\bar{\theta}_1$.

$$\tan\bar{\theta}_1 = \frac{v_1 \sin\theta_1}{v_1\cos\theta_1 + v_0}$$

$$\Rightarrow \quad \cos\theta_1 = -\frac{v_0}{v_1}\sin^2\bar{\theta}_1 \pm \cos\bar{\theta}_1\sqrt{1 - \frac{v_0^2}{v_1^2}\sin^2\bar{\theta}_1}.$$

Task #2: Find the relation between $\bar{\theta}_1$ and $\bar{\theta}_2$. →[mex238]

Task #3: Find the range of the angle $\bar{\theta} \doteq \bar{\theta}_1 + \bar{\theta}_2$. →[mex239]

[mex237] Decay of particle: maximum kinetic energy.

A particle of mass M at rest decays into three particles of masses m_1, m_2, m_3 by releasing a total decay energy ϵ. Assume that mass-energy conversion is negligible ($M = m_1 + m_2 + m_3$) and that the resulting momenta of the decay products are nonrelativistic. What is the maximum kinetic energy $T_1^{\max}$ of the emerging particle with mass m_1?

Solution:

[mex238] Decay of particle: directions in lab frame I

A particle of mass M and velocity $\mathbf{v}_0$ (in the lab frame) decays into two particles of masses m_1, m_2 by releasing a total decay energy ϵ. Assume that mass-energy conversion is negligible ($M = m_1 + m_2$) and that the resulting momenta of the decay products are nonrelativistic. Show that the angles of the directions of the decay products relative to the forward direction of the original particle satisfy the following relation:

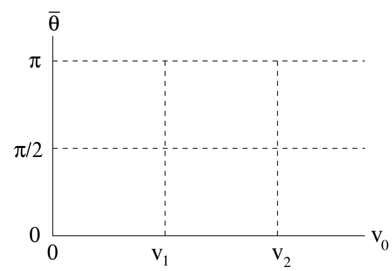
$$\frac{2\epsilon \sin^2(\bar{\theta}_1 + \bar{\theta}_2)}{(m_1 + m_2)v_0^2} = \frac{m_1}{m_2}\sin^2\bar{\theta}_1 + \frac{m_2}{m_1}\sin^2\bar{\theta}_2 - 2\sin\bar{\theta}_1 \sin\bar{\theta}_2 \cos(\bar{\theta}_1 + \bar{\theta}_2).$$

Explain what this relation implies in the two limits $v_0 \to 0$ and $v_0 \to \infty$.

Solution:

[mex239] Decay of particle: directions in lab frame II

A particle of mass M and velocity $\mathbf{v}_0$ (in the lab frame) decays into two particles of masses m_1, m_2 by releasing a total decay energy ϵ. Assume that mass-energy conversion is negligible ($M = m_1 + m_2$) and that the resulting momenta of the decay products are nonrelativistic.
(a) Calculate the angle $\bar{\theta} \doteq \bar{\theta}_1 + \bar{\theta}_2$ between the two emerging particles in the lab frame as a function of v_0, v_1, v_2 and θ_1, where $\theta_1, \theta_2 = \pi - \theta_1$ are the corresponding angles in the center-of-mass frame.
(b) Determine the range of $\bar{\theta}$ as a function of v_0 under the assumption that $v_1 < v_2$ on a map as follows:



Solution: