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09. Central Force Motion II

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Abstract
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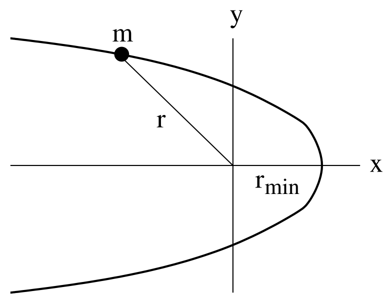
[mex44] Cometary motion on parabolic orbit

Determine a parametric representation $x(\eta), y(\eta), t(\eta), \vartheta(\eta)$ for the parabolic motion in time of a comet with mass m in the central force potential $V(r) = -\kappa/r$. Start from the general integral expression for $t(r)$ and use the parametrization $r = r_{min}(1 + \eta^2)$.



Solution:

Motion in time on elliptic Kepler orbit [mln19]

Use the formal solution with $E < 0,\ V(r) = -\dfrac{\kappa}{r},\ \kappa = GmM$:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left[E + \frac{\kappa}{r} - \frac{\ell^2}{2mr^2}\right]}} = \sqrt{\frac{m}{2|E|}} \int \frac{rdr}{\sqrt{-r^2 + \frac{\kappa}{|E|}\,r - \frac{\ell^2}{2m|E|}}}.$$

Introduce semi-major axis $a = \dfrac{\kappa}{2|E|}$ and eccentricity $e = \sqrt{1 - \dfrac{2|E|\ell^2}{m\kappa^2}}$:

$$\Rightarrow\ t = \sqrt{\frac{ma}{\kappa}} \int \frac{rdr}{\sqrt{a^2e^2 - (a-r)^2}}.$$

Substitute $a - r = ae\cos\psi$: $\ \Rightarrow\ r(\psi) = a(1 - e\cos\psi), \quad dr = ae\sin\psi d\psi.$

$$\Rightarrow\ t = \sqrt{\frac{ma}{\kappa}} \int d\psi\, a(1 - e\cos\psi) = \sqrt{\frac{ma^3}{\kappa}}(\psi - e\sin\psi).$$

For the angular coordinate ϑ, use $r(\psi)$ and the orbital equation $r(\vartheta) = p/(1+\cos\vartheta)$ with $p = a(1-e^2)$. Then eliminate r from $r(\vartheta)$ and $r(\psi)$. For the Cartesian coordinates use $ex = p - r$ and $x^2 + y^2 = r^2$.

Parametric representation for the motion in time: $0 \leq \psi \leq 2\pi$

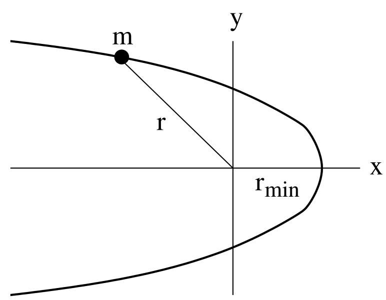
$$\begin{aligned}
r(\psi) &= a(1 - e\cos\psi) && (r_{min} \leq r \leq r_{max}) \\
\tan\frac{\vartheta(\psi)}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2} && (0 \leq \vartheta \leq 2\pi) \\
x(\psi) &= a(\cos\psi - e) \\
y(\psi) &= a\sqrt{1-e^2}\sin\psi \\
t(\psi) &= \sqrt{\frac{ma^3}{\kappa}}(\psi - e\sin\psi) && (0 \leq t \leq \tau)
\end{aligned}$$

Period of motion: $\tau = 2\pi\sqrt{\dfrac{ma^3}{\kappa}}$.

Circular limit: $e = 0\ \Rightarrow\ r = a = \text{const},\ \vartheta = \psi,\ t = \tau\,\dfrac{\vartheta}{2\pi}$.

[mex234] Cometary motion on hyperbolic orbit

Determine a parametric representation $x(\xi), y(\xi), t(\xi), \vartheta(\xi)$ for the hyperbolic motion in time of a comet with mass m in the central force potential $V(r) = -\kappa/r$. Start from the general integral expression for $t(r)$ and use the parametrization $\tilde{a} + r = \tilde{a}e\cosh\xi$ with $\tilde{a} = \kappa/2E$ and $e^2 = 1 + 2E\ell^2/m\kappa^2$, where E is the energy and ℓ the angular momentum.



Solution:

[mex145] Close encounter of the first kind

A rock of mass m approaches the solar system with a velocity v_0, and if it had not been attracted toward the sun it would have missed the sun by a distance d. Neglect the influence of the planets. Show that the closest approach of the orbit is

$$a = \sqrt{d^2 + d_0^2} - d_0, \quad d_0 \doteq \frac{Gm_\odot}{v_0^2}.$$

Solution:

[mex43] Kepler's second and third laws

Derive Kepler's second and third laws of planetary motion from the results established in class for central force motion. Use the case of an elliptic orbit ($0 < e < 1$). Specifically: (a) Show that the rate at which area is swept by the position vector of the planet, dA/dt, is a constant. Determine that constant. (b) Integrate the result for dA/dt over one period of revolution τ to show that the following relation holds between τ and the semi-major axis a: $\tau^2 = 4\pi^2(m/\kappa)a^3$, where $\kappa = GMm, M = m_S + m_P, m = m_S m_P/(m_S + m_P)$.

Solution:

[mex164] Circular and radial motion in inverse-square law potential

A particle of mass m is subject to the central force $F(r) = -mk^2/r^3$, where k is a constant. (a) Find the time T it takes the particle to move once around a circular orbit of radius r_0. (b) Find the time τ it takes the particle to reach the center of force if released from rest at radius r_0.

Solution:

[mex54] Circular orbits of the Yukawa potential

A particle of mass m moves in the Yukawa potential $V(r) = -(k/r)e^{-r/\rho}$. Circular orbits exist only if the angular momentum ℓ does not exceed a certain value ℓ_{max}. For any value $\ell < \ell_{max}$, there exist two circular orbits, one stable orbit at radius $R_S(\ell)$ and one unstable orbit at radius $R_U(\ell)$. (a) Determine the value of $\ell_{max}/\sqrt{mk\rho}$. (b) Determine the value $R_S(\ell_{max})/\rho = R_U(\ell_{max})/\rho$. (c) Sketch the two functions $R_S(\ell)$ and $R_U(\ell)$ and label them clearly.

Solution:

Orbital Differential Equation [mln46]

Equation of motion for radial motion: $m\ddot{r} - \frac{\ell^2}{mr^3} = F(r), \quad F(r) = -V'(r).$

Angular velocity: $\dot{\vartheta} = \frac{\ell}{mr^2}.$

Use new radial variable: $u \equiv \frac{1}{r}.$

$$\Rightarrow \frac{dr}{dt} = \frac{d}{dt}\frac{1}{u} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\vartheta}\frac{d\vartheta}{dt} = -\frac{\ell}{m}\frac{du}{d\vartheta}.$$

$$\Rightarrow \frac{d^2r}{dt^2} = -\frac{\ell}{m}\frac{d}{dt}\left(\frac{du}{d\vartheta}\right) = -\frac{\ell}{m}\frac{d^2u}{d\vartheta^2}\frac{d\vartheta}{dt} = -\left(\frac{\ell}{m}\right)^2 u^2\frac{d^2u}{d\vartheta^2}.$$

$$\Rightarrow \quad -\frac{\ell^2}{m}u^2\frac{d^2u}{d\vartheta^2} - \frac{\ell^2}{m}u^3 = F\left(\frac{1}{u}\right).$$

Orbital differential equation: $\frac{d^2u}{d\vartheta^2} + u = -\frac{m}{\ell^2u^2}F\left(\frac{1}{u}\right).$

Initial conditions: $u(0) = 1/r_{min},\ 1/r_{max}, \quad u'(0) = 0.$

Like the orbital integral, the orbital differential equation describes the relation between the radial and angular coordinates of an orbit, a relation from which the variable 'time' has been eliminated.

While the orbital integral is most useful for calculating orbits of a given central force potential, the orbital differential equation is particularly useful for finding central force potentials in which given orbits are realized.

Applications:

- Kepler problem [mex48]
- Exponential spiral orbit [mex49]
- Circular orbit through center of force [mex50]
- Linear spiral orbit [mex52]

[mex49] Exponential spiral orbit

A particle of mass m moves along an exponential spiral orbit $r(\vartheta) = r_0 e^{\vartheta}$ under the influence of a central force potential $V(r)$. (a) Use the orbital differential equation

$$\frac{d^2u}{d\vartheta^2} + u = -\frac{m}{\ell^2 u^2} F(u^{-1}),$$

where $u \equiv 1/r$, $F(r) = -dV/dr$ to determine the potential $V(r)$. (b) Determine the energy E of this orbit. (c) Determine the motion in time $r(t), \vartheta(t)$ of the particle on this orbit.

Solution:

[mex48] Orbital differential equation applied to Kepler problem

Derive the orbital relation $p/r = 1 + e\cos(\vartheta - \vartheta_0)$ with $p = \ell^2/m\kappa$ and $e = \sqrt{1 + 2E\ell^2/m\kappa^2}$, which describes all orbits of the Kepler problem, from the orbital differential equation

$$\frac{d^2u}{d\vartheta^2} + u = -\frac{m}{\ell^2 u^2} F(u^{-1}),$$

where $u \equiv 1/r$, $F(r) = -dV/dr$, and $V(r) = -\kappa/r$. Do not reason backward. Pretend you do not know the solution.

Solution:

[mex52] Linear spiral orbit

A particle of mass m moves along a linear spiral orbit $r(\vartheta) = k\vartheta$ under the influence of a central force potential $V(r)$. (a) Use the orbital differential equation

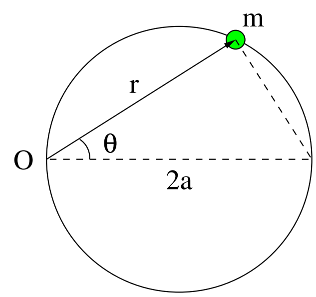
$$\frac{d^2u}{d\vartheta^2} + u = -\frac{m}{\ell^2 u^2} F(u^{-1}),$$

where $u \equiv 1/r$, $F(r) = -dV/dr$ to determine the potential $V(r)$. (b) Determine the energy E of this orbit. (c) Determine the motion in time $r(t), \vartheta(t)$ of the particle on this orbit.

Solution:

[mex50] Crash course on circular orbit

A particle of mass m moves on a circular orbit of radius a passing through the center O of a power-law central force potential $V(r) = -\kappa/r^\alpha$. (a) Determine the exponent α for which such an orbit exists. (b) Find the angular momentum ℓ and the energy E of this orbit. (c) Determine the period τ of this circular orbit as a function of a, m, κ.



Solution:

Laplace-Runge-Lenz Vector [mln45]

Central force: $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$.

Equation of motion: $\dot{\mathbf{p}} = F(r)\dfrac{\mathbf{r}}{r}$.

Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}$.

Conservation law: $\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = 0 \;\Rightarrow\; \mathbf{L} = \text{const.}$

$$\begin{aligned}
\Rightarrow\; \frac{d}{dt}(\mathbf{p} \times \mathbf{L}) &= \dot{\mathbf{p}} \times \mathbf{L} = \frac{mF(r)}{r}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{mF(r)}{r}\left[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - r^2\dot{\mathbf{r}}\right] \\
&= -mF(r)r^2\left[\frac{1}{r}\dot{\mathbf{r}} - \frac{\dot{r}}{r^2}\mathbf{r}\right] = -mF(r)r^2\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right].
\end{aligned}$$

We have used $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, $\quad \mathbf{r} \cdot \dot{\mathbf{r}} = \dfrac{1}{2}\dfrac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = r\dot{r}$.

Kepler system: $F(r) = -\dfrac{\kappa}{r^2} \;\Rightarrow\; \dfrac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \dfrac{d}{dt}\left[\dfrac{m\kappa\mathbf{r}}{r}\right]$.

Laplace-Runge-Lenz vector: $\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\kappa\dfrac{\mathbf{r}}{r} = \text{const.}$

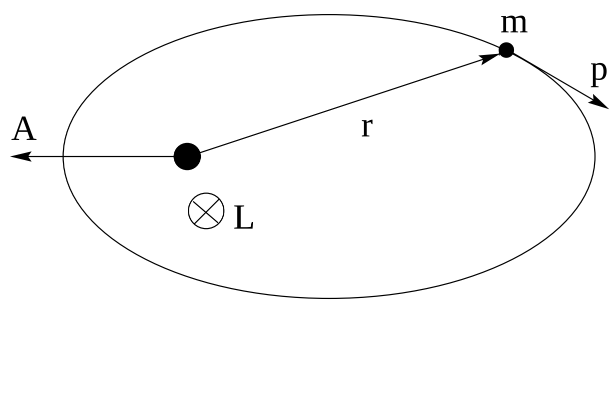
The vector $\mathbf{A}$ lies in the plane of the orbit, points to the pericenter, and has magnitude $A = m\kappa e$, where e is the eccentricity of the orbit.

$$\mathbf{L} \perp \mathbf{p} \times \mathbf{L} \text{ and } \mathbf{L} \perp \mathbf{r} \;\Rightarrow\; \mathbf{A} \perp \mathbf{L}.$$

$$\mathbf{A} \cdot \mathbf{r} \equiv Ar\cos\vartheta = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - m\kappa r = \mathbf{L} \cdot (\mathbf{r} \times \mathbf{p}) - m\kappa r = \ell^2 - m\kappa r.$$

$\mathbf{A} \cdot \dfrac{\mathbf{r}}{r}$ assumes its maximum value for $r = r_{min}$ (pericenter).

$$\frac{\ell^2/m\kappa}{r} = 1 + \frac{A}{m\kappa}\cos\vartheta \quad \text{(conic section)}.$$



Precession of the Perihelion [mln21]

Gravitation $\Rightarrow$ inverse-square force $\Rightarrow$ bounded orbits are closed (stationary ellipses with sun at one focus) $\Rightarrow$ apsidal angle: $\Delta\vartheta = \pi$.

Any slight deviation from the $1/r^2$-force law causes a precession of the perihelion in planetary orbits.

The perihelion of the planets Mercury, Venus, and Mars, was long observed to precess. The main cause is the presence of other planets in the solar system.

For Mercury the total precession observed is 531" per century, of which 43" are unexplained by many-body effects.

The residual 43" per century can be accounted for as a combination of relativistic effects:

- $\frac{1}{3}$ due to time dilation (special relativity),
- $\frac{1}{6}$ due to mass increase (special relativity),
- $\frac{1}{2}$ due to the fact that force is not instantaneous (general relativity).

In the framework of a pertubation calculation, the relativistic effects can be taken into account as a correction to the gravitational potential:

$$V(r) = -\frac{\kappa}{r} - \frac{\gamma}{r^3}, \qquad \kappa = GmM, \quad \gamma = \frac{G\ell^2 M}{c^2 m}.$$

Here m is the mass of the planet, M is the solar mass, G is the universal gravitational constant, ℓ is the angular momentum of the orbit, and c is the speed of light.

The angle $\delta\vartheta$ of precession per cycle can be calculated by different means including the following:

- perturbative correction to the orbital integral $\Rightarrow$ [mex165],
- perturbative solution of the orbital integral equation $\Rightarrow$ [mex166].

The result of both methods is $\delta\vartheta = 6\pi \left(\frac{GmM}{c\ell}\right)^2 \simeq \frac{6\pi GM}{c^2 a(1-e^2)}$.

The effect is enhanced when the semi-major axis a is small and the eccentricity e is large. Hence it is most prominent in Mercury's orbit.

[mex165] Precession of the perihelion: orbital integral

Consider the Kepler problem with a correction term reflecting relativistic effects:

$$V(r) = -\frac{\kappa}{r} - \frac{\gamma}{r^3}, \qquad \kappa = GmM, \quad \gamma = \frac{G\ell^2 M}{c^2 m}.$$

Calculate the angle $\delta\vartheta$ of precession per cycle of the perihelion as a correction to the known apsidal angle of the Kepler problem in a perturbative treatment of the orbital integral:

$$\delta\vartheta = 2\int_{r_1}^{r_2} dr\, \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} - 2\pi.$$

Solution:

[mex166] Precession of the perihelion: orbital differential equation

Consider the Kepler problem with a correction term reflecting relativistic effects:

$$V(r) = -\frac{\kappa}{r} - \frac{\gamma}{r^3}, \qquad \kappa = GmM, \quad \gamma = \frac{G\ell^2 M}{c^2 m}.$$

(a) Show that the resulting orbital differential equation acquires a nonlinear term as follows:

$$\frac{d^2u}{d\vartheta^2} + u = \frac{1}{p} + \alpha u^2, \qquad \frac{1}{p} = \frac{m\kappa}{\ell^2}, \quad \alpha = \frac{3GM}{c^2}.$$

(b) Calculate the angle $\delta\vartheta$ of precession per cycle of the perihelion by a perturbative solution.

Solution:

[mex45] The comet and the planet

A comet moves along a parabolic orbit which brings it to a distance d from the sun at its closest point. A planet circles the sun at radius R in the same plane. (a) Find the fraction of a planetary year which the comet spends inside the planetary orbit as a function of d/R. (b) Show that this fraction cannot exceed the value $2/3\pi$ no matter what the value of d/R is. Use the results of [mex44] for the parabolic motion. The masses of the comet and the planet are very small compared to the mass of the sun.

Solution:

[mex42] Free fall with or without angular momentum

Two particles of mass m_1 and m_2 under the influence of their mutual gravitational attraction describe circular orbits with period τ about their center of mass. Now they are abruptly stopped in their orbits and allowed to gravitate toward each other. Show that they will collide after a time $T = \tau/4\sqrt{2}$.

Solution:

[mex169] Elliptic and hyperbolic orbits

Calculate the orbital integral (a) for an orbit with energy $E > 0$ and angular momentum ℓ of the attractive central-force potential $V(r) = \frac{1}{2}kr^2$ and (b) for an orbit with energy $E < 0$ and angular momentum ℓ of the repulsive central-force potential $V(r) = -\frac{1}{2}kr^2$. Show that the solutions (a) and (b) can be cast into the form $x^2/a^2 \pm y^2/b^2 = 1$, respectively, if the Cartesian axes are suitably oriented. Find the parameters a and b in each case as functions of E, ℓ, m, k. Express E and ℓ in terms of a, b, m, k in each case.

Solution: