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On the Global Behavior of Some Systems of Difference

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ON THE GLOBAL BEHAVIOR OF SOME SYSTEMS OF DIFFERENCE EQUATIONS

BY

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ABSTRACT

This dissertation is an exposition of systems of difference equations. I examine multiple examples of both piecewise and rational difference equations.

In the first two manuscripts, I share the published results of two members of the following family of 81 systems of piecewise linear difference equations:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, \qquad n = 0, 1, \ldots$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$, and where the parameters a, b, c and d are integers between -1 and 1, inclusively. Since each parameter can be one of three values, there are 81 members. Each system is designated a number. The system's number $\mathcal{N}$ is given by

$$\mathcal{N} = 27(a + 1) + 9(b + 1) + 3(c + 1) + (d + 1) + 1.$$

The first manuscript is a study of System(2). System(2) results when $a = b = c = -1$ and $d = 0$. For System(2), I show that there exists a unique equilibrium solution and exactly two prime period-5 solutions, and that every solution of the system is eventually one of the two prime period-5 solutions or the unique equilibrium solution.

The second manuscript is a study of System(8). System(8) results when $a = b = -1$, $c = 1$ and $d = 0$. For System(8), I show that there exists a unique equilibrium solution and exactly two prime period-3 solutions, and that except for the equilibrium solution, every solution of the system is eventually one of the two prime period-3 solutions.

Of the 81 systems, 65 have been studies thoroughly. In Appendix .1, I give the unpublished results of the 21 systems that I studied. In Appendix .2, I list all 81 systems (studied by W. Tikjha, E. Grove, G. Ladas, and E. Lapierre) each with a theorem or conjecture about its global behavior.

In the third manuscript, I give the published results of the following system of rational difference equations:

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{x_n + y_n} \\ \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + y_n}{y_n} \end{cases}, \qquad n = 0, 1, ...$$

where the parameters and initial conditions are positive real values. I show that the system is permanent and has a unique positive equilibrium which is locally asymptotically stable. I also find sufficient conditions to insure that the unique positive equilibrium is globally asymptotically stable.

In Appendix .3, I give the unpublished results of the following system of rational difference equations:

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{x_n + y_n} \\ \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + y_n}{B_2 x_n + y_n} \end{cases}, \qquad n = 0, 1, ...$$

where the parameters and initial conditions are positive real values. I show that the system is permanent. I also find sufficient conditions to insure that the unique positive equilibrium is globally asymptotically stable.

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The systems of difference equations studied for this dissertation were first called to my attention by my adviser, Dr. Gerasimos Ladas. I would like to thank him for his advice, encouragement and support throughout this project and more generally throughout out all of my studies. I also thank Dr. Mustafa Kulenovic for serving on my examination committee and my dissertation committee. I must acknowledge with gratitude and respect Dr. Edward Grove, for his time and continuous coaching, all while he was recovering from a substantial medical condition. My list of acknowledgments cannot be complete without including the memory of Dr. Louis Pakula. A fellow New Yorker, he was the first person I spoke with at the University of Rhode Island about enrolling in the PhD program. He was very encouraging and gave me hope that I would be able to complete the program.

I would like to thank Emmanouil Drymonis , Chris Lynd and Wirot Tikjha for introducing me to several computer algebra systems, graphic software and LaTex, all of which became essential tools for building this dissertation.

I would also like to thank my colleagues at Johnson and Wales University who for the last six years rearranged their schedules and committee meetings to accommodate my class schedule at the University of Rhode Island.

This dissertation would not be possible without all of those I mentioned above.

DEDICATION

To my husband, Robert L. Lapierre.

PREFACE

This dissertation is written in manuscript format.

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On the Global Behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n - |y_n|$

Published in Advances in Difference Equations, 2010

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1.1 Abstract

In this paper we consider the system in the title where the initial condition $(x_0, y_0) \in \mathbf{R}^2$. We show that the system has exactly two prime period-5 solutions and a unique equilibrium point $(0, -1)$. We also show that every solution of the system is eventually one of the two prime period-5 solutions or else the unique equilibrium point.

1.2 Introduction

In this paper we consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases} , \qquad n = 0, 1, ... \tag{2}$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$. We show that every solution of System(2) is eventually either one of two prime period-5 solutions or else the unique equilibrium point $(0, -1)$.

System(2) was motivated by Devanney's Gingerbread man map [1, 2]

$$x_{n+1} = |x_n| - x_{n-1} + 1$$

or its equivalent system of piecewise linear difference equations [3, 4]

$$\begin{cases} x_{n+1} & = & |x_n| - y_n + 1 \\ \\ y_{n+1} & = & x_n \end{cases} , \quad n = 0, 1, 2, \ldots .$$

We believe that the methods and techniques used in this paper will be useful in discovering the global character of solutions of similar systems, including the Gingerbread man map.

In this paper we consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases} , \qquad n = 0, 1, ... \tag{2}$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$. We show that every solution of System(2) is eventually either one of two prime period-5 solutions or else the unique equilibrium point $(0, -1)$.

1.3 Global Results

System(2) has the equilibrium point $(\overline{x}, \overline{y}) \in \mathbf{R}^2$ given by

$$(\overline{x}, \overline{y}) = (0, -1).$$

System(2) has two prime period-5 solutions,

$$P_5^1 = \begin{pmatrix} x_0 = 0, & y_0 = 1 \\ x_1 = -2, & y_1 = -1 \\ x_2 = 2, & y_2 = -3 \\ x_3 = 4, & y_3 = -1 \\ x_4 = 4, & y_4 = 3 \end{pmatrix} \quad \text{and} \quad P_5^2 = \begin{pmatrix} x_0 = 0, & y_0 = \frac{1}{7} \\ x_1 = -\frac{8}{7}, & y_1 = -\frac{1}{7} \\ x_2 = \frac{2}{7}, & y_2 = -\frac{9}{7} \\ x_3 = \frac{4}{7}, & y_3 = -1 \\ x_4 = \frac{4}{7}, & y_4 = -\frac{3}{7} \end{pmatrix}.$$

Set

$$\begin{aligned} l_1 &= \{(x, y) : x \geq 0, y = 0\} \\ l_2 &= \{(x, y) : x = 0, y \geq 0\} \\ l_3 &= \{(x, y) : x < 0, y = 0\} \\ l_4 &= \{(x, y) : x = 0, y < 0\} \\ Q_1 &= \{(x, y) : x > 0, y > 0\} \\ Q_2 &= \{(x, y) : x < 0, y > 0\} \\ Q_3 &= \{(x, y) : x < 0, y < 0\} \\ Q_4 &= \{(x, y) : x > 0, y < 0\}. \end{aligned}$$

Theorem 1.3.1 *Let $(x_0, y_0) \in \mathbf{R}^2$. Then there exists an integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually either the prime period-5 solution P_5^1, the prime period-5 solution P_5^2 or else the unique equilibrium point $(0, -1)$.*

The proof is a direct consequence of the following lemmas.

Lemma 1.3.2 *Suppose there exists an integer $M \geq 0$ such that $-1 \leq x_M \leq 0$ and $y_M = -x_M - 1$. Then $(x_{M+1}, y_{M+1}) = (0, -1)$, and so $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is the equilibrium solution.*

Proof: Note that

$$\begin{array}{rclclcl} x_{M+1} & = & |x_M| - y_M - 1 & = & -x_M - (-x_M - 1) - 1 & = & 0 \\ y_{M+1} & = & x_M - |y_M| & = & x_M - (x_M + 1) & = & -1, \end{array}$$

and so the proof is complete. □

Lemma 1.3.3 *Suppose there exists an integer $M \geq 0$ such that $x_M \geq 1$ and $y_M = x_M - 1$. Then $(x_{M+1}, y_{M+1}) = (0, 1)$, and so $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^1.*

Proof: We have

$$\begin{array}{rclclcl} x_{M+1} & = & |x_M| - y_M - 1 & = & x_M - (x_M - 1) - 1 & = & 0 \\ y_{M+1} & = & x_M - |y_M| & = & x_M - (x_M - 1) & = & 1, \end{array}$$

and so the proof is complete. □

Lemma 1.3.4 *Suppose there exists an integer $M \geq 0$ such that $x_M = 0$ and $y_M \geq 0$. Then the following statements are true:*

1. $x_{M+5} = 0$.
2. *If* $y_M > \dfrac{1}{4}$, *then* $\{(x_n, y_n)\}_{n=M+5}^{\infty}$ *is* P_5^1.
3. *If* $0 \leq y_M \leq \dfrac{1}{4}$, *then* $y_{M+5} = 8y_M - 1$.

Proof: We have $x_M = 0$ and $y_M \geq 0$. Then

$$\begin{array}{rclcl} x_{M+1} & = & |x_M| - y_M - 1 & = & -y_M - 1 < 0 \\ y_{M+1} & = & x_M - |y_M| & = & -y_M \leq 0 \\ x_{M+2} & = & |x_{M+1}| - y_{M+1} - 1 & = & 2y_M \geq 0 \\ y_{M+2} & = & x_{M+1} - |y_{M+1}| & = & -2y_M - 1 < 0 \\ x_{M+3} & = & |x_{M+2}| - y_{M+2} - 1 & = & 4y_M \geq 0 \\ y_{M+3} & = & x_{M+2} - |y_{M+2}| & = & -1 \\ x_{M+4} & = & |x_{M+3}| - y_{M+3} - 1 & = & 4y_M \geq 0 \\ y_{M+4} & = & x_{M+3} - |y_{M+3}| & = & 4y_M - 1 \\ x_{M+5} & = & |x_{M+4}| - y_{M+4} - 1 & = & 0, \end{array}$$

and so Statement 1 is true.

If $y_M > \dfrac{1}{4}$, then $y_{M+5} = x_{M+4} - |y_{M+4}| = 1$. That is $(x_{M+5}, y_{M+5}) = (0, 1)$ and so Statement 2 is true.

If $0 \leq y_M \leq \dfrac{1}{4}$, then $y_{M+5} = x_{M+4} - |y_{M+4}| = 8y_M - 1$, and so Statement 3 is true. □

Lemma 1.3.5 *Suppose there exists an integer $M \geq 0$ such that $x_M = 0$ and $y_M < -1$. Then the following statements are true:*

1. *$x_{M+4} = 0$.*
2. *If $-\dfrac{3}{2} < y_M < -1$, then $y_{M+4} = -4y_M - 5$.*
3. *If $y_M \leq -\dfrac{3}{2}$, then $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1.*

Proof: We have $x_M = 0$ and $y_M < -1$. Then

$$\begin{array}{rcl cl}
x_{M+1} & = & |x_M| - y_M - 1 & = & -y_M - 1 > 0 \\
y_{M+1} & = & x_M - |y_M| & = & y_M < 0 \\
x_{M+2} & = & |x_{M+1}| - y_{M+1} - 1 & = & -2y_M - 2 > 0 \\
y_{M+2} & = & x_{M+1} - |y_{M+1}| & = & -1 \\
x_{M+3} & = & |x_{M+2}| - y_{M+2} - 1 & = & -2y_M - 2 > 0 \\
y_{M+3} & = & x_{M+2} - |y_{M+2}| & = & -2y_M - 3 \\
x_{M+4} & = & |x_{M+3}| - y_{M+3} - 1 & = & 0,
\end{array}$$

and so Statement 1 is true.

Now if $-\dfrac{3}{2} < y_M < -1$, then $y_{M+3} = -2y_M - 3 < 0$. Thus $y_{M+4} = x_{M+3} - |y_{M+3}| = -4y_M - 5$, and so Statement 2 is true.

Lastly, if $y_M \leq -\dfrac{3}{2}$, then $y_{M+3} = -2y_M - 3 \geq 0$. Thus $y_{M+4} = x_{M+3} - |y_{M+3}| = 1$; that is $(x_{M+4}, y_{M+4}) = (0, 1)$ and so Statement 3 is true. □

Lemma 1.3.6 *Suppose there exists an integer $M \geq 0$ such that $x_M \geq 0$ and $y_M = 0$. Then the following statements are true:*

1. *If $x_M \geq 1$ then $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^1.*

2. *If $\frac{1}{4} < x_M < 1$, then $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is P_5^1.*

3. *If $0 \leq x_M \leq \frac{1}{4}$, then $x_{M+6} = 0$ and $y_{M+6} = 8x_M - 1$.*

Proof: First consider the case $x_M \geq 1$ and $y_M = 0$. Then

$$\begin{array}{rcllcl} x_{M+1} &=& |x_M| - y_M - 1 &=& x_M - 1 \geq 0 \\ y_{M+1} &=& x_M - |y_M| &=& x_M > 0 \\ x_{M+2} &=& |x_{M+1}| - y_{M+1} - 1 &=& -2 \\ y_{M+2} &=& x_{M+1} - |y_{M+1}| &=& -1, \end{array}$$

and so Statement 1 is true.

Next consider the case $0 \leq x_M < 1$ and $y_M = 0$. Then

$$\begin{array}{rcllcl} x_{M+1} &=& |x_M| - y_M - 1 &=& x_M - 1 < 0 \\ y_{M+1} &=& x_M - |y_M| &=& x_M \geq 0 \\ x_{M+2} &=& |x_{M+1}| - y_{M+1} - 1 &=& -2x_M \leq 0 \\ y_{M+2} &=& x_{M+1} - |y_{M+1}| &=& -1 \\ x_{M+3} &=& |x_{M+2}| - y_{M+2} - 1 &=& 2x_M \geq 0 \\ y_{M+3} &=& x_{M+2} - |y_{M+2}| &=& -2x_M - 1 < 0 \\ x_{M+4} &=& |x_{M+3}| - y_{M+3} - 1 &=& 4x_M \geq 0 \\ y_{M+4} &=& x_{M+3} - |y_{M+3}| &=& -1 \\ x_{M+5} &=& |x_{M+4}| - y_{M+4} - 1 &=& 4x_M \geq 0 \\ y_{M+5} &=& x_{M+4} - |y_{M+4}| &=& 4x_M - 1 \\ x_{M+6} &=& |x_{M+5}| - y_{M+5} - 1 &=& 0. \end{array}$$

If $\frac{1}{4} < x_M < 1$, then $y_{M+5} = 4x_M - 1 > 0$ and so $y_{M+6} = x_{M+5} - |y_{M+5}| = 1$. That is $(x_{M+6}, y_{M+6}) = (0, 1)$ and so Statement 2 is true.

If $0 \leq x_M \leq \frac{1}{4}$, then $y_{M+5} = 4x_M - 1 \leq 0$. Thus $y_{M+6} = x_{M+5} - |y_{M+5}| = 8x_M - 1$, and so Statement 3 is true.

□

Lemma 1.3.7 *Suppose there exists an integer $M \geq 0$ such that $x_M < -1$ and $y_M = 0$. Then the following statements are true:*

1. $x_{M+4} = 0$.

2. *If* $-\frac{3}{2} \leq x_M < -1$, *then* $y_{M+4} = -4x_M - 5$.

3. *If* $x_M < -\frac{3}{2}$, *then* $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ *is* P_5^1.

Proof: Let $x_M < -1$ and $y_M = 0$. Then

$$\begin{array}{rclcl} x_{M+1} & = & |x_M| - y_M - 1 & = & -x_M - 1 > 0 \\ y_{M+1} & = & x_M - |y_M| & = & x_M < 0 \\ x_{M+2} & = & |x_{M+1}| - y_{M+1} - 1 & = & -2x_M - 2 > 0 \\ y_{M+2} & = & x_{M+1} - |y_{M+1}| & = & -1 \\ x_{M+3} & = & |x_{M+2}| - y_{M+2} - 1 & = & -2x_M - 2 > 0 \\ y_{M+3} & = & x_{M+2} - |y_{M+2}| & = & -2x_M - 3 \\ x_{M+4} & = & |x_{M+3}| - y_{M+3} - 1 & = & 0, \end{array}$$

and so Statement 1 is true.

If $-\frac{3}{2} \leq x_M < -1$, then $y_{M+3} = -2x_M - 3 \leq 0$. Thus $y_{M+4} = x_{M+3} - |y_{M+3}| = -4x_M - 5$, and so Statement 2 is true.

If $x_M < -\frac{3}{2}$, then $y_{M+3} = -2x_M - 3 > 0$ and $y_{M+4} = x_{M+3} - |y_{M+3}| = 1$. That is $(x_{M+4}, y_{M+4}) = (0, 1)$ and so $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1 and the proof is complete. □

We now give the proof of Theorem 1.3.1 when (x_M, y_M) is in $l_2 = \{(x, y) : x = 0, y \geq 0\}$.

Lemma 1.3.8 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_2$. Then the following statements are true:*

1. *If* $0 \leq y_M < \frac{1}{7}$, *then* $\{(x_n, y_n)\}_{n=M}^{\infty}$ *is eventually the equilibrium solution.*

2. *If* $y_M = \frac{1}{7}$, *then the solution* $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ *is* P_5^2.

3. *If* $y_M > \dfrac{1}{7}$, *then the solution* $\{(x_n, y_n)\}_{n=M}^{\infty}$ *is eventually* P_5^1.

Proof:

1. We shall first show Statement 1 is true. Suppose $0 \leq y_M < \dfrac{1}{7}$, for each $\mathrm{n} \geq 0$, let

$$a_{\mathrm{n}} = \frac{2^{3\mathrm{n}} - 1}{7 \cdot 2^{3\mathrm{n}}}.$$

Observe that

$$0 = a_0 < a_1 < a_2 < \ldots < \frac{1}{7} \quad \text{and} \quad \lim_{\mathrm{n} \to \infty} a_{\mathrm{n}} = \frac{1}{7}.$$

Thus there exists a unique integer $K \geq 0$ such that $y_M \in [a_K, a_{K+1})$.

We first consider the case $K = 0$; that is $y_M \in \left[0, \dfrac{1}{8}\right)$. By Statements 1 and 3 of Lemma 1.3.4, $x_{M+5} = 0$ and $y_{M+5} = 8y_M - 1$. Clearly $y_{M+5} < 0$, and so

$$\begin{aligned} x_{M+6} &= |x_{M+5}| - y_{M+5} - 1 &= -8y_M \leq 0 \\ y_{M+6} &= x_{M+5} - |y_{M+5}| &= 8y_M - 1. \end{aligned}$$

Now $-1 < x_{M+6} \leq 0$ and $y_{M+6} = -x_{M+6} - 1$, and so by Lemma 1.3.2, $\{(x_n, y_n)\}_{n=M+7}^{\infty}$ is the equilibrium solution.

Without loss of generality we may assume $K \geq 1$.

For each integer n such that $n \geq 0$, let $\mathcal{P}(n)$ be the following statement:

$$\begin{aligned} x_{M+5n+5} &= 0 \\ y_{M+5n+5} &= 2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7} \right) \geq 0. \end{aligned}$$

Claim: $\mathcal{P}(n)$ is true for $0 \leq n \leq K - 1$.

The proof of the Claim will be by induction on n. We shall first show that $\mathcal{P}(0)$ is true.

Recall that $x_M = 0$ and $y_M \in [a_K, a_{K+1}) \subset \left[\frac{1}{8}, \frac{1}{7}\right)$. Then by Statements 1 and 3 of Lemma 1.3.4, we have $x_{M+5(0)+5} = 0$ and $y_{M+5(0)+5} = 8y_M - 1$.

Note that,

$$y_{M+5(0)+5} = 8y_M - 1 = 2^{3(0+1)} y_M - \left(\frac{2^{3(0+1)} - 1}{7}\right) \geq 0$$

and so $\mathcal{P}(0)$ is true. Thus if $K = 1$, then we have shown that for $0 \leq n \leq K - 1$, $\mathcal{P}(n)$ is true. It remains to consider the case $K \geq 2$. So assume that $K \geq 2$. Let n be an integer such that $0 \leq n \leq K - 2$ and suppose $\mathcal{P}(n)$ is true. We shall show that $\mathcal{P}(n + 1)$ is true.

Since $\mathcal{P}(n)$ is true we know

$$x_{M+5n+5} = 0 \quad \text{and} \quad y_{M+5n+5} = 2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right) \geq 0.$$

It is easy to verify that for $y_M \in [a_K, a_{K+1}) = \left[\frac{2^{3K} - 1}{7 \cdot 2^{3K}}, \frac{2^{3(K+1)} - 1}{7 \cdot 2^{3(K+1)}}\right)$

$$y_{M+5n+5} = 2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right) < \frac{1}{7} < \frac{1}{4}.$$

Thus by Statements 1 and 3 of Lemma 1.3.4,

$$\begin{aligned}
x_{M+5(n+1)+5} &= 0 \quad \text{and} \\
y_{M+5(n+1)+5} &= 8(y_{M+5n+5}) - 1 \\
&= 2^3 \left[2^{3(n+1)} y_M - \left(\frac{2^{3(n+1)} - 1}{7}\right)\right] - 1 \\
&= 2^{3n+6} y_M - \frac{2^{3n+6}}{7} + \frac{2^3}{7} - 1 \\
&= 2^{3(n+2)} y_M - \left(\frac{2^{3(n+2)} - 1}{7}\right).
\end{aligned}$$

Recall that $y_M \in [a_K, a_{K+1}) = \left[\frac{2^{3K}-1}{7\cdot 2^{3K}}, \frac{2^{3(K+1)}-1}{7\cdot 2^{3(K+1)}}\right)$.

In particular,

$$\begin{aligned}
y_{M+5(n+1)+5} &= 2^{3(n+2)}y_M - \left(\frac{2^{3(n+2)}-1}{7}\right) \\
&\geq 2^{3(n+2)}\left(\frac{2^{3K}-1}{7\cdot 2^{3K}}\right) - \left(\frac{2^{3(n+2)}-1}{7}\right) \\
&= \frac{2^{3n+3K+6}}{7\cdot 2^{3K}} - \frac{2^{3n+6}}{7\cdot 2^{3K}} - \frac{2^{3n+6}}{7} + \frac{1}{7} \\
&= \frac{1}{7}\left(1 - 2^{3[n-(K-2)]}\right) \geq \frac{1}{7}(1-1) \\
&= 0,
\end{aligned}$$

and so $\mathcal{P}(n+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{P}(n)$ is true for $0 \leq n \leq K-1$. Specifically, $\mathcal{P}(K-1)$ is true, and so

$$x_{M+5(K-1)+5} = 0 \quad \text{and} \quad y_{M+5(K-1)+5} = 2^{3K}y_M - \left(\frac{2^{3K}-1}{7}\right) \geq 0.$$

In particular,

$$2^{3K}\left(\frac{2^{3K}-1}{7\cdot 2^{3K}}\right) - \left(\frac{2^{3K}-1}{7}\right) \;\leq\; y_{M+5(K-1)+5} \;<\; 2^{3K}\left(\frac{2^{3K+3}-1}{7\cdot 2^{3K+3}}\right) - \left(\frac{2^{3K}-1}{7}\right).$$

That is, $0 \leq y_{M+5(K-1)+5} < \frac{1}{8}$, and so by case $K=0$, $\{(x_n, y_n)\}_{n=M+5K+7}^{\infty}$ is the equilibrium solution, and the proof of Statement 1 is complete.

2. We shall next show that Statement 2 is true. Suppose $(x_M, y_M) = \left(0, \frac{1}{7}\right)$. Note that $\left(0, \frac{1}{7}\right) \in P_5^2$. Thus the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is P_5^2.

3. Finally, we shall show that Statement 3 is true. Suppose $y_M > \frac{1}{7}$.

First consider $y_M > \frac{1}{4}$. By Statement 2 of Lemma 1.3.4, the solution $\{(x_n, y_n)\}_{n=M+5}^{\infty}$ is P_5^1.

Next consider the case $y_M \in \left(\frac{1}{7}, \frac{1}{4}\right]$. For each n $\geq$ 1 let

$$b_{\mathrm{n}} = \frac{2^{3\mathrm{n}-1}+3}{7\cdot 2^{3\mathrm{n}-1}}.$$

Observe that

$$\frac{1}{4} = b_1 > b_2 > b_3 > \ldots > \frac{1}{7} \quad \text{and} \quad \lim_{\mathrm{n}\to\infty} b_{\mathrm{n}} = \frac{1}{7}.$$

Thus there exists a unique integer $K \geq 1$ such that $y_M \in (b_{K+1}, b_K]$.

If we slightly augment the proof of Statement 1 of this lemma then the statement $\mathcal{P}(n)$ still holds. First note that it is easy to determine through direct computations that the base case of the inductive argument still holds. All that is needed to complete the proof for $y_M \in (b_{K+1}, b_K]$ is the following Claim.

Claim: Let n be an integer such that $0 \leq n \leq K-2$ and suppose $\mathcal{P}(n)$ is true. We shall show that $\mathcal{P}(n+1)$ is true.

Proof: Since $\mathcal{P}(n)$ is true we know

$$x_{M+5n+5} = 0 \quad \text{and} \quad y_{M+5n+5} = 2^{3(n+1)}y_M - \left(\frac{2^{3(n+1)}-1}{7}\right) \geq 0.$$

Recall that $y_M \in (b_{K+1}, b_K] = \left[\frac{2^{3(K+1)-1}+3}{7\cdot 2^{3(K+1)-1}}, \frac{2^{3K-1}+3}{7\cdot 2^{3K-1}}\right)$, then

$$\begin{aligned} y_{M+5n+5} &= 2^{3(n+1)}y_M - \left(\frac{2^{3(n+1)}-1}{7}\right) \\ &\leq 2^{3n+3}\left(\frac{2^{3K-1}+3}{7\cdot 2^{3K-1}}\right) - \left(\frac{2^{3n+3}-1}{7}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{3n+3}2^{3K-1}}{7\cdot 2^{3K-1}} + \frac{2^{3n+3}3}{7\cdot 2^{3K-1}} - \frac{2^{3n+3}}{7} + \frac{1}{7} \\
&= \frac{1}{7}\left(2^{3n-3K+4}3+1\right) \leq \frac{1}{7}\left(\frac{3}{4}+1\right) \\
&= \frac{1}{4}.
\end{aligned}
$$

Thus by Statements 1 and 3 of Lemma 1.3.4,

$$
\begin{aligned}
x_{M+5(n+1)+5} &= 0 \quad \text{and} \\
y_{M+5(n+1)+5} &= 8(y_{M+5n+5}) - 1 \\
&= 2^{3(n+2)}y_M - \left(\frac{2^{3(n+2)}-1}{7}\right).
\end{aligned}
$$

Again recall that $y_M \in (b_{K+1}, b_K] = \left[\frac{2^{3(K+1)-1}+3}{7\cdot 2^{3(K+1)-1}}, \frac{2^{3K-1}+3}{7\cdot 2^{3K-1}}\right)$.

In particular,

$$
\begin{aligned}
y_{M+5(n+1)+5} &= 2^{3(n+2)}y_M - \left(\frac{2^{3(n+2)}-1}{7}\right) \\
&\geq 2^{3(n+2)}\left(\frac{2^{3(K+1)-1}+3}{7\cdot 2^{3(K+1)-1}}\right) - \left(\frac{2^{3(n+2)}}{7}\right) + \left(\frac{1}{7}\right) \\
&= \frac{2^{3n+3K+8}}{7\cdot 2^{3K+2}} + \frac{2^{3n+6}3}{7\cdot 2^{3K+2}} - \frac{2^{3n+6}}{7} + \frac{1}{7} \\
&= \frac{1}{7}\left(2^{3n-3K+4}+1\right) \\
&\geq 0.
\end{aligned}
$$

The proof of the Claim is complete and this completes the proof that $\mathcal{P}(n)$ is true for Statement 3 of this lemma for $0 \leq n \leq K-1$. Specifically $\mathcal{P}(K-1)$ is true, and so

$$
x_{M+5(K-1)+5} = 0 \text{ and } y_{M+5(K-1)+5} = 2^{3K}y_M - \left(\frac{2^{3K}-1}{7}\right) \geq 0.
$$

Recall that for $y_M \in (b_{K+1}, b_K]$.

In particular,

$$y_{M+5K} = 2^{3K} y_M - \left(\frac{2^{3K}-1}{7}\right) > 2^{3K}\left(\frac{2^{3K+2}+3}{7\cdot 2^{3K+2}}\right) - \left(\frac{2^{3K}-1}{7}\right) = \frac{1}{4}.$$

By Statement 2 of Lemma 1.3.4, the solution $\{(x_n, y_n)\}_{n=M+5K+5}^{\infty}$ is P_5^1.

□

We now give the proof of Theorem 1.3.1 when (x_M, y_M) is in $l_4 = \{(x,y) : x = 0, y < 0\}$.

Lemma 1.3.9 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_4$. Then the following statements are true:*

1. *If $-\frac{9}{7} < y_M < 0$, then $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually the equilibrium solution.*
2. *If $y_M = -\frac{9}{7}$, then the solution $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.*
3. *If $y_M < -\frac{9}{7}$, then the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually P_5^1.*

Proof:

1. We shall first show that Statement 1 is true. So suppose $-\frac{9}{7} < y_M < 0$.

Case 1: Suppose $-1 \leq y_M < 0$. Then

$$\begin{aligned} x_{M+1} &= |x_M| - y_M - 1 &&= -y_M - 1 \leq 0 \\ y_{M+1} &= x_M - |y_M| &&= y_M. \end{aligned}$$

In particular, $-1 < x_{M+1} \leq 0$ and $y_{M+1} = -x_{M+1} - 1$, and so by Lemma 1.3.2, $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution.

Case 2: Suppose $-\frac{5}{4} \leq y_M < -1$. By Statements 1 and 2 of Lemma 1.3.5, $x_{M+4} = 0$ and $y_{M+4} = -4y_M - 5$. Then

$$\begin{aligned} x_{M+5} &= |x_{M+4}| - y_{M+4} - 1 &&= 4y_M + 4 < 0 \\ y_{M+5} &= x_{M+4} - |y_{M+4}| &&= -4y_M - 5. \end{aligned}$$

Thus $-1 \leq x_{M+5} < 0$ and $y_{M+5} = -x_{M+5} - 1$, and so by Lemma 1.3.2, $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is the equilibrium solution.

Case 3: Suppose $-\frac{9}{7} < y_M < -\frac{5}{4}$. By Statements 1 and 2 of Lemma 1.3.5, $x_{M+4} = 0$ and $y_{M+4} = -4y_M - 5$. Note that $0 < y_{M+4} < \frac{1}{7}$ and so by Statement 1 of Lemma 1.3.8, $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually equilibrium solution.

2. We shall next show that Statement 2 is true. Suppose $y_M = -\frac{9}{7}$. By direct calculations we have $(x_{M+1}, y_{M+1}) = \left(\frac{2}{7}, -\frac{9}{7}\right)$. So the solution $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.

3. Finally, we shall show that Statement 3 is true. Suppose $x_M = 0$ and $y_M < -\frac{9}{7}$.

Case 1: Suppose $-\frac{3}{2} < y_M < -\frac{9}{7}$. By Statements 1 and 2 of Lemma 1.3.5, we have $x_{M+4} = 0$ and $y_{M+4} = -4y_M - 5$. Note that $\frac{1}{7} < y_{M+4} < 1$ and so by Statement 3 of Lemma 1.3.8, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually P_5^1.

Case 2 : Suppose $y_M \leq -\frac{3}{2}$. By Statement 3 of Lemma 1.3.5, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1.

□

We now give the proof of Theorem 1.3.1 when (x_M, y_M) is in $l_1 = \{(x, y) : x \geq 0, y = 0\}$.

Lemma 1.3.10 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_1$. Then the following statements are true:*

1. *If $0 \leq x_M < \frac{1}{7}$, then $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually the equilibrium solution.*

2. *If $x_M = \frac{1}{7}$, then the solution $\{(x_n, y_n)\}_{n=M+3}^{\infty}$ is P_5^2.*

3. If $x_M > \frac{1}{7}$, then the solution $\{(x_n, y_n)\}_{n=M}^{\infty}$ is eventually P_5^1.

Proof:

1. We shall first show Statement 1 is true. So suppose $0 \leq x_M < \frac{1}{7}$ and $y_M = 0$. By Statement 3 of Lemma 1.3.6, $x_{M+6} = 0$ and $y_{M+6} = 8x_M - 1$. In particular, $-1 < y_{M+6} < \frac{1}{7}$ and so by Statement 1 of Lemma 1.3.8 and Statement 1 of Lemma 1.3.9, $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is eventually the equilibrium solution.

2. We shall next show Statement 2 is true. Suppose $x_M = \frac{1}{7}$. By direct calculations we have $(x_{M+3}, y_{M+3}) = \left(\frac{2}{7}, -\frac{9}{7}\right)$. Thus the solution $\{(x_n, y_n)\}_{n=M+3}^{\infty}$ is P_5^2.

3. Finally, we shall show Statement 3 is true.

   First consider the case $\frac{1}{7} < x_M \leq \frac{1}{4}$. By Statement 3 of Lemma 1.3.6, $x_{M+6} = 0$ and $y_{M+6} = 8x_M - 1$. Now, $\frac{1}{7} < y_{M+6} \leq 1$ and so by Statement 3 of Lemma 1.3.8, the solution $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is eventually P_5^1.

   Next consider the case $x_M > \frac{1}{4}$. Then by Statements 1 and 2 of Lemma 1.3.6, if $x_M \geq 1$ then $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^1 and if $\frac{1}{4} < x_M < 1$ then $\{(x_n, y_n)\}_{n=M+6}^{\infty}$ is P_5^1.

□

We next give the proof of Theorem 1.3.1 when (x_M, y_M) is in $l_3 = \{(x, y) : x < 0, y = 0\}$.

Lemma 1.3.11 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in l_3$. Then the following statements are true:*

1. *If* $-\frac{9}{7} < x_M < 0$, *then* $\{(x_n, y_n)\}_{n=M}^{\infty}$ *is eventually the equilibrium solution.*

2. *If* $x_M = -\frac{9}{7}$, *then the solution* $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ *is* P_5^2.

3. *If* $x_M < -\frac{9}{7}$, *then the solution* $\{(x_n, y_n)\}_{n=M}^{\infty}$ *is eventually* P_5^1.

Proof:

1. We will first prove Statement 1 is true. Suppose $-\frac{9}{7} < x_M < 0$.

   First consider the case $-1 \leq x_M < 0$. Then

$$\begin{array}{rcccl} x_{M+1} & = & |x_M| - y_M - 1 & = & -x_M - 1 \\ y_{M+1} & = & x_M - |y_M| & = & x_M. \end{array}$$

   In particular, $-1 < x_{M+1} \leq 0$ and $y_{M+1} = -x_M - 1$ and so by Lemma 1.3.2, $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution.

   Next consider the case $-\frac{9}{7} < x_M < -1$. By Statements 1 and 2 of Lemma 1.3.7, $x_{M+4} = 0$ and $y_{M+4} = -4x_M - 5$. In particular, $-1 < y_{M+4} < \frac{1}{7}$ and so by Statement 1 of Lemma 1.3.8 and Statement 1 of Lemma 1.3.9, $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually the equilibrium solution.

2. We shall next show Statement 2 is true. Suppose $x_M = -\frac{9}{7}$. By direct calculations we have $(x_{M+1}, y_{M+1}) = \left(\frac{2}{7}, -\frac{9}{7}\right)$. That is, $\{(x_n, y_n)\}_{n=M+1}^{\infty}$ is P_5^2.

3. Lastly, we shall show that Statement 3 is true. Suppose $x_M < -\frac{9}{7}$.

   First consider the case $-\frac{3}{2} \leq x_M < -\frac{9}{7}$. By Statements 1 and 2 of Lemma 1.3.7, $x_{M+4} = 0$ and $y_{M+4} = -4x_M - 5$. In particular, $\frac{1}{7} < y_{M+4} \leq 1$ and so by Statement 3 of Lemma 1.3.8, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is eventually P_5^1.

Next consider the case $x_M < -\frac{3}{2}$. By Statement 3 of Lemma 1.3.7, the solution $\{(x_n, y_n)\}_{n=M+4}^{\infty}$ is P_5^1.

□

We next give the proof of Theorem 1.3.1 when (x_M, y_M) is in $Q_1 = \{(x, y) : x > 0, y > 0\}$.

Lemma 1.3.12 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in Q_1$. Then the following statements are true:*

1. *If $y_M \leq x_M - 1$, then the solution $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is P_5^1.*
2. *If $y_M > x_M - 1$, then there exists an integer N such that $(x_{M+N}, y_{M+N}) \in l_2 \cup l_4$.*

Proof: Suppose $x_M > 0$ and $y_M > 0$.

Then

$$\begin{array}{lllll} x_{M+1} & = & |x_M| - y_M - 1 & = & x_M - y_M - 1 \\ y_{M+1} & = & x_M - |y_M| & = & x_M - y_M. \end{array}$$

Case 1: Suppose $y_M \leq x_M - 1$. Then, in particular, $x_{M+1} = x_M - y_M - 1 \geq 0$ and $y_{M+1} = x_M - y_M > 0$. Thus

$$\begin{array}{lllll} x_{M+2} & = & |x_{M+1}| - y_{M+1} - 1 & = & -2 \\ y_{M+2} & = & x_{M+1} - |y_{M+1}| & = & -1, \end{array}$$

and so Statement 1 true.

Case 2: Suppose $y_M > x_M - 1$. Then, in particular, $x_{M+1} = x_M - y_M - 1 < 0$.

Case 2a: Suppose $x_M - y_M < 0$.

Then $y_{M+1} = x_M - y_M < 0$. It follows by a straight forward computation, which will be omitted, that $x_{M+5} = 0$. Hence $(x_{M+5}, y_{M+5}) \in l_2 \cup l_4$.

Case 2b: Suppose $x_M - y_M \geq 0$.

Then $y_{M+1} = x_M - y_M \geq 0$. It follows by a straight forward computation, which will be omitted, that $x_{M+6} = 0$. Hence $(x_{M+6}, y_{M+6}) \in l_2 \cup l_4$, and the proof is complete. □

We next give the proof of Theorem 1.3.1 when (x_M, y_M) is in $Q_3 = \{(x, y) : x < 0, y < 0\}$.

Lemma 1.3.13 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in Q_3$. Then the following statements are true:*

1. *If $y_M \geq -x_M - 1$, then the solution $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution.*

2. *If $y_M < -x_M - 1$, then $(x_{M+4}, y_{M+4}) \in l_2 \cup l_4$.*

Proof: By assumption, we have $x_M < 0$ and $y_M < 0$.

If $y_M \geq -x_M - 1$. Then

$$\begin{array}{rclcl} x_{M+1} &=& |x_M| - y_M - 1 &=& -x_M - y_M - 1 \leq 0 \\ y_{M+1} &=& x_M - |y_M| &=& x_M + y_M < 0 \\ x_{M+2} &=& |x_{M+1}| - y_{M+1} - 1 &=& 0 \\ y_{M+2} &=& x_{M+1} - |y_{M+1}| &=& -1. \end{array}$$

Hence $\{(x_n, y_n)\}_{n=M+2}^{\infty}$ is the equilibrium solution and Statement 1 is true.

If $y_M < -x_M - 1$ then it follows by a straight forward computation, which will be omitted, that $x_{M+4} = 0$. Thus $(x_{M+4}, y_{M+4}) \in l_2 \cup l_4$ and Statement 2 is true. □

We next give the proof of Theorem 1.3.1 when (x_M, y_M) is in $Q_2 = \{(x, y) : x < 0, y > 0\}$.

Lemma 1.3.14 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in Q_2$. Then the following statements are true:*

1. *If $y_M \geq -x_M - 1$, then $(x_{M+1}, y_{M+1}) \in Q_3 \cup l_4$.*
2. *If $y_M \leq -x_M - \frac{3}{2}$, then $(x_{M+3}, y_{M+3}) \in Q_1 \cup l_1$.*
3. *If $y_M < -x_M - 1, y_M > -x_M - \frac{3}{2}$ and $x_M \leq -\frac{5}{4}$, then $(x_{M+4}, y_{M+4}) \in Q_1 \cup l_1$.*
4. *If $y_M < -x_M - 1, y_M > -x_M - \frac{3}{2}, x_M > -\frac{5}{4}$ and $y_M \leq x_M + \frac{5}{4}$, then $(x_{M+5}, y_{M+5}) \in Q_3 \cup l_4$.*
5. *If $y_M < -x_M - 1, y_M > -x_M - \frac{3}{2}, x_M > -\frac{5}{4}$ and $y_M > x_M + \frac{5}{4}$, then $(x_{M+6}, y_{M+6}) \in Q_3 \cup l_4$.*

Proof: Now $x_M < 0$ and $y_M > 0$.

1. If $y_M \geq -x_M - 1$, then

$$\begin{aligned} x_{M+1} &= -x_M - y_M - 1 \leq 0 \\ y_{M+1} &= x_M - y_M < 0. \end{aligned}$$

Thus $(x_{M+1}, y_{M+1}) \in Q_3 \cup l_4$.

2. If $y_M \leq -x_M - \frac{3}{2}$, then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+3} &= -2x_M + 2y_M - 2 > 0 \\ y_{M+3} &= -2x_M - 2y_M - 3 \geq 0. \end{aligned}$$

Hence $(x_{M+3}, y_{M+3}) \in Q_1 \cup l_1$.

3. If $y_M < -x_M - 1, y_M > -x_M - \frac{3}{2}$ and $x_M \leq -\frac{5}{4}$, then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+4} &= 4y_M > 0 \\ y_{M+4} &= -4x_M - 5 \geq 0. \end{aligned}$$

Thus $(x_4, y_4) \in Q_1 \cup l_1$.

4. If $y_M < -x_M - 1, y_M > -x_M - \dfrac{3}{2}, x_M > -\dfrac{5}{4}$ and $y_M \leq x_M + \dfrac{5}{4}$, then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+5} &= 4x_M + 4y_M + 4 < 0 \\ y_{M+5} &= -4x_M + 4y_M - 5 \leq 0. \end{aligned}$$

Thus $(x_{M+5}, y_{M+5}) \in Q_3 \cup l_4$.

5. Finally, suppose that $y_M < -x_M - 1, y_M > -x_M - \dfrac{3}{2}, x_M > -\dfrac{5}{4}$ and $y_M > x_M + \dfrac{5}{4}$. Then $x_{M+1} = -x_M - y_M - 1 > 0$. It follows by a straight forward computation, which will be omitted, that

$$\begin{aligned} x_{M+5} &= 4x_M + 4y_M + 4 < 0 \\ y_{M+5} &= -4x_M + 4y_M - 5 > 0. \end{aligned}$$

Note that

$$y_{M+5} = -4x_M + 4y_M - 5 > -4x_M - 4y_M - 5 = -x_{M+5} - 1$$

and so by the first statement of this Lemma, $(x_{M+6}, y_{M+6}) \in Q_3 \cup l_4$.

□

Thus we see that if there exists an integer $N \geq 0$ such that $(x_N, y_N) \notin Q_4$ then the proof of Theorem 1.3.1 is complete. Finally, we consider the case where the initial condition $(x_M, y_M) \in Q_4 = \{(x, y) : x > 0, y < 0\}$.

Lemma 1.3.15 *Suppose there exists an integer $M \geq 0$ such that $(x_M, y_M) \in Q_4$. Then there exists a positive integer $N \leq 4$ such that $(x_{M+N}, y_{M+N}) \notin Q_4$.*

Proof: Without loss of generality, it suffices to consider the case where

$$(x_{M+n}, y_{M+n}) \in Q_4 \text{ for } 0 \leq n \leq 3.$$

Now $(x_M, y_M) \in Q_4$, and hence $x_M > 0$ and $y_M < 0$.

Thus

$$\begin{array}{lllll} x_{M+1} & = & |x_M| - y_M - 1 & = & x_M - y_M - 1 \\ y_{M+1} & = & x_M - |y_M| & = & x_M + y_M. \end{array}$$

We have $(x_{M+1}, y_{M+1}) \in Q_4$, and thus

$$\begin{array}{lllll} x_{M+2} & = & |x_{M+1}| - y_{M+1} - 1 & = & -2y_M - 2, \\ y_{M+2} & = & x_{M+1} - |y_{M+1}| & = & 2x_M - 1. \end{array}$$

We also have $(x_2, y_2) \in Q_4$, and hence

$$\begin{array}{lllll} x_{M+3} & = & |x_{M+2}| - y_{M+2} - 1 & = & -2x_M - 2y_M - 2, \\ y_{M+3} & = & x_{M+2} - |y_{M+2}| & = & 2x_M - 2y_M - 3. \end{array}$$

Finally, we have $(x_{M+3}, y_{M+3}) \in Q_4$, and so

$$\begin{array}{lllll} x_{M+4} & = & |x_{M+3}| - y_{M+3} - 1 & = & -4x_M < 0, \\ y_{M+4} & = & x_{M+3} - |y_{M+3}| & = & -4y_M - 5. \end{array}$$

In particular $x_{M+4} < 0$ and hence $(x_{M+4}, y_{M+4}) \notin Q_4$. □

Conclusion

We have presented the complete results concerning the global character of the solutions to System(2). We divided the real plane into 8 sections and utilized mathematical induction, proof by iteration, and direct computations to show that every solution of System(2) is eventually either the prime period-5 solution P_5^1, the prime period-5 solution P_5^2 or else the unique equilibrium point $(0, -1)$. The proofs involve careful consideration of the various cases and subcases.

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1.4 References

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MANUSCRIPT 2

On the Global Behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$

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2.1 Abstract

In this paper we consider the system of piecewise linear difference equations in the title, where the initial conditions x_0 and y_0 are real numbers. We show that there exists a unique equilibrium solution and exactly two prime period-3 solutions, and that except for the unique equilibrium solution, every solution of the system is eventually one of the two prime period-3 solutions.

2.2 Introduction

In this paper we consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} &= |x_n| - y_n - 1 \\ y_{n+1} &= x_n + |y_n| \end{cases}, \quad n = 0, 1, \ldots \tag{1}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers. We show that every solution of System(1) is either (from the beginning) the unique equilibrium point

$$(\bar{x}, \bar{y}) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$$

or else is eventually one of the following period-3 cycles:

$$\mathbf{P}_3^1 = \begin{pmatrix} x_0 = 0 & , & y_0 = -1 \\ x_1 = 0 & , & y_1 = 1 \\ x_2 = -2 & , & y_2 = 1 \end{pmatrix} \text{ or } \mathbf{P}_3^2 = \begin{pmatrix} x_0 = 0 & , & y_0 = -\frac{1}{3} \\ x_1 = -\frac{2}{3} & , & y_1 = \frac{1}{3} \\ x_2 = -\frac{2}{3} & , & y_2 = -\frac{1}{3} \end{pmatrix}.$$

This study of System(1) was motivated by Devaney's celebrated Gingerbreadman map

$$\begin{cases} x_{n+1} &= |x_n| - y_n + 1 \\ y_{n+1} &= x_n \end{cases}, \quad n = 0, 1, \ldots .$$

See Ref. [2, 4, 5, 9].

We believe that the methods and techniques used in this paper will be useful in discovering the global behavior of similar piecewise linear systems of the form

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b \\ \\ y_{n+1} = x_n + c\,|y_n| + d \end{cases}, \qquad n = 0, 1, 2...$$

For another system of this form see [10].

2.3 Global Results

Set

$$\begin{aligned} l_1 &= \{(x, y) : x \geq 0, y = 0\} \\ l_2 &= \{(x, y) : x = 0, y \geq 0\} \\ l_3 &= \{(x, y) : x \leq 0, y = 0\} \\ l_4 &= \{(x, y) : x = 0, y \leq 0\} \\ Q_1 &= \{(x, y) : x > 0, y > 0\} \\ Q_2 &= \{(x, y) : x < 0, y > 0\} \\ Q_3 &= \{(x, y) : x < 0, y < 0\} \\ Q_4 &= \{(x, y) : x > 0, y < 0\}. \end{aligned}$$

Theorem 2.3.1 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(8) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then either* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is the unique equilibrium* $(\bar{x}, \bar{y})$, *or else there exists a non-negative integer* $N \geq 0$ *such that the solution* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *of System(8) is either the prime period-3 cycle* $\mathbf{P}_3^1$ *or the prime period-3 cycle* $\mathbf{P}_3^2$.

The proof of Theorem 2.3.1 is a direct consequence of the following lemmas.

Lemma 2.3.2 *Suppose there exists a non-negative integer* $N \geq 0$ *such that*

$$y_N = -x_N - 1 \qquad \text{and} \qquad y_N \geq 0.$$

Then $(x_{N+1}, y_{N+1}) = (0, -1)$, *and so* $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ *is the period-3 cycle* $\mathbf{P}_3^1$.

Proof: Note that $x_N = -y_N - 1 \leq -1$, and so

$$\begin{array}{rcccccc} x_{N+1} & = & |x_N| - y_N - 1 & = & -x_N - (-x_N - 1) - 1 & = & 0 \\ y_{N+1} & = & x_N + |y_N| & = & x_N + (-x_N - 1) & = & -1. \end{array}$$

The proof is complete. □

Lemma 2.3.3 *Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in l_2$. Then $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.*

Proof: I have

$$\begin{array}{rcccccccc} x_{N+1} & = & |x_N| - y_N - 1 & = & 0 - y_N - 1 & = & -y_N - 1 & < & 0 \\ y_{N+1} & = & x_N + |y_N| & = & 0 + y_N & = & y_N & \geq & 0 \end{array}$$

and so it follows by Lemma 2.3.2 that $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. □

Lemma 2.3.4 *Suppose there exists a non-negative integer $N \geq 0$ such that $x_N = 0$ and $y_N < -1$. Then*

1. $x_{N+3} = 2y_N + 2 < 0$.
2. *If* $-\dfrac{3}{2} \leq y_N < -1$, *then* $y_{N+3} = -2y_N - 3 \leq 0$.
3. *If* $y_N < -\dfrac{3}{2}$, *then* $\{(x_n, y_n)\}_{n=N+4}^{\infty}$ *is the period-3 cycle* $\mathbf{P}_3^1$.

Proof: I have

$$\begin{array}{rcccccc} x_{N+1} & = & |x_N| - y_N - 1 & = & -y_N - 1 & > & 0 \\ y_{N+1} & = & x_N + |y_N| & = & -y_N & > & 0 \\ x_{N+2} & = & |x_{N+1}| - y_{N+1} - 1 & = & -2 & & \\ y_{N+2} & = & x_{N+1} + |y_{N+1}| & = & -2y_N - 1 & > & 0 \\ x_{N+3} & = & |x_{N+2}| - y_{N+2} - 1 & = & 2y_N + 2 & < & 0 \\ y_{N+3} & = & x_{N+2} + |y_{N+2}| & = & -2y_N - 3. & & \end{array}$$

If $-\frac{3}{2} \leq y_N < -1$, then $y_{N+3} = -2y_N - 3 \leq 0$. If $y_N < -\frac{3}{2}$, then $y_{N+3} = -2y_N - 3 > 0$ and so by Lemma 2.3.2 $\{(x_n, y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. The proof is complete. □

Lemma 2.3.5 *Suppose there exists a non-negative integer $N \geq 0$ such that $x_N = 0$ and $-1 < y_N \leq 0$. Then*

1. *If $-\frac{1}{4} < y_N \leq 0$, then $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.*
2. *If $-\frac{1}{2} < y_N \leq -\frac{1}{4}$, then $x_{N+5} = 8y_N + 2$, $y_{N+5} = -8y_N - 3$, and $x_{N+6} = 0$.*
3. *If $-1 < y_N \leq -\frac{1}{2}$, then $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.*

Proof: I have

$$
\begin{array}{rcccccc}
x_{N+1} & = & |x_N| - y_N - 1 & = & -y_N - 1 & < & 0 \\
y_{N+1} & = & x_N + |y_N| & = & -y_N & \geq & 0 \\
x_{N+2} & = & |x_{N+1}| - y_{N+1} - 1 & = & 2y_N & \leq & 0 \\
y_{N+2} & = & x_{N+1} + |y_{N+1}| & = & -2y_N - 1 & & \\
x_{N+3} & = & |x_{N+2}| - y_{N+2} - 1 & = & 0. & &
\end{array}
$$

If $-\frac{1}{4} < y_N \leq 0$, then $y_{N+2} < 0$ and $y_{N+3} = x_{N+2} + |y_{N+2}| = 4y_N + 1 > 0$. It follows by Lemma 2.3.3 that $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$, and so Statement 1 is true.

If $-\frac{1}{2} < y_N \leq -\frac{1}{4}$, then $y_{N+2} < 0$ and

$$
\begin{array}{rcccccc}
y_{N+3} & = & x_{N+2} + |y_{N+2}| & = & 4y_N + 1 & \leq & 0 \\
x_{N+4} & = & |x_{N+3}| - y_{N+3} - 1 & = & -4y_N - 2 & < & 0 \\
y_{N+4} & = & x_{N+3} + |y_{N+3}| & = & -4y_N - 1 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
x_{N+5} &= |x_{N+4}| - y_{N+4} - 1 &&= 8y_N + 2 \leq 0 \\
y_{N+5} &= x_{N+4} + |y_{N+4}| &&= -8y_N - 3 \\
x_{N+6} &= |x_{N+5}| - y_{N+5} - 1 &&= 0
\end{aligned}
$$

and so Statement 2 is true.

If $-1 < y_N \leq -\frac{1}{2}$, then $y_{N+6} = x_{N+5} + |y_{N+5}| = -1$ and so $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. The proof is complete. □

Lemma 2.3.6 *Suppose there exists a non-negative integer* $N \geq 0$ *such that* $(x_N, y_N) \in l_4$. *Then the following five statements are true:*

1. *Suppose* $-\frac{1}{3} < y_N \leq 0$. *Then* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *is eventually the period-3 cycle* $\mathbf{P}_3^1$.
2. *Suppose* $y_N = -\frac{1}{3}$. *Then* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *is the period-3 cycle* $\mathbf{P}_3^2$.
3. *Suppose* $-\frac{4}{3} < y_N < -\frac{1}{3}$. *Then* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *is eventually the period-3 cycle* $\mathbf{P}_3^1$.
4. *Suppose* $y_N = -\frac{4}{3}$. *Then* $\{(x_n, y_n)\}_{n=N+3}^{\infty}$ *is the period-3 cycle* $\mathbf{P}_3^2$.
5. *Suppose* $y_N < -\frac{4}{3}$. *Then* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *is eventually the period-3 cycle* $\mathbf{P}_3^1$.

Proof: I have $x_N = 0$ and $y_N \leq 0$.

1. Suppose $-\frac{1}{3} < y_N \leq 0$. Note that by Statement 1 of Lemma 2.3.5, that if $-\frac{1}{4} < y_N \leq 0$, then $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

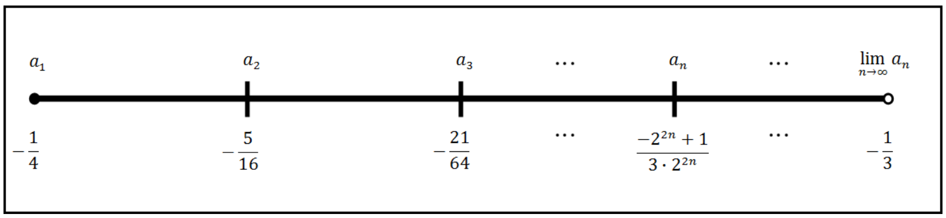
   So suppose $-\frac{1}{3} < y_N \leq -\frac{1}{4}$. For each integer $n \geq 1$, let

$$a_n = \frac{-2^{2n}+1}{3 \cdot 2^{2n}}.$$

Observe that

$$-\frac{1}{4} = a_1 > a_2 > a_3 > \ldots > -\frac{1}{3} \quad \text{and} \quad \lim_{n\to\infty} a_n = -\frac{1}{3}.$$

See diagram below:



Thus there exists a unique integer $K \geq 1$ such that $y_N \in (a_{K+1}, a_K]$. I first consider the case $K = 1$; that is, $y_N \in \left(-\frac{5}{16}, -\frac{1}{4}\right]$. It follows from Statement 2 of Lemma 2.3.5 that $x_{N+5} = 8y_N + 2 \leq 0$, $y_{N+5} = -8y_N - 3 < 0$, and $x_{N+6} = 0$. Thus $y_{N+6} = x_{N+5} + |y_{N+5}| = 16y_N + 5 > 0$, and so by Lemma 2.3.3 I have $\{(x_n, y_n)\}_{n=N+8}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Hence without loss of generality, I may assume $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{P}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+3} &= 0 \\ y_{N+3m+3} &= 2^{2m+2}y_N + \frac{2^{2m+2}-1}{3} \leq 0. \end{aligned}$$

Claim: $\mathcal{P}(m)$ is true for $1 \leq m \leq K-1$.

The proof of the Claim will be by induction on m. I shall first show that $\mathcal{P}(1)$ is true.

Recall that $x_N = 0$ and $y_N \in (a_{K+1}, a_K] \subset \left(-\frac{1}{3}, -\frac{5}{16}\right]$, and so by Statement

2 of Lemma 2.3.5 I have $x_{N+5} = 8y_N + 2 < 0$ and $y_{N+5} = -8y_N - 3 < 0$. Then

$$x_{N+3(1)+3} = 0$$

$$y_{N+3(1)+3} = 16y_N + 5 = 2^{2(1)+2}y_N + \frac{2^{2(1)+2}-1}{3} \leq 0$$

and so $\mathcal{P}(1)$ is true. Thus if $K = 2$, then I have shown that for $1 \leq m \leq K-1$, $\mathcal{P}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{P}(m)$ is true. I shall show that $\mathcal{P}(m+1)$ is true.

Since $\mathcal{P}(m)$ is true I know

$$x_{N+3m+3} = 0$$

$$y_{N+3m+3} = 2^{2m+2}y_N + \frac{2^{2m+2}-1}{3} \leq 0.$$

Recall that $y_N \in (a_{K+1}, a_K] = \left(\frac{-2^{2(K+1)}+1}{3 \cdot 2^{2(K+1)}}, \frac{-2^{2K}+1}{3 \cdot 2^{2K}} \right]$.

Then

$$x_{N+3m+4} = |x_{N+3m+3}| - y_{N+3m+3} - 1 = -2^{2m+2}y_N - \left(\frac{2^{2m+2}-1}{3}\right) - 1.$$

Note that $x_{N+3m+4} = -y_{N+3m+3} - 1$.

In particular,

$$\begin{aligned} x_{N+3m+4} &= -2^{2m+2}y_N - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\ &< -2^{2m+2}\left(\frac{-2^{2(K+1)}+1}{3 \cdot 2^{2(K+1)}}\right) - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{2m+2K+4}}{3\cdot 2^{2K+2}} - \frac{2^{2m+2}}{3\cdot 2^{2K+2}} - \frac{2^{2m+2}}{3} + \frac{1}{3} - 1 \\
&= -\frac{2^{2m-2K}}{3} - \frac{2}{3} \\
&< 0
\end{aligned}
$$

and

$$
y_{N+3m+4} = x_{N+3m+3} + |y_{N+3m+3}| = -y_{N+3m+3} \geq 0.
$$

Thus

$$
\begin{aligned}
x_{N+3m+5} &= |x_{N+3m+4}| - y_{N+3m+4} - 1 \\
&= y_{N+3m+3} + 1 - (-y_{N+3m+3}) - 1 \\
&= 2y_{N+3m+3} \leq 0
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+3m+5} &= x_{N+3m+4} + |y_{N+3m+4}| = -y_{N+3m+3} - 1 + (-y_{N+3m+3}) \\
&= -2y_{N+3m+3} - 1.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
y_{N+3m+5} &= -2\left(2^{2m+2}y_N + \frac{2^{2m+2}-1}{3}\right) - 1 \\
&< -2\left[2^{2m+2}\left(\frac{-2^{2(K+1)}+1}{3\cdot 2^{2(K+1)}}\right) + \frac{2^{2m+2}-1}{3}\right] - 1 \\
&= \frac{2^{2m+2K+5}}{3\cdot 2^{2K+2}} - \frac{2^{2m+3}}{3\cdot 2^{2K+2}} - \frac{2^{2m+3}}{3} + \frac{2}{3} - 1 \\
&= -\frac{2^{2m-2K+1}}{3} - \frac{1}{3} \\
&< 0.
\end{aligned}
$$

Finally,

$$\begin{aligned}
x_{N+3(m+1)+3} &= x_{N+3m+6} \\
&= |x_{N+3m+5}| - y_{N+3m+5} - 1 \\
&= -2y_{N+3m+3} - (-2y_{N+3m+3} - 1) - 1 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
y_{N+3(m+1)+3} &= y_{N+3m+6} \\
&= x_{N+3m+5} + |y_{N+3m+5}| \\
&= 2y_{N+3m+3} + 2y_{N+3m+3} + 1 \\
&= 4y_{N+3m+3} + 1 \\
&= 2^2 \left(2^{2m+2} y_N + \frac{2^{2m+2} - 1}{3} \right) + 1 \\
&= 2^{2m+4} y_N + \frac{2^{2m+4} - 4}{3} + 1 \\
&= 2^{2(m+1)+2} y_N + \frac{2^{2(m+1)+2} - 1}{3}.
\end{aligned}$$

In particular,

$$\begin{aligned}
y_{N+3(m+1)+3} &\leq 2^{2(m+1)+2} \left(\frac{-2^{2K} + 1}{3 \cdot 2^{2K}} \right) + \frac{2^{2(m+1)+2} - 1}{3} \\
&= -\frac{2^{2m+2K+4}}{3 \cdot 2^{2K}} + \frac{2^{2m+4}}{3 \cdot 2^{2K}} + \frac{2^{2m+4}}{3} - \frac{1}{3} \\
&= -\frac{1}{3} \left(1 - 2^{2m-2K+4} \right) \\
&\leq 0
\end{aligned}$$

and so $\mathcal{P}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{P}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{P}(K-1)$ is true, and so

$$\begin{aligned}
x_{N+3(K-1)+3} &= x_{N+3K} &&= 0 \\
y_{N+3(K-1)+3} &= y_{N+3K} &&= 2^{2K} y_N + \frac{2^{2K} - 1}{3} \quad < \quad 0.
\end{aligned}$$

Note that

$$2^{2K}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K+2}}\right)+\frac{2^{2K}-1}{3} < y_{N+3K} \leq 2^{2K}\left(\frac{-2^{2K}+1}{3\cdot 2^{2K}}\right)+\frac{2^{2K}-1}{3}.$$

So as

$$\begin{aligned} 2^{2K}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K+2}}\right)+\frac{2^{2K}-1}{3} &= \frac{-2^{4K+2}}{3\cdot 2^{2K+2}}+\frac{2^{2K}}{3\cdot 2^{2K+2}}+\frac{2^{2K}}{3}-\frac{1}{3} \\ &= \frac{1}{3}\left(\frac{1}{2^2}-1\right)=-\frac{1}{4} \end{aligned}$$

and

$$2^{2K}\left(\frac{-2^{2K}+1}{3\cdot 2^{2K}}\right)+\frac{2^{2K}-1}{3}=\frac{-2^{2K}+1}{3}+\frac{2^{2K}-1}{3}=0$$

I have

$$-\frac{1}{4} \quad < \quad y_{N+3K} \quad \leq 0$$

and so it follows from Statement 1 of Lemma 2.3.5 that $\{(x_n, y_n)\}_{n=N+3K+5}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

2. Suppose $y_n = -\frac{1}{3}$. Note that $(0, -\frac{1}{3}) \in \mathbf{P}_3^1$ and so $\{(x_n, y_n)\}_{n=N}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

3. Suppose $-\frac{4}{3} < y_N \leq -\frac{1}{3}$.

   I shall first consider the case where $-\frac{4}{3} < y_N \leq -1$.

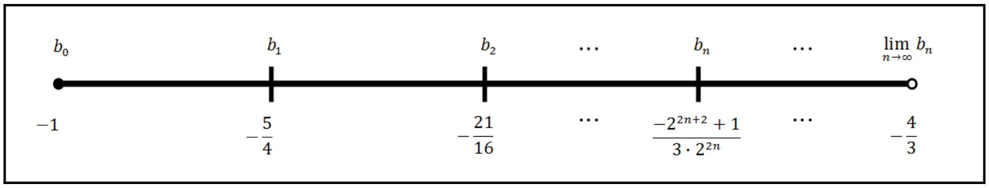
   So suppose $-\frac{4}{3} < y_N \leq -1$. For each integer $n \geq 0$, let

$$b_n = \frac{-2^{2n+2}+1}{3\cdot 2^{2n}}.$$

Observe that

$$-1 = b_0 > b_1 > b_2 > \ldots > -\frac{4}{3} \quad \text{and} \quad \lim_{n\to\infty} b_n = -\frac{4}{3}.$$

See diagram below:



Thus there exists a unique integer $K \geq 1$ such that $y_N \in (b_K, b_{K-1}]$. I first consider the case $K = 1$; that is, $y_N \in \left(-\frac{5}{4}, -1\right]$. Note that if $y_N = -1$ then $(x_N, y_N) = (0, -1)$ and $\{(x_n, y_n)\}_{n=N}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. So assume $y_N \in \left(-\frac{5}{4}, -1\right)$. By Statements 1 and 2 of Lemma 2.3.4, I have $x_{N+3} = 2y_N + 2 < 0$ and $y_{N+3} = -2y_N - 3 \leq 0$. Then

$$\begin{aligned} x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 &&= 0 \\ y_{N+4} &= x_{N+3} + |y_{N+3}| &&= 4y_N + 5 > 0 \end{aligned}$$

and so it follows by Lemma 2.3.3 that $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Hence without loss of generality, I may assume $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{Q}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+1} &= 0 \\ y_{N+3m+1} &= 2^{2m} y_N + \frac{2^{2m+2} - 1}{3} \leq 0. \end{aligned}$$

Claim: $\mathcal{Q}(m)$ is true for $1 \leq m \leq K - 1$.

The proof of the Claim will be by induction on m. I shall first show that $\mathcal{Q}(1)$ is true.

Recall that $x_N = 0$ and $y_N \in (b_K, b_{K-1}] \subset \left(-\frac{4}{3}, -\frac{5}{4}\right]$, and so by Statements 1 and 2 of Lemma 2.3.4 I have

$$\begin{aligned} x_{N+3} &= 2y_N + 2 < 0 \\ y_{N+3} &= -2y_N - 3 < 0 \end{aligned}$$

$$\begin{aligned} x_{N+3(1)+1} &= |x_{N+3}| - y_{N+3} - 1 = 0 \\ y_{N+3(1)+1} &= x_{N+3} + |y_{N+3}| \\ &= 4y_N + 5 \leq 0 \\ &= 2^{2(1)}y_N + \frac{2^{2(1)+2} - 1}{3} \leq 0 \end{aligned}$$

and so $\mathcal{Q}(1)$ is true. Thus if $K = 2$, then I have shown that for $1 \leq m \leq K - 1$, $\mathcal{Q}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K - 2$, and suppose $\mathcal{Q}(m)$ is true. I shall show that $\mathcal{Q}(m+1)$ is true.

Since $\mathcal{Q}(m)$ is true I know

$$\begin{aligned} x_{N+3m+1} &= 0 \\ y_{N+3m+1} &= 2^{2m}y_N + \frac{2^{2m+2} - 1}{3} \leq 0 \end{aligned}$$

and so

$$x_{N+3m+2} = |x_{N+3m+1}| - y_{N+3m+1} - 1 = 0 - y_{N+3m+1} - 1.$$

Recall that $y_N \in (b_K, b_{K-1}] = \left(\frac{-2^{2K+2} + 1}{3 \cdot 2^{2K}}, \frac{-2^{2K} + 1}{3 \cdot 2^{2K-2}}\right]$.

In particular,

$$\begin{aligned} x_{N+3m+2} &= -2^{2m}y_N - \left(\frac{2^{2m+2} - 1}{3}\right) - 1 \\ &< -2^{2m}\left(\frac{-2^{2K+2} + 1}{3 \cdot 2^{2K}}\right) - \left(\frac{2^{2m+2} - 1}{3}\right) - 1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{2K+2m+2}}{3\cdot 2^{2K}} - \frac{2^{2m}}{3\cdot 2^{2K}} - \frac{2^{2m+2}}{3} + \frac{1}{3} - 1 \\
&= -\frac{1}{3}\left(2^{2m-2K+2} + 2\right) \\
&< 0
\end{aligned}
$$

and

$$
y_{N+3m+2} \;=\; x_{N+3m+1} + |y_{N+3m+1}| \;=\; 0 - y_{N+3m+1} \;\geq\; 0.
$$

Hence

$$
\begin{aligned}
x_{N+3m+3} &= |x_{N+3m+2}| - y_{N+3m+2} - 1 \\
&= y_{N+3m+1} + 1 - (-y_{N+3m+1}) - 1 \\
&= 2y_{N+3m+1} \;\leq\; 0
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+3m+3} &= x_{N+3m+2} + |y_{N+3m+2}| \;=\; -y_{N+3m+1} - 1 + (-y_{N+3m+1}) \\
&= -2y_{N+3m+1} - 1.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
y_{N+3m+3} &= -2\left[2^{2m}y_N + \frac{2^{2m+2}-1}{3}\right] - 1 \\
&< -2\left[2^{2m}\left(\frac{-2^{2K+2}+1}{3\cdot 2^{2K}}\right) + \frac{2^{2m+2}-1}{3}\right] - 1 \\
&= \frac{2^{2K+2m+3}}{3\cdot 2^{2K}} - \frac{2^{2m+1}}{3\cdot 2^{2K}} - \frac{2^{2m+3}}{3} + \frac{2}{3} - 1 \\
&= -\frac{1}{3}\left(2^{2m-2K+1} + 1\right) \\
&< 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
x_{N+3(m+1)+1} &= x_{N+3m+4} \\
&= |x_{N+3m+3}| - y_{N+3m+1} - 1 \\
&= -2y_{N+3m+1} - (-2y_{N+3m+1} - 1) - 1 \;=\; 0
\end{aligned}
$$

and

$$\begin{aligned} y_{N+3(m+1)+1} &= y_{N+3m+4} \\ &= x_{N+3m+3} + |y_{N+3m+3}| \\ &= 2y_{N+3m+1} + 2y_{N+3m+1} + 1 \\ &= 4y_{N+3m+1} + 1 \\ &= 2^{2(m+1)} y_N + \frac{2^{2(m+1)+2} - 1}{3}. \end{aligned}$$

In particular,

$$\begin{aligned} y_{N+3(m+1)+1} &\leq 2^{2m+2} \left(\frac{-2^{2K} + 1}{3 \cdot 2^{2K-2}} \right) + \frac{2^{2m+4} - 1}{3} \\ &= -\frac{2^{2K+2m+2}}{3 \cdot 2^{2K-2}} + \frac{2^{2m+2}}{3 \cdot 2^{2K-2}} + \frac{2^{2m+4}}{3} - \frac{1}{3} \\ &= \frac{1}{3} \left(2^{2m-2K+4} - 1 \right) \\ &\leq 0 \end{aligned}$$

and so $\mathcal{Q}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{Q}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{Q}(K-1)$ is true, and so

$$\begin{aligned} x_{N+3(K-1)+1} &= 0 \\ y_{N+3(K-1)+1} &= 2^{2(K-1)} y_N + \frac{2^{2(K-1)+2} - 1}{3} \leq 0. \end{aligned}$$

Note that

$$\begin{aligned} 0 \geq y_{N+3(K-1)+1} &> 2^{2(K-1)} \left(\frac{-2^{2K+2} + 1}{3 \cdot 2^{2K}} \right) + \frac{2^{2K} - 1}{3} \\ &= -\frac{2^{4K}}{3 \cdot 2^{2K}} + \frac{2^{2K-2}}{3 \cdot 2^{2K}} + \frac{2^{2K}}{3} - \frac{1}{3} \\ &= \frac{1}{3} \left(\frac{1}{4} - 1 \right) \\ &= -\frac{1}{4} \end{aligned}$$

and so it follows by Statement 1 of Lemma 2.3.5 that $\{(x_n, y_n)\}_{n=N+3K+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Suppose $-1 < y_N < -\frac{1}{2}$. By Statement 3 of Lemma 2.3.5 I have $\{(x_n, y_n)\}_{n=N+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

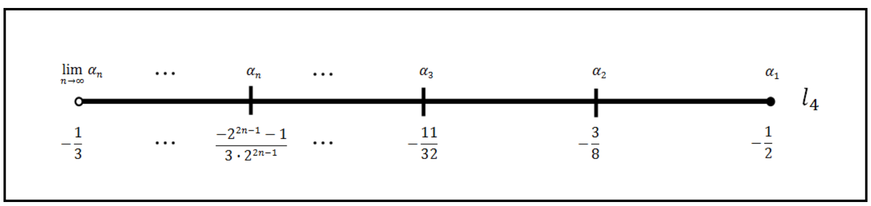
To complete the proof of Statement 3 I shall now suppose that $-\frac{1}{2} \leq y_N < -\frac{1}{3}$. For each integer $n \geq 1$, let

$$\alpha_n = \frac{-2^{2n-1} - 1}{3 \cdot 2^{2n-1}}.$$

Observe that

$$-\frac{1}{2} = \alpha_1 < \alpha_2 < \alpha_3 < \ldots < -\frac{1}{3} \quad \text{and} \quad \lim_{n \to \infty} \alpha_n = -\frac{1}{3}.$$

See diagram below:



Thus there exists a unique integer $K \geq 1$ such that $y_N \in [\alpha_K, \alpha_{K+1})$. I first consider the case $K = 1$; that is, $y_N \in \left[-\frac{1}{2}, -\frac{3}{8}\right)$. By Statement 2 of Lemma 2.3.5 I have $x_{N+5} = 8y_N + 2 \leq 0$, $y_{N+5} = -8y_N - 3 > 0$, and so it follows by Lemma 2.3.2 that $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$. Without loss of generality I may assume $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{R}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+2} &= 2^{2m+1} y_N + \frac{2^{2m+1} - 2}{3} &< 0 \\ y_{N+3m+2} &= -2^{2m+1} y_N - \left(\frac{2^{2m+1} + 1}{3}\right) &\leq 0. \end{aligned}$$

Claim: $\mathcal{R}(m)$ is true for $1 \leq m \leq K-1$.

The proof of the Claim will be by induction on m. I shall first show that $\mathcal{R}(1)$ is true.

Recall that $x_N = 0$ and $y_N \in [\alpha_K, \alpha_{K+1}) \subset \left[-\frac{3}{8}, -\frac{1}{3}\right)$, and so it follows from Statement 2 of Lemma 2.3.5 that

$$\begin{aligned} x_{N+3(1)+2} &= 8y_N + 2 &=& \quad 2^{2(1)+1}y_N + \frac{2^{2(1)+1}-2}{3} &<& \quad 0 \\ y_{N+3(1)+2} &= -8y_N - 3 &=& \quad -2^{2(1)+1}y_N - \left(\frac{2^{2(1)+1}+1}{3}\right) &\leq& \quad 0 \end{aligned}$$

and so $\mathcal{R}(1)$ is true. Thus if $K = 2$, then I have shown that for $1 \leq m \leq K-1$, $\mathcal{R}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{R}(m)$ is true. I shall show that $\mathcal{R}(m+1)$ is true.

Since $\mathcal{R}(m)$ is true I know

$$\begin{aligned} x_{N+3m+2} &= 2^{2m+1}y_N + \frac{2^{2m+1}-2}{3} &<& \quad 0 \\ y_{N+3m+2} &= -2^{2m+1}y_N - \left(\frac{2^{2m+1}+1}{3}\right) &\leq& \quad 0. \end{aligned}$$

Then

$$\begin{aligned} x_{N+3m+3} &= |x_{N+3m+2}| - y_{N+3m+2} - 1 \\ &= -2^{2m+1}y_N - \frac{2^{2m+1}-2}{3} - \left(-2^{2m+1}y_N - \frac{2^{2m+1}+1}{3}\right) - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} y_{N+3m+3} &= x_{N+3m+2} + |y_{N+3m+2}| \\ &= 2^{2m+1}y_N + \frac{2^{2m+1}-2}{3} + 2^{2m+1}y_N + \frac{2^{2m+1}+1}{3} \\ &= 2^{2m+2}y_N + \frac{2^{2m+2}-1}{3}. \end{aligned}$$

Recall that $y_N \in [\alpha_K, \alpha_{K+1}) = \left[\frac{-2^{2K-1}-1}{3 \cdot 2^{2K-1}}, \frac{-2^{2(K+1)-1}-1}{3 \cdot 2^{2(K+1)-1}}\right)$.

In particular,

$$\begin{aligned} y_{N+3m+3} &< 2^{2m+2}\left(\frac{-2^{2(K+1)-1}-1}{3 \cdot 2^{2(K+1)-1}}\right) + \frac{2^{2m+2}-1}{3} \\ &= -\frac{2^{2K+2m+3}}{3 \cdot 2^{2K+1}} - \frac{2^{2m+2}}{3 \cdot 2^{2K+1}} + \frac{2^{2m+2}}{3} - \frac{1}{3} \\ &= -\frac{1}{3}\left(1 + 2^{2m-2K+1}\right) \\ &< 0. \end{aligned}$$

Then

$$x_{N+3m+4} = |x_{N+3m+3}| - y_{N+3m+3} - 1 = 0 - y_{N+3m+3} - 1 = -y_{N+3m+3} - 1.$$

In particular,

$$\begin{aligned} x_{N+3m+4} &= -2^{2m+2}y_N - \frac{2^{2m+2}-1}{3} - 1 \\ &\leq -2^{2m+2}\left(\frac{-2^{2K-1}-1}{3 \cdot 2^{2K-1}}\right) - \left(\frac{2^{2m+2}-1}{3}\right) - 1 \\ &= \frac{2^{2m+2K+1}}{3 \cdot 2^{2K-1}} + \frac{2^{2m+2}}{3 \cdot 2^{2K-1}} - \frac{2^{2m+2}}{3} + \frac{1}{3} - 1 \\ &= -\frac{2}{3}\left(1 - 2^{2m-2K+2}\right) \\ &< 0. \end{aligned}$$

Hence

$$y_{N+3m+4} = x_{N+3m+3} + |y_{N+3m+3}| = 0 + (-y_{N+3m+3}) = -y_{N+3m+3} > 0.$$

Finally,

$$\begin{aligned}
x_{N+3(m+1)+2} &= x_{N+3m+5} \\
&= |x_{N+3m+4}| - y_{N+3m+4} - 1 \\
&= y_{N+3m+3} + 1 - (-y_{N+3m+3}) - 1 \\
&= 2y_{N+3m+3} && < \quad 0 \\
&= 2^{2(m+1)+1} y_N + \frac{2^{2(m+1)+1} - 2}{3} && < \quad 0
\end{aligned}$$

and

$$\begin{aligned}
y_{N+3(m+1)+2} &= y_{N+3m+5} \\
&= x_{N+3m+4} + |y_{N+3m+4}| \\
&= -y_{N+3m+3} - 1 + (-y_{N+3m+3}) \\
&= -2y_{N+3m+3} - 1 \\
&= -2^{2(m+1)+1} y_N - \left(\frac{2^{2(m+1)+1} + 1}{3} \right).
\end{aligned}$$

In particular,

$$\begin{aligned}
y_{N+3(m+1)+2} &\leq -2^{2m+3} \left(\frac{-2^{2K-1} - 1}{3 \cdot 2^{2K-1}} \right) - \left(\frac{2^{2m+3} + 1}{3} \right) \\
&= \frac{2^{2m+2K+2}}{3 \cdot 2^{2K-1}} + \frac{2^{2m+3}}{3 \cdot 2^{2K-1}} - \frac{2^{2m+3}}{3} - \frac{1}{3} \\
&= \frac{1}{3} \left(2^{2m-2K+4} - 1 \right) \\
&\leq 0
\end{aligned}$$

and so $\mathcal{R}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{R}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{R}(K-1)$ is true, and so

$$\begin{aligned}
x_{N+3(K-1)+2} &= 2^{2(K-1)+1} y_N + \frac{2^{2(K-1)+1} - 2}{3} && < \quad 0 \\
y_{N+3(K-1)+2} &= -2^{2(K-1)+1} y_N - \left(\frac{2^{2(K-1)+1} + 1}{3} \right) && \leq \quad 0.
\end{aligned}$$

Then

$$\begin{aligned} x_{N+3K} &= x_{N+3(K-1)+3} \\ &= |x_{N+3(K-1)+2}| - y_{N+3(K-1)+2} - 1 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} y_{N+3K} &= y_{N+3(K-1)+3} \\ &= x_{N+3(K-1)+2} + |y_{N+3(K-1)+2}| \\ &= 2^{2K} y_N + \frac{2^{2K}-1}{3}. \end{aligned}$$

Note that

$$2^{2K}\left(\tfrac{-2^{2K-1}-1}{3\cdot 2^{2K-1}}\right) + \tfrac{2^{2K}-1}{3} \quad \leq \quad y_{N+3K} \quad < \quad 2^{2K}\left(\tfrac{-2^{2K+1}-1}{3\cdot 2^{2K+1}}\right) + \tfrac{2^{2K}-1}{3}.$$

So as

$$2^{2K}\left(\frac{-2^{2K-1}-1}{3\cdot 2^{2K-1}}\right) + \frac{2^{2K}-1}{3} = \frac{-2^{4K-1}}{3\cdot 2^{2K-1}} + \frac{2^{2K}}{3\cdot 2^{2K-1}} + \frac{2^{2K}}{3} - \frac{1}{3} = -1$$

and

$$2^{2K}\left(\frac{-2^{2K+1}-1}{3\cdot 2^{2K+1}}\right) + \frac{2^{2K}-1}{3} = \frac{-2^{4K+1}}{3\cdot 2^{2K+1}} + \frac{2^{2K}}{3} - \frac{1}{3} = -\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$$

I have

$$-1 \leq y_{N+3K} < -\frac{1}{2}$$

and so it follows by Statement 3 of Lemma 2.3.5 and the fact $(0,-1) \in \mathbf{P}_3^1$ that the solution $\{(x_n, y_n)\}_{n=N+3K+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

4. Suppose $y_N = -\frac{4}{3}$. By direct computations I have $(x_{N+3}, y_{N+3}) = (-\frac{2}{3}, -\frac{1}{3}) \in \mathbf{P}_3^2$, and so $\{(x_n, y_n)\}_{n=N+3}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^2$.

5. Suppose $y_N < -\frac{4}{3}$.

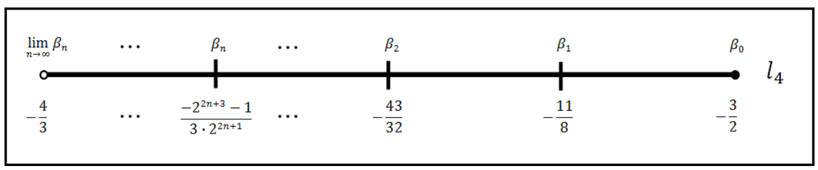
First consider the case $-\frac{3}{2} \leq y_N < -\frac{4}{3}$. For each integer $n \geq 0$, let

$$\beta_n = \frac{-2^{2n+3} - 1}{3 \cdot 2^{2n+1}}.$$

Observe that

$$-\frac{3}{2} = \beta_0 < \beta_1 < \beta_2 < \ldots < -\frac{4}{3} \quad \text{and} \quad \lim_{n\to\infty} \beta_n = -\frac{4}{3}.$$

See diagram below:



Thus there exists a unique integer $K \geq 1$ such that $y_N \in [\beta_{K-1}, \beta_K)$. I first consider the case $K = 1$; that is, $y_N \in \left[-\frac{3}{2}, -\frac{11}{8}\right)$. By Statements 1 and 2 of Lemma 2.3.4 I have

$$\begin{aligned} x_{N+3} &= 2y_N + 2 < 0 \\ y_{N+3} &= -2y_N - 3 \leq 0 \end{aligned}$$

and so

$$\begin{aligned} x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 = 0 \\ y_{N+4} &= x_{N+3} + |y_{N+3}| = 4y_N + 5 < 0. \end{aligned}$$

In particular, $-1 \leq y_{N+4} < -\frac{1}{2}$. It follows by Statement 3 of Lemma 2.3.5 that the solution $\{(x_n, y_n)\}_{n=N+7}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Thus without loss of generality, I may assume that $K \geq 2$. For each integer $m \geq 1$, let $\mathcal{S}(m)$ be the following statement:

$$\begin{aligned} x_{N+3m+3} &= 2^{2m+1}y_N + \frac{2^{2m+3}-2}{3} &< 0 \\ y_{N+3m+3} &= -2^{2m+1}y_N - \left(\frac{2^{2m+3}-2}{3}\right) - 1 &\leq 0. \end{aligned}$$

Claim: $\mathcal{S}(m)$ is true for $1 \leq m \leq K-1$.

The proof of the Claim will be by induction on m. I shall first show that $\mathcal{S}(1)$ is true.

Recall that $x_N = 0$ and $y_N \in [\beta_{K-1}, \beta_K) \subset \left[-\frac{11}{8}, -\frac{4}{3}\right)$, and so by Statements 1 and 2 of Lemma 2.3.4 I have

$$\begin{aligned} x_{N+3} &= 2y_N + 2 < 0 \\ y_{N+3} &= -2y_N - 3 < 0 \end{aligned}$$

$$\begin{aligned} x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 &= 0 \\ y_{N+4} &= x_{N+3} + |y_{N+3}| &= 4y_N + 5 < 0 \\ x_{N+5} &= |x_{N+4}| - y_{N+4} - 1 &= -4y_N - 6 < 0 \\ y_{N+5} &= x_{N+4} + |y_{N+4}| &= -4y_N - 5 > 0. \end{aligned}$$

Finally,

$$\begin{aligned} x_{N+3(1)+3} &= x_{N+6} = |x_{N+5}| - y_{N+5} - 1 = 8y_N + 10 < 0 \\ y_{N+3(1)+3} &= y_{N+6} = x_{N+5} + |y_{N+5}| = -8y_N - 11 \leq 0. \end{aligned}$$

It follows that $\mathcal{S}(1)$ is true. Thus if $K = 2$, then I have shown that for $1 \leq m \leq K-1$, $\mathcal{S}(m)$ is true. It remains to consider the case $K \geq 3$. So assume that $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$, and suppose $\mathcal{S}(m)$ is true. I shall show that $\mathcal{S}(m+1)$ is true.

Since $\mathcal{S}(m)$ is true, I know

$$
\begin{aligned}
x_{N+3m+3} &= \qquad 2^{2m+1}y_N + \frac{2^{2m+3}-2}{3} \qquad &< \quad 0 \\
y_{N+3m+3} &= -2^{2m+1}y_N - \left(\frac{2^{2m+3}-2}{3}\right) - 1 &\leq \quad 0.
\end{aligned}
$$

Note that $y_{N+3m+3} = -x_{N+3m+3} - 1$, and so $-1 \leq x_{N+3m+3} < 0$.

Thus

$$
\begin{aligned}
x_{N+3m+4} &= |x_{N+3m+3}| - y_{N+3m+3} - 1 \\
&= -x_{N+3m+3} - (-x_{N+3m+3} - 1) - 1 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+3m+4} &= x_{N+3m+3} + |y_{N+3m+3}| \\
&= x_{N+3m+3} + x_{N+3m+3} + 1 \\
&= 2x_{N+3m+3} + 1.
\end{aligned}
$$

Recall that $y_N \in [\beta_{K-1}, \beta_K) = \left[\frac{-2^{2(K-1)+3}-1}{3 \cdot 2^{2(K-1)+1}}, \frac{-2^{2K+3}-1}{3 \cdot 2^{2K+1}}\right)$.

In particular,

$$
\begin{aligned}
y_{N+3m+4} &= 2\left[2^{2m+1}y_N + \frac{2^{2m+3}-2}{3}\right] + 1 \\
&< 2\left[2^{2m+1}\left(\frac{-2^{2K+3}-1}{3 \cdot 2^{2K+1}}\right) + \frac{2^{2m+3}-2}{3}\right] + 1 \\
&= -\frac{2^{2K+2m+5}}{3 \cdot 2^{2K+1}} - \frac{2^{2m+2}}{3 \cdot 2^{2K+1}} + \frac{2^{2m+4}}{3} - \frac{1}{3} \\
&= -\frac{1}{3}\left(2^{2m-2K+1} + 1\right) \\
&< 0.
\end{aligned}
$$

Also note that $-1 < x_{N+3m+3} < -\frac{1}{2}$.

Thus

$$\begin{aligned} x_{N+3m+5} &= |x_{N+3m+4}| - y_{N+3m+4} - 1 \\ &= 0 - (2x_{N+3m+3} + 1) - 1 \\ &= -2x_{N+3m+3} - 2 \quad < \quad 0 \end{aligned}$$

and

$$\begin{aligned} y_{N+3m+5} &= x_{N+3m+4} + |y_{N+3m+4}| \\ &= 0 + (-2x_{N+3m+3} - 1) \\ &= -2x_{N+3m+3} - 1 \quad > \quad 0. \end{aligned}$$

Finally,

$$\begin{aligned} x_{N+3(m+1)+3} &= x_{N+3m+6} \\ &= |x_{N+3m+5}| - y_{N+3m+5} - 1 \\ &= 2x_{N+3m+3} + 2 - (-2x_{N+3m+3} - 1) - 1 \\ &= 4x_{N+3m+3} + 2 && < \quad 0 \\ &= 4\left[2^{2m+1}y_N + \left(\frac{2^{2m+3} - 2}{3}\right)\right] + 2 && < \quad 0 \\ &= 2^{2(m+1)+1}y_N + \left(\frac{2^{2(m+1)+3} - 2}{3}\right) + 2 && < \quad 0 \end{aligned}$$

and

$$\begin{aligned} y_{N+3(m+1)+3} &= y_{N+3m+6} \\ &= x_{N+3m+5} + |y_{N+3m+5}| \\ &= -2x_{N+3m+3} - 2 + (-2x_{N+3m+3} - 1) \\ &= -4x_{N+3m+3} - 3 \\ &= -4\left[2^{2m+1}y_N + \left(\frac{2^{2m+3} - 2}{3}\right)\right] - 3 \\ &= -2^{2(m+1)+1}y_N - \left(\frac{2^{2(m+1)+3} - 2}{3}\right) - 1. \end{aligned}$$

In particular,

$$
\begin{aligned}
y_{N+3m+6} &\leq -4\left[2^{2m+1}\left(\frac{-2^{2(K-1)+3}-1}{3\cdot 2^{2(K-1)+1}}\right)+\frac{2^{2m+3}-2}{3}\right]-3 \\
&= \frac{2^{2K+2m+4}}{3\cdot 2^{2K-1}}+\frac{2^{2m+3}}{3\cdot 2^{2K-1}}-\frac{2^{2m+5}}{3}-\frac{1}{3} \\
&= \frac{1}{3}\left(2^{2m-2K+4}-1\right) \\
&< 0
\end{aligned}
$$

and so $\mathcal{S}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{S}(m)$ is true for $1 \leq m \leq K-1$. Specifically, $\mathcal{S}(K-1)$ is true, and so

$$
x_{N+3(K-1)+3} = x_{N+3K} = 2^{2K-1}y_N + \frac{2^{2K+1}-2}{3} < 0
$$

$$
y_{N+3(K-1)+3} = y_{N+3K} = -2^{2K-1}y_N - \left(\frac{2^{2K+1}-2}{3}\right) - 1 < 0.
$$

Note that $y_{N+3K} = -x_{N+3K} - 1$.

Thus

$$
\begin{aligned}
x_{N+3K+1} &= |x_{N+3K}| - y_{N+3K} - 1 \\
&= -x_{N+3K} - (-x_{N+3K} - 1) - 1 \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+3K+1} &= x_{N+3K} + |y_{N+3K}| \\
&= x_{N+3K} + x_{N+3K} + 1 \\
&= 2x_{N+3K} + 1 \\
&= 2\left(2^{2K-1}y_N + \frac{2^{2K+1}-2}{3}\right) + 1.
\end{aligned}
$$

Note that

$$2\left[2^{2K-1}\left(\frac{-2^{2(K-1)+3}-1}{3\cdot 2^{2(K-1)+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1 \quad\leq\quad y_{N+3K+1}$$
$$< 2\left[2^{2K-1}\left(\frac{-2^{2K+3}-1}{3\cdot 2^{2K+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1.$$

So as

$$2\left[2^{2K-1}\left(\frac{-2^{2(K-1)+3}-1}{3\cdot 2^{2(K-1)+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1 \quad=$$
$$\frac{-2^{4K+1}}{3\cdot 2^{2K-1}}-\frac{2^{2K}}{3\cdot 2^{2K-1}}+\frac{2^{2K+2}}{3}-\frac{1}{3}=-\frac{1}{3}(2+1) \quad=\quad -1$$

and

$$2\left[2^{2K-1}\left(\frac{-2^{2K+3}-1}{3\cdot 2^{2K+1}}\right)+\frac{2^{2K+1}-2}{3}\right]+1 \quad=\quad \frac{-2^{2K+3}}{3\cdot 2}-\frac{1}{6}+\frac{2^{2K+2}}{3}-\frac{1}{3}$$
$$=\quad -\frac{1}{6}-\frac{1}{3}=-\frac{1}{2}.$$

I have

$$-1\leq y_{N+3K+1}<-\frac{1}{2}$$

and hence it follows from case 3 of this Lemma and the fact that $(0,-1)\in\mathbf{P}_3^1$ that the solution $\{(x_n,y_n)\}_{n=N+3K+5}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$.

Finally, suppose $y_N<-\frac{3}{2}$. Then by Statement 3 of Lemma 2.3.4 the solution $\{(x_n,y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

□

Lemma 2.3.7 *Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in Q_1$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.*

Proof: I have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &&= x_N - y_N - 1 \\ y_{N+1} &= x_N + |y_N| &&= x_N + y_N > 0. \end{aligned}$$

If $x_{N+1} \geq 0$ then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &&= -2y_N - 2 < 0 \\ y_{N+2} &= x_{N+1} + |y_{N+1}| &&= 2x_N - 1 > 0 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &&= -2x_N + 2y_N + 2 \leq 0 \\ y_{N+3} &= x_{N+2} + |y_{N+2}| &&= 2x_N - 2y_N - 3 \\ x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 &&= 0 \end{aligned}$$

and so $(x_{N+4}, y_{N+4}) \in l_2 \cup l_4$. By Lemmas 2.3.3 and 2.3.6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

If $x_{N+1} < 0$ then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &&= -2x_N < 0 \\ y_{N+2} &= x_{N+1} + |y_{N+1}| &&= 2x_N - 1 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &&= 0 \end{aligned}$$

and so $(x_{N+3}, y_{N+3}) \in l_2 \cup l_4$. By Lemmas 2.3.3 and 2.3.6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

Lemma 2.3.8 *Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in Q_2$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.*

Proof: I have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= -x_N - y_N - 1 \\ y_{N+1} &= x_N + |y_N| &= x_N + y_N. \end{aligned}$$

Case 1: Suppose $y_{N+1} \geq 0$. Then by Lemma 2.3.2, the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 2: Suppose $y_{N+1} < 0$ and $x_{N+1} \leq 0$. Then $x_{N+2} = |x_{N+1}| - y_{N+1} - 1 = 0$ and so $(x_{N+2}, y_{N+2}) \in l_2 \cup l_4$. By Lemmas 2.3.3 and 2.3.6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 3: Suppose $y_{N+1} < 0$ and $x_{N+1} > 0$. Then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} - 1 &= -2x_N - 2y_N - 2 &> 0 \\ y_{N+2} &= x_{N+1} + |y_{N+1}| &= -2x_N - 2y_N - 1 &> 0 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} - 1 &= -2 & \\ y_{N+3} &= x_{N+2} + |y_{N+2}| &= -4x_N - 4y_N - 3 &> 0 \\ x_{N+4} &= |x_{N+3}| - y_{N+3} - 1 &= 4x_N + 4y_N + 4 &< 0 \\ y_{N+4} &= x_{N+3} + |y_{N+3}| &= -4x_N - 4y_N - 5 & \\ x_{N+5} &= |x_{N+4}| - y_{N+4} - 1 &= 0 & \end{aligned}$$

and so $(x_{N+5}, y_{N+5}) \in l_2 \cup l_4$. By Lemmas 2.3.3 and 2.3.6, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

Lemma 2.3.9 *Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in Q_4$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.*

Proof: I have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= x_N - y_N - 1 \\ y_{N+1} &= x_N + |y_N| &= x_N - y_N &> 0 \end{aligned}$$

Case 1: Suppose $x_{N+1} > 0$. Then $(x_{N+1}, y_{N+1}) \in Q_1$ and so by Lemma 2.3.7, the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 2: Suppose $x_{N+1} = 0$. Then $(x_{N+1}, y_{N+1}) \in l_2$ and so by Lemma 2.3.3, the solution $\{(x_n, y_n)\}_{n=N+4}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 3: Suppose $x_{N+1} < 0$. Then $(x_{N+1}, y_{N+1}) \in Q_2$ and so by Lemma 2.3.8, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

□

Lemma 2.3.10 *Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in l_1$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or $\mathbf{P}_3^2$.*

Proof: I have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 &= x_N - 1 \\ y_{N+1} &= x_N + |y_N| &= x_N \end{aligned}$$

Case 1: Suppose $x_N = 0$. Then $(x_{N+1}, y_{N+1}) = (-1, 0)$, and so $(x_{N+2}, y_{N+2}) = (0, -1)$. Hence the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 2: Suppose $0 < x_N \leq 1$. Then $x_{N+1} \leq 0$ and $y_{N+1} > 0$. Thus $(x_{N+1}, y_{N+1}) \in Q_2 \cup l_2$, and hence by Lemmas 2.3.3 and 2.3.8, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 3: Suppose $x_N > 1$. Then $x_{N+1} > 0$ and $y_{N+1} > 0$. Thus $(x_{N+1}, y_{N+1}) \in Q_1$ and by Lemma 2.3.7, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

Lemma 2.3.11 *Suppose there exists a non-negative integer $N \geq 0$ such that $(x_N, y_N) \in l_3$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.*

Proof: I have

$$\begin{aligned} x_{N+1} &= |x_N| - y_N - 1 = -x_N - 1 \\ y_{N+1} &= x_N + |y_N| = x_N < 0. \end{aligned}$$

Case 1: Suppose $-1 < x_N \leq 0$. Then $x_{N+2} = |x_{N+1}| - y_{N+1} - 1 = 0$, and so $(x_{N+2}, y_{N+2}) \in l_2 \cup l_4$. It follows by Lemmas 2.3.3 and 2.3.6, that the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

Case 2: Suppose $x_N = -1$. Then $(x_{N+1}, y_{N+1}) = (0, -1) \in \mathbf{P}_3^1$, and so the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is the period-3 cycle $\mathbf{P}_3^1$.

Case 3: Suppose $x_N < -1$. Then $(x_{N+1}, y_{N+1}) \in Q_4 \cup l_1$. It follows by Lemmas 2.3.9 and 2.3.10, the solution $\{(x_n, y_n)\}_{n=N+2}^{\infty}$ is eventually the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. □

To complete the proof of Theorem 2.1 it remains to consider the case where the initial condition $(x_0, y_0) \in Q_3$.

Lemma 2.3.12 *Suppose $(x_0, y_0) \in Q_3$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium solution $(\bar{x}, \bar{y}) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$, or else is eventually either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.*

Proof: If $(x_0, y_0) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$, then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the equilibrium. So suppose $(x_0, y_0) \in Q_3 \setminus \left\{\left(-\frac{2}{5}, -\frac{1}{5}\right)\right\}$. It suffices to show that there exists an integer $N \geq 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$.

For the sake of contradiction, assume that it is false that there exists an integer $N \geq 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$. It follows from the previous lemmas that $x_n < 0$ and $y_n < 0$ for every integer $n \geq 0$.

Case 1: Suppose $x_0 \leq -2$ and $y_0 < 0$. Then

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1 > 0$$

which is a contradiction, and the proof is complete.

Case 2: Suppose $-2 < x_0 < 0$ and $y_0 \leq -1$. Then

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1 > 0$$

which is a contradiction, and the proof is complete.

Case 3: It remains to consider the case $(x_0, y_0) \in (-2, 0) \times (-1, 0)$. For each integer $n \geq 0$, let

$$a_n = \frac{-2^{4n-2} - 1}{5 \cdot 2^{4n-3}}, \quad b_n = \frac{-2^{4n} + 1}{5 \cdot 2^{4n-1}}, \quad c_n = \frac{-2^{4n-2} - 1}{5 \cdot 2^{4n-2}}, \quad d_n = \frac{-2^{4n} + 1}{5 \cdot 2^{4n}}$$

$$\text{and } D_n = \frac{2^{4n} - 1}{5}.$$

Observe that

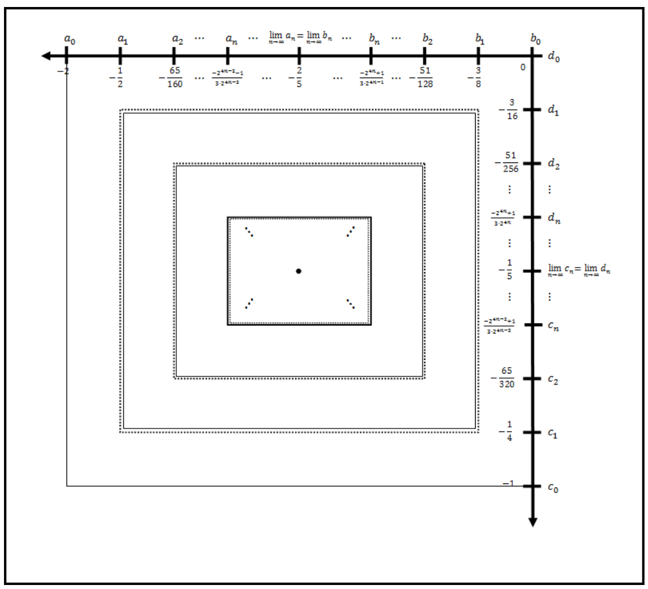
$$-2 = a_0 < a_1 < a_2 < \ldots < -\frac{2}{5} \qquad \text{and} \qquad \lim_{n\to\infty} a_n = -\frac{2}{5}$$

$$0 = b_0 > b_1 > b_2 > \ldots > -\frac{2}{5} \qquad \text{and} \qquad \lim_{n\to\infty} b_n = -\frac{2}{5}$$

$$-1 = c_0 < c_1 < c_2 < \ldots < -\frac{1}{5} \qquad \text{and} \qquad \lim_{n\to\infty} c_n = -\frac{1}{5}$$

$$0 = d_0 > d_1 > d_2 > \ldots > -\frac{1}{5} \qquad \text{and} \qquad \lim_{n\to\infty} d_n = -\frac{1}{5}.$$

See diagram below:



There exists a unique integer $K \geq 0$ such that

$$(x_0, y_0) \in [a_K, b_K] \times [c_K, d_K] \setminus [a_{K+1}, b_{K+1}] \times [c_{K+1}, d_{K+1}].$$

I first consider the case $K = 0$; that is, $(x_0, y_0) \in [-2, 0] \times [-1, 0] \setminus [-\frac{1}{2}, -\frac{3}{8}] \times [-\frac{1}{4}, -\frac{3}{16}]$. Note that by Lemmas 2.3.6 and 2.3.11, and by Case 1 and Case 2 of this lemma, I know that the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually either the period-3 cycle $\mathbf{P}_3^1$ or the period-3 cycle $\mathbf{P}_3^2$ when (x_0, y_0) is an element of the outer boundaries of $[-2,0] \times [-1,0]$.

Recall by assumption that $x_n < 0$ and $y_n < 0$ for every integer $n \geq 0$.

So suppose $(x_0, y_0) \in (-2,0) \times (-1,0) \setminus \left[-\frac{1}{2}, -\frac{3}{8}\right] \times \left[-\frac{1}{4}, -\frac{3}{16}\right]$. Then

$$x_1 = |x_0| - y_0 - 1 = -x_0 - y_0 - 1$$

$$y_1 = x_0 + |y_0| = x_0 - y_0$$

$$x_2 = |x_1| - y_1 - 1 = (x_0 + y_0 + 1) - (x_0 - y_0) - 1 = 2y_0$$

$$y_2 = x_1 + |y_1| = (-x_0 - y_0 - 1) + (-x_0 + y_0) = -2x_0 - 1.$$

If $-2 < x_0 < -\frac{1}{2}$, then $y_2 > 0$ which is a contradiction.

Thus $-\frac{1}{2} \leq x_0 < 0$. Then

$$x_3 = |x_2| - y_2 - 1 = (-2y_0) - (-2x_0 - 1) - 1 = 2x_0 - 2y_0$$

$$y_3 = x_2 + |y_2| = (2y_0) + (2x_0 + 1) = 2x_0 + 2y_0 + 1$$

$$x_4 = |x_3| - y_3 - 1 = (-2x_0 + 2y_0) - (2x_0 + 2y_0 + 1) - 1 = -4x_0 - 2$$

$$y_4 = x_3 + |y_3| = (2x_0 - 2y_0) + (-2x_0 - 2y_0 - 1) = -4y_0 - 1.$$

If $-1 < y_0 < -\frac{1}{4}$, then $y_4 > 0$ which is a contradiction.

Hence $-\frac{1}{4} \leq y_0 < 0$. Then

$$x_5 = |x_4| - y_4 - 1 = (4x_0 + 2) - (-4y_0 - 1) - 1 = 4x_0 + 4y_0 + 2$$

$$y_5 = x_4 + |y_4| = (-4x_0 - 2) + (4y_0 + 1) = -4x_0 + 4y_0 - 1$$

$$x_6 = |x_5| - y_5 - 1 = (-4x_0 - 4y_0 - 2) - (-4x_0 + 4y_0 - 1) - 1 = -8y_0 - 2$$

$$y_6 = x_5 + |y_5| = (4x_0 + 4y_0 + 2) + (4x_0 - 4y_0 + 1) = 8x_0 + 3.$$

If $-\frac{3}{8} < x_0 < 0$, then $y_6 > 0$ which is a contradiction.

Hence $-\frac{1}{2} < x_0 \leq -\frac{3}{8}$. Thus

$$x_7 = |x_6| - y_6 - 1 = (8y_0 + 2) - (8x_0 + 3) - 1 = -8x_0 + 8y_0 - 2$$

$$y_7 = x_6 + |y_6| = (-8y_0 - 2) + (-8x_0 - 3) = -8x_0 - 8y_0 - 5$$

$$x_8 = |x_7| - y_7 - 1 = (8x_0 - 8y_0 + 2) - (-8x_0 - 8y_0 - 5) - 1 = 16x_0 + 6$$

$$y_8 = x_7 + |y_7| = (-8x_0 + 8y_0 - 2) + (8x_0 + 8y_0 + 5) = 16y_0 + 3 > 0,$$

which is a contradiction. Thus the case $K = 0$ is complete.

Next consider the case $K \geq 1$. Recall that $x_n < 0$ and $y_n < 0$ for all $n \geq 0$.

For each integer m such that $0 \leq m \leq K - 1$, let $\mathcal{P}(m)$ be the following proposition:

$$\begin{aligned}
x_{8m+1} &= -2^{4m}x_0 - 2^{4m}y_0 - 3D_m - 1 \\
y_{8m+1} &= 2^{4m}x_0 - 2^{4m}y_0 + D_m \\
\\
x_{8m+2} &= 2^{4m+1}y_0 + 2D_m \\
y_{8m+2} &= -2^{4m+1}x_0 - 4D_m - 1 \\
\\
x_{8m+3} &= 2^{4m+1}x_0 - 2^{4m+1}y_0 + 2D_m \\
y_{8m+3} &= 2^{4m+1}x_0 + 2^{4m+1}y_0 + 6D_m + 1
\end{aligned}$$

$$\begin{aligned}
x_{8m+4} &= -2^{4m+2}x_0 - 8D_m - 2\\
y_{8m+4} &= -2^{4m+2}y_0 - 4D_m - 1\\[1em]
x_{8m+5} &= 2^{4m+2}x_0 + 2^{4m+2}y_0 + 12D_m + 2\\
y_{8m+5} &= -2^{4m+2}x_0 + 2^{4m+2}y_0 - 4D_m - 1\\[1em]
x_{8m+6} &= -2^{4m+3}y_0 - 8D_m - 2\\
y_{8m+6} &= 2^{4m+3}x_0 + 16D_m + 3\\[1em]
x_{8m+7} &= -2^{4m+3}x_0 + 2^{4m+3}y_0 - 8D_m - 2\\
y_{8m+7} &= -2^{4m+3}x_0 - 2^{4m+3}y_0 - 24D_m - 5\\[1em]
x_{8m+8} &= 2^{4m+4}x_0 + 32D_m + 6\\
y_{8m+8} &= 2^{4m+4}x_0 + 16D_m + 3.
\end{aligned}$$

Claim: $\mathcal{P}(m)$ is true for $0 \le m \le K-1$. The proof of the Claim will be by induction on m. I shall first show that $\mathcal{P}(0)$ is true.

$$\begin{aligned}
x_{8(0)+1} &= -x_0 - y_0 - 1 &&= -2^{4(0)}x_0 - 2^{4(0)}y_0 - 3D_0 - 1\\
y_{8(0)+1} &= x_0 - y_0 &&= 2^{4(0)}x_0 - 2^{4(0)}y_0 - D_0\\[1em]
x_{8(0)+2} &= 2y_0 &&= 2^{4(0)+1}y_0 + 2D_0\\
y_{8(0)+2} &= -2x_0 - 1 &&= -2^{4(0)+1}x_0 - 4D_0 - 1\\[1em]
x_{8(0)+3} &= 2x_0 - 2y_0 &&= 2^{4(0)+1}x_0 - 2^{4(0)+1}y_0 + 2D_0\\
y_{8(0)+3} &= 2x_0 + 2y_0 + 1 &&= 2^{4(0)+1}x_0 + 2^{4(0)+1}y_0 + 6D_0 + 1\\[1em]
x_{8(0)+4} &= -4x_0 - 2 &&= -2^{4(0)+2}x_0 - 8D_0 - 2\\
y_{8(0)+4} &= -4y_0 - 1 &&= -2^{4(0)+2}x_0 - 4D_0 - 1\\[1em]
x_{8(0)+5} &= 4x_0 + 4y_0 + 2 &&= 2^{4(0)+1}x_0 - 2^{4(0)+2}y_0 + 12D_0 + 2\\
y_{8(0)+5} &= -4x_0 + 4y_0 - 1 &&= -2^{4(0)+2}x_0 + 2^{4(0)+2}y_0 - 4D_0 - 1\\[1em]
x_{8(0)+6} &= -8x_0 - 2 &&= -2^{4(0)+3}x_0 - 8D_0 - 2\\
y_{8(0)+6} &= 8y_0 + 3 &&= 2^{4(0)+3}x_0 + 16D_0 + 3\\[1em]
x_{8(0)+7} &= -8x_0 + 8y_0 - 2 &&= -2^{4(0)+3}x_0 + 2^{4(0)+3}y_0 - 8D_0 - 2\\
y_{8(0)+7} &= -8x_0 - 8y_0 - 5 &&= -2^{4(0)+3}x_0 - 2^{4(0)+3}y_0 - 24D_0 + 5\\[1em]
x_{8(0)+8} &= 16x_0 + 6 &&= 2^{4(0)+4}x_0 + 32D_0 + 6\\
y_{8(0)+8} &= 16y_0 + 3 &&= 2^{4(0)+4}x_0 + 16D_0 + 3
\end{aligned}$$

and so $\mathcal{P}(0)$ is true. Thus if $K = 1$, then I have shown that for $0 \le m \le K - 1$, $\mathcal{P}(m)$ is true. It remains to consider the case $K \ge 2$. So assume that $K \ge 2$. Suppose that m is an integer such that $0 \le m \le K - 2$, and that $\mathcal{P}(m)$ is true. I shall show that $\mathcal{P}(m+1)$ is true.

Since $\mathcal{P}(m)$ is true, I know

$$x_{8m+8} = 2^{4m+4}x_0 + 32D_m + 6$$

$$y_{8m+8} = 2^{4m+4}x_0 + 16D_m + 3.$$

Hence

$$\begin{aligned}
x_{8(m+1)+1} &= x_{8m+9} \\
&= |x_{8m+8}| - y_{8m+8} - 1 \\
&= -(2^{4m+4}x_0 + 32D_m + 6) - (2^{4m+4}y_0 + 16D_m + 3) - 1 \\
&= -2^{4m+4}x_0 - 2^{4m+4}y_0 - 48D_m - 10 \\
&= -2^{4m+4}x_0 - 2^{4m+4}y_0 - 48\left(\frac{2^{4m}-1}{5}\right) - 10 \\
&= -2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3\left(\frac{2^{4(m+1)}-1}{5}\right) - 1 \\
&= -2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3D_{m+1} - 1
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+1} &= y_{8m+9} \\
&= x_{8m+8} + |y_{8m+8}| \\
&= 2^{4m+4}x_0 + 32D_m + 6 + (-2^{4m+4}y_0 - 16D_m - 3) \\
&= 2^{4m+4}x_0 - 2^{4m+4}y_0 + 16D_m + 3 \\
&= 2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 + 16\left(\frac{2^{4m}-1}{5}\right) + 3 \\
&= 2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 + D_{m+1}.
\end{aligned}$$

Thus

$$\begin{aligned}
x_{8(m+1)+2} &= x_{8m+10} \\
&= |x_{8m+9}| - y_{8m+9} - 1 \\
&= -(-2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3D_{m+1} - 1) \\
&\quad -(2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 + D_{m+1}) - 1 \\
&= 2^{4(m+1)+1}y_0 + 2D_{m+1}
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+2} &= y_{8m+10} \\
&= x_{8m+9} + |y_{8m+9}| \\
&= -2^{4(m+1)}x_0 - 2^{4(m+1)}y_0 - 3D_{m+1} - 1+ \\
&\qquad (-2^{4(m+1)}x_0 + 2^{4(m+1)}y_0 - D_{m+1}) \\
&= -2^{4(m+1)+1}x_0 - 4D_{m+1} - 1.
\end{aligned}$$

Then

$$\begin{aligned}
x_{8(m+1)+3} &= x_{8m+11} \\
&= |x_{8m+10}| - y_{8m+10} - 1 \\
&= -2^{4(m+1)+1}y_0 - 2D_{m+1} + 2^{4(m+1)+1}x_0 + 4D_{m+1} + 1 - 1 \\
&= 2^{4(m+1)+1}x_0 - 2^{4(m+1)+1}y_0 + 2D_{m+1}
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+3} &= y_{8m+11} \\
&= x_{8m+10} + |y_{8m+10}| \\
&= 2^{4(m+1)+1}y_0 + 2D_{m+1} + 2^{4(m+1)+1}x_0 + 4D_{m+1} + 1 \\
&= 2^{4(m+1)+1}x_0 + 2^{4(m+1)+1}y_0 + 6D_{m+1} + 1.
\end{aligned}$$

Hence

$$\begin{aligned}
x_{8(m+1)+4} &= x_{8m+12} \\
&= |x_{8m+11}| - y_{8m+11} - 1 \\
&= -2^{4(m+1)+1}x_0 + 2^{4(m+1)+1}y_0 - 2D_{m+1} - 2^{4(m+1)+1}x_0
\end{aligned}$$

$$
\begin{aligned}
&-2^{4(m+1)+1}y_0 - 6D_{m+1} - 2 \\
&= -2^{4(m+1)+2}x_0 - 8D_{m+1} - 2
\end{aligned}
$$

and

$$
\begin{aligned}
y_{8(m+1)+4} &= y_{8m+12} \\
&= x_{8m+11} + |y_{8m+11}| \\
&= 2^{4(m+1)+1}x_0 - 2^{4(m+1)+1}y_0 + 2D_{m+1} - 2^{4(m+1)+1}x_0 \\
&\quad -2^{4(m+1)+1}y_0 - 6D_{m+1} - 1 \\
&= -2^{4(m+1)+2}y_0 - 4D_{m+1} - 1.
\end{aligned}
$$

Thus

$$
\begin{aligned}
x_{8(m+1)+5} &= x_{8m+13} \\
&= |x_{8m+12}| - y_{8m+12} - 1 \\
&= 2^{4(m+1)+2}x_0 + 8D_{m+1} + 2 + 2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 - 1 \\
&= 2^{4(m+1)+2}x_0 + 2^{4(m+1)+2}y_0 + 12D_{m+1} + 2
\end{aligned}
$$

and

$$
\begin{aligned}
y_{8(m+1)+5} &= y_{8m+13} \\
&= x_{8m+12} + |y_{8m+12}| \\
&= -2^{4(m+1)+2}x_0 - 8D_{m+1} - 2 + 2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 \\
&= -2^{4(m+1)+2}x_0 + 2^{4(m+1)+2}y_0 - 4D_{m+1} - 1.
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_{8(m+1)+6} &= x_{8m+14} \\
&= |x_{8m+13}| - y_{8m+13} - 1 \\
&= -2^{4(m+1)+2}x_0 - 2^{4(m+1)+2}y_0 - 12D_{m+1} - 2 \\
&\quad +2^{4(m+1)+2}x_0 - 2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 - 1 \\
&= -2^{4(m+1)+3}y_0 - 8D_{m+1} - 2
\end{aligned}
$$

and

$$\begin{aligned}
y_{8(m+1)+6} &= y_{8m+14} \\
&= x_{8m+13} + |y_{8m+13}| \\
&= 2^{4(m+1)+2}x_0 + 2^{4(m+1)+2}y_0 + 12D_{m+1} + 2 + 2^{4(m+1)+2}x_0 \\
&\quad -2^{4(m+1)+2}y_0 + 4D_{m+1} + 1 \\
&= 2^{4(m+1)+3}x_0 + 16D_{m+1} + 3.
\end{aligned}$$

Then

$$\begin{aligned}
x_{8(m+1)+7} &= x_{8m+15} \\
&= |x_{8m+14}| - y_{8m+14} - 1 \\
&= 2^{4(m+1)+3}y_0 + 8D_{m+1} + 2 - 2^{4(m+1)+3}x_0 - 16D_{m+1} - 3 - 1 \\
&= -2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 - 8D_{m+1} - 2
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+7} &= y_{8m+15} \\
&= x_{8m+14} + |y_{8m+14}| \\
&= -2^{4(m+1)+3}y_0 - 8D_{m+1} - 2 - 2^{4(m+1)+3}x_0 - 16D_{m+1} - 3 \\
&= -2^{4(m+1)+3}x_0 - 2^{4(m+1)+3}y_0 - 24D_{m+1} - 5.
\end{aligned}$$

Thus

$$\begin{aligned}
x_{8(m+1)+8} &= x_{8m+16} \\
&= |x_{8m+15}| - y_{8m+15} - 1 \\
&= 2^{4(m+1)+3}x_0 - 2^{4(m+1)+3}y_0 + 8D_{m+1} + 2 \\
&\quad +2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 + 24D_{m+1} + 5 - 1 \\
&= 2^{4(m+1)+4}x_0 + 32D_{m+1} + 6
\end{aligned}$$

and

$$\begin{aligned}
y_{8(m+1)+8} &= y_{8m+16} \\
&= x_{8m+15} + |y_{8m+15}| \\
&= -2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 - 8D_{m+1} - 2 \\
&\quad +2^{4(m+1)+3}x_0 + 2^{4(m+1)+3}y_0 + 24D_{m+1} + 5 \\
&= 2^{4(m+1)+4}y_0 + 16D_{m+1} + 3
\end{aligned}$$

and so $\mathcal{P}(m+1)$ is true. Thus the proof of the Claim is complete. That is, $\mathcal{P}(m)$ is true for $0 \le m \le K-1$. In particular, $\mathcal{P}(K-1)$ is true. Thus

$$x_{8K} = x_{8(K-1)+8} = 2^{4(K-1)+4}x_0 + 32D_{K-1} + 6$$

and

$$y_{8K} = y_{8(K-1)+8} = 2^{4(K-1)+4}y_0 + 16D_{K-1} + 3.$$

Hence

$$\begin{aligned}
x_{8K+1} &= |x_{8K}| - y_{8K} - 1 \\
&= -2^{4K}x_0 - 32D_{K-1} - 6 - 2^{4K}y_0 - 16D_{K-1} - 3 - 1 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 48D_{K-1} - 10 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 48\left(\frac{2^{4(K-1)} - 1}{5}\right) - 10 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - \frac{3 \cdot 2^{4K}}{5} + \frac{3}{5} - 1 \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 3D_K - 1
\end{aligned}$$

and

$$\begin{aligned}
y_{8K+1} &= x_{8K} + |y_{8K}| \\
&= 2^{4K}x_0 + 32D_{K-1} + 6 - 2^{4K}y_0 - 16D_{K-1} - 3 \\
&= 2^{4K}x_0 - 2^{4K}y_0 + 16D_{K-1} + 3
\end{aligned}$$

$$
\begin{aligned}
&= 2^{4K}x_0 - 2^{4K}y_0 + 16\left(\frac{2^{4(K-1)}-1}{5}\right) + 3 \\
&= 2^{4K}x_0 - 2^{4K}y_0 + \frac{2^{4K}}{5} - \frac{2^4}{5} + 3 \\
&= 2^{4K}x_0 - 2^{4K}y_0 + D_K.
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_{8K+2} &= |x_{8K+1}| - y_{8K+1} - 1 \\
&= 2^{4K}x_0 + 2^{4K}y_0 + 3D_K + 1 - 2^{4K}x_0 + 2^{4K}y_0 - D_K - 1 \\
&= 2^{4K+1}y_0 + 2D_K
\end{aligned}
$$

and

$$
\begin{aligned}
y_{8K+2} &= x_{8K+1} + |y_{8K+1}| \\
&= -2^{4K}x_0 - 2^{4K}y_0 - 3D_K - 1 - 2^{4K}x_0 + 2^{4K}y_0 - D_K \\
&= -2^{4K+1}x_0 - 4D_K - 1.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
(x_0, y_0) \in \; & [a_K, b_K] \times [c_K, d_K] \setminus [a_{K+1}, b_{K+1}] \times [c_{K+1}, d_{K+1}] \\
= \; & \left[\frac{-2^{4K-2}-1}{5\cdot 2^{4K-3}}, \frac{-2^{4K}+1}{5\cdot 2^{4K-1}}\right] \times \left[\frac{-2^{4K-2}-1}{5\cdot 2^{4K-2}}, \frac{-2^{4K}+1}{5\cdot 2^{4K}}\right] \\
& \setminus \left[\frac{-2^{4(K+1)-2}-1}{5\cdot 2^{4(K+1)-3}}, \frac{-2^{4(K+1)}+1}{5\cdot 2^{4(K+1)-1}}\right] \times \left[\frac{-2^{4(K+1)-2}-1}{5\cdot 2^{4(K+1)-2}}, \frac{-2^{4(K+1)}+1}{5\cdot 2^{4(K+1)}}\right].
\end{aligned}
$$

Suppose $(x_0, y_0) \in [a_K, a_{K+1}) \times [c_K, d_K]$.

Hence

$$
\begin{aligned}
y_{8K+2} &> -2^{4K+1}(a_{K+1}) - 4D_K - 1 \\
&= -2^{4K+1}\left(\frac{-2^{4(K+1)-2}-1}{5\cdot 2^{4(K+1)-3}}\right) - 4D_K - 1 \\
&= \frac{2^{8K+3}}{5\cdot 2^{4K+1}} + \frac{2^{4(K+1)}-1}{5\cdot 2^{4(K+1)-3}} - \frac{2^{4(K+2)}}{5} + \frac{4}{5} - 1 \\
&= 0
\end{aligned}
$$

which is a contradiction.

Next suppose $(x_0, y_0) \in [a_{K+1}, b_K] \times [c_K, c_{K+1})$.

Then

$$\begin{aligned} x_{8K+3} &= |x_{8K+2}| - y_{8K+2} - 1 \\ &= -2^{4K+1}y_0 - 2D_K + 2^{4K+1}x_0 + 4D_K + 1 - 1 \\ &= 2^{4K+1}x_0 - 2^{4K+1}y_0 + 2D_K \end{aligned}$$

and

$$\begin{aligned} y_{8K+3} &= x_{8K+2} + |y_{8K+2}| \\ &= 2^{4K+1}y_0 + 2D_K + 2^{4K+1}x_0 + 4D_K + 1 \\ &= 2^{4K+1}x_0 + 2^{4K+1}y_0 + 6D_K + 1. \end{aligned}$$

Hence

$$\begin{aligned} x_{8K+4} &= |x_{8K+3}| - y_{8K+3} - 1 \\ &= -2^{4K+1}x_0 + 2^{4K+1}y_0 - 2D_K - 2^{4K+1}x_0 - 2^{4K+1}y_0 - 6D_K - 1 - 1 \\ &= -2^{4K+2}x_0 - 8D_K - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+4} &= x_{8K+3} + |y_{8K+3}| \\ &= 2^{4K+1}x_0 - 2^{4K+1}y_0 + 2D_K - 2^{4K+1}x_0 - 2^{4K+1}y_0 - 6D_K - 1 \\ &= -2^{4K+2}y_0 - 4D_K - 1. \end{aligned}$$

Recall that $(x_0, y_0) \in [a_{K+1}, b_K] \times [c_K, c_{K+1})$.

Thus

$$\begin{aligned} y_{8K+4} &> -2^{4K+2}(c_{K+1}) - 4D_K - 1 \\ &= -2^{4K+2}\left(\frac{-2^{4K+2} - 1}{5 \cdot 2^{4K+2}}\right) - 4\left(\frac{2^{4K} - 1}{5}\right) - 1 \\ &= \frac{2^{8K+4}}{5 \cdot 2^{4K+2}} + \frac{2^{4K+2}}{5 \cdot 2^{4K+2}} - \frac{2^{4K+2}}{5} + \frac{4}{5} - 1 \\ &= 0 \end{aligned}$$

which is a contradiction.

Now suppose that $(x_0, y_0) \in (b_{K+1}, b_K] \times [c_{K+1}, d_K]$.

Hence

$$\begin{aligned} x_{8K+5} &= |x_{8K+4}| - y_{8K+4} - 1 \\ &= 2^{4K+2}x_0 + 8D_K + 2 + 2^{4K+2}y_0 + 4D_K + 1 - 1 \\ &= 2^{4K+2}x_0 + 2^{4K+2}y_0 + 12D_K + 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+5} &= x_{8K+4} + |y_{8K+4}| \\ &= -2^{4K+2}x_0 - 8D_K - 2 + 2^{4K+2}y_0 + 4D_K + 1 \\ &= -2^{4K+2}x_0 + 2^{4K+2}y_0 - 4D_K - 1. \end{aligned}$$

Then

$$\begin{aligned} x_{8K+6} &= |x_{8K+5}| - y_{8K+5} - 1 \\ &= -2^{4K+2}x_0 - 2^{4K+2}y_0 - 12D_K - 2 + 2^{4K+2}x_0 - 2^{4K+2}y_0 + 4D_K + 1 - 1 \\ &= -2^{4K+3}y_0 - 8D_K - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+6} &= x_{8K+5} + |y_{8K+5}| \\ &= 2^{4K+2}x_0 + 2^{4K+2}y_0 + 12D_K + 2 + 2^{4K+2}x_0 - 2^{4K+2}y_0 + 4D_K + 1 \\ &= 2^{4K+3}x_0 + 16D_K + 3. \end{aligned}$$

Recall that $(x_0, y_0) \in (b_{K+1}, b_K] \times [c_{K+1}, d_K]$.

Thus

$$\begin{aligned} y_{8K+6} &> 2^{4K+3}\left(b_{K+1}\right) + 16\left(\frac{2^{4K}-1}{5}\right) + 3 \\ &= 2^{4K+3}\left(\frac{-2^{4(K+1)}+1}{5 \cdot 2^{4(K+1)-1}}\right) + 16\left(\frac{2^{4K}-1}{5}\right) + 3 \end{aligned}$$

$$= \frac{-2^{4K+4}}{5} + \frac{1}{5} + \frac{2^{4K+4}}{5} - \frac{16}{5} + 3$$

$$= 0$$

which is a contradiction.

Finally, suppose $(x_0, y_0) \in [a_{K+1}, b_{K+1}] \times (d_{K+1}, d_K]$.

Thus

$$\begin{aligned} x_{8K+7} &= |x_{8K+6}| - y_{8K+6} - 1 \\ &= 2^{4K+3}y_0 + 8D_K + 2 - 2^{4K+3}x_0 - 16D_K - 3 - 1 \\ &= -2^{4K+3}x_0 + 2^{4K+3}y_0 - 8D_K - 2 \end{aligned}$$

and

$$\begin{aligned} y_{8K+7} &= x_{8K+6} + |y_{8K+6}| \\ &= -2^{4K+3}y_0 - 8D_K - 2 - 2^{4K+3}x_0 - 16D_K - 3 \\ &= -2^{4K+3}x_0 - 2^{4K+3}y_0 - 24D_K - 5. \end{aligned}$$

Hence

$$\begin{aligned} x_{8K+8} &= |x_{8K+7}| - y_{8K+7} - 1 \\ &= 2^{4K+3}x_0 - 2^{4K+3}y_0 + 8D_K + 2 + 2^{4K+3}x_0 + 2^{4K+3}y_0 + 24D_K + 5 - 1 \\ &= 2^{4K+3}x_0 + 32D_K + 6 \end{aligned}$$

and

$$\begin{aligned} y_{8K+8} &= x_{8K+7} + |y_{8K+7}| \\ &= -2^{4K+3}x_0 + 2^{4K+3}y_0 - 8D_K - 2 + 2^{4K+3}x_0 + 2^{4K+3}y_0 + 24D_K + 5 \\ &= 2^{4K+4}y_0 + 16D_K + 3. \end{aligned}$$

Recall that $(x_0, y_0) \in [a_{K+1}, b_{K+1}] \times (d_{K+1}, d_K]$.

Thus

$$\begin{aligned} y_{8K+8} &> 2^{4K+4}(d_{K+1}) + 16\left(\frac{2^{4K} - 1}{5}\right) + 3 \\ &> 2^{4K+4}\left(\frac{-2^{4(K+1)} + 1}{5 \cdot 2^{4(K+1)}}\right) + 16\left(\frac{2^{4K} - 1}{5}\right) + 3 \end{aligned}$$

$$= -\frac{2^{4K+4}}{5} + \frac{1}{5} + \frac{2^{4K+4}}{5} - \frac{16}{5} + 3$$

$$= 0$$

which is a contradiction. The proof is complete.

□

2.4 References

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MANUSCRIPT 3

On the Global Behavior of $x_{n+1} = \dfrac{\alpha_1}{x_n + y_n}$ and $y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + y_n}{y_n}$

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3.1 Abstract

We investigate the system of rational difference equations in the title, where the parameters and the initial conditions are positive real numbers. We show that the system is permanent and has a unique positive equilibrium which is locally asymptotically stable. We also find sufficient conditions to insure that the unique positive equilibrium is globally asymptotically stable.

3.2 Introduction

We show that the system of rational difference equations

$$\left\{ \begin{array}{rcl} x_{n+1} & = & \dfrac{\alpha_1}{x_n + y_n} \\ \\ y_{n+1} & = & \dfrac{\alpha_2 + \beta_2 x_n + y_n}{y_n} \end{array} \right. , \quad n = 0, 1, \ldots \tag{2}$$

is permanent, where the parameters $\alpha_1, \alpha_2, \beta_2$ and the initial conditions x_0, y_0 of the system are positive real numbers. We actually show that there exist positive real numbers l_1, l_2, L_1, L_2 such that for every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system 2, we have

$$l_1 < x_n < L_1 \quad \text{and} \quad l_2 < y_n < L_2 \qquad \text{for n} \geq 3.$$

We show that the system has a unique positive equilibrium which is locally asymptotically stable. We also find sufficient conditions to insure that the unique positive equilibrium is globally asymptotically stable.

For the last four years we have been interested in the boundedness character and the global behavior of systems of rational difference equations. This paper is part of a general project which involves the system of rational difference equations

$$\left\{ \begin{array}{rcl} x_{n+1} & = & \dfrac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\ \\ y_{n+1} & = & \dfrac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{array} \right. , \quad n = 0, 1, \ldots \tag{3}$$

which includes 2401 special cases. In the numbering system which was introduced by Camouzis, Kulenović, Ladas, and Merino in ([6]), system 2 is referred to as System(12, 41). Related work has recently been given in ([1]-[11]) and ([14]-[19]).

The following well-known result is needed for the local asymptotic stability analysis of the equilibrium of System(12, 41).

Theorem 3.2.1 *Let $F = (f, g)$ be a continuously differentiable function defined on an open set W in R^2, and let $(\bar{x}, \bar{y})$ in W be a fixed point of F.*

1. *If all the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ have modulus less than one, then the equilibrium point $(\bar{x}, \bar{y})$ is locally asymptotically stable.*

2. *If at least one of the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ has modulus greater than one, then the equilibrium point $(\bar{x}, \bar{y})$ is unstable.*

The following theorem gives necessary and sufficient conditions for the two roots of a quadratic equation to have modulus less than one.

Theorem 3.2.2 *([13]) Assume p and q are real numbers. Then a necessary and sufficient condition for both roots of the equation*

$$\lambda^2 + p\lambda + q = 0$$

to have modulus less than 1 is that

$$|p| < 1 + q < 2.$$

The next theorem gives a sufficient condition to insure that there exists a unique positive equilibrium, and it is a global attractor. Let k be a positive integer. For $i \in \{1, \ldots, k\}$, assume $[a_i, b_i]$ is a closed and bounded interval, and let $F^i : [a_1, b_1] \times \ldots \times [a_k, b_k] \to [a_i, b_i]$ be a continuous function. For each $i, j \in \{1, \ldots, k\}$, let $M_{i,j} : [a_i, b_i] \to [a_i, b_i]$ and $m_{i,j} : [a_i, b_i] \to [a_i, b_i]$ be defined as follows: given $m_i, M_i \in [a_i, b_i]$

set

$$M_{i,j}(m_i, M_i) = \begin{cases} M_i, & \text{if } \mathrm{F^j} \text{ is increasing in } z_i \\ m_i, & \text{if } \mathrm{F^j} \text{ is non}-\text{increasing in } z_i \end{cases}$$

and

$$m_{i,j}(m_i, M_i) = M_{i,j}(M_i, m_i).$$

Theorem 3.2.3 *([12]) Assume that each $i \in \{1, \dots, k\}$, $[a_i, b_i]$ is a closed and bounded interval of real numbers, and the function*

$$F^i : C([a_1, b_1] \times \ldots \times [a_k, b_k], [a_i, b_i]),$$

satisfies the following conditions:

1. *$F^i(z_1, \dots, z_k)$ is weakly monotonic in each of its arguments.*

2. *If $M_1, \dots, M_k, m_1, \dots, m_k$, where $m_i \leq M_i$ for each $i \in \{1, \dots, k\}$, is a solution of the system of 2k equations:*

$$\begin{cases} M_i &= F^i(M_{1,i}(m_1, M_1), \dots, M_{k,i}(m_k, M_k)) \\ m_i &= F^i(m_{1,i}(m_1, M_1), \dots, m_{k,i}(m_k, M_k)) \end{cases}, \quad i \in \{1, \dots, k\}$$

   then

$$M_i = m_i, \text{for all } i \in \{1, \dots, k\}.$$

Then the system of k difference equations:

$$\begin{cases} x^1_{n+1} &= F^1(x^1_n, \dots, x^k_n) \\ x^2_{n+1} &= F^2(x^1_n, \dots, x^k_n) \\ &\vdots \\ x^k_{n+1} &= F^k(x^1_n, \dots, x^k_n) \end{cases}, \quad n = 0, 1, \dots (*)$$

with initial condition $(x^1_0, \dots, x^k_0) \in [a_1, b_1] \times \ldots \times [a_k, b_k]$, has exactly one equilibrium point $(\bar{x}^1, \dots, \bar{x}^k)$, and it is a global attractor.

3.3 Local Stability Analyses

Lemma 3.3.1 *System(12,41) has a unique equilibrium* $(\bar{x}, \bar{y})$*. Moreover,* $(\bar{x}, \bar{y})$ *is locally asymptotically stable.*

Proof: Suppose $(\bar{x}, \bar{y})$ is a feasible equilibrium of System(12,41). That is

$$\bar{x} = \frac{\alpha_1}{\bar{x} + \bar{y}} \qquad \text{and} \qquad \bar{y} = \frac{\alpha_2 + \beta_2\bar{x} + \bar{y}}{\bar{y}}.$$

Note that $\bar{x} < \sqrt{\alpha_1}$ and $\bar{y} = \dfrac{\alpha_1 - \bar{x}^2}{\bar{x}}$

and so

$$\frac{\alpha_1 - \bar{x}^2}{\bar{x}} = \bar{y} = \frac{\alpha_2 + \beta_2\bar{x} + \bar{y}}{\bar{y}} = \frac{\alpha_2 + \beta_2\bar{x} + \frac{\alpha_1 - \bar{x}^2}{\bar{x}}}{\frac{\alpha_1 - \bar{x}^2}{\bar{x}}}.$$

After simplifying we have

$$\alpha_2\bar{x}^2 + \beta_2\bar{x}^3 + \alpha_1\bar{x} - \bar{x}^3 - (\alpha_1 - \bar{x}^2)^2 = 0.$$

Set

$$f(x) \quad = \quad x^4 + (1 - \beta_2)x^3 - (2\alpha_1 + \alpha_2)x^2 - \alpha_1 x + \alpha_1^2. \tag{4}$$

Thus in order to show that there exists a unique equilibrium $(\bar{x}, \bar{y})$, it suffices to show $f(x) = 0$ has a unique positive solution less than $\sqrt{\alpha_1}$. By Descartes' rule of signs we know (4) has at most two positive roots. We also see that $f(0) = \alpha_1^2 > 0$ and $f(\sqrt{\alpha_1}) = -\alpha_1(\sqrt{\alpha_1}\beta_2 + \alpha_2) < 0$. Since $f(x)$ is a fourth degree polynomial with a positive leading coefficient we know that it has a minimum of two positive roots. Therefore there are exactly two positive roots; one root is less than $\sqrt{\alpha_1}$, and the other is greater than $\sqrt{\alpha_1}$. Thus the proof is complete.

We shall now investigate the linearized stability of the equilibrium $(\bar{x}, \bar{y})$ of System(12,41).

Let

$$f(x,y) = \frac{\alpha_1}{x+y} \qquad \text{and} \qquad g(x,y) = \frac{\alpha_2 + \beta_2 x + y}{y}.$$

Then

$$\mathcal{J}_{(\bar{x},\bar{y})} = \begin{pmatrix} \frac{\partial f}{\partial \bar{x}} & \frac{\partial f}{\partial \bar{y}} \\ \\ \frac{\partial g}{\partial \bar{x}} & \frac{\partial g}{\partial \bar{y}} \end{pmatrix} = \begin{pmatrix} \frac{-\alpha_1}{(\bar{x}+\bar{y})^2} & \frac{-\alpha_1}{(\bar{x}+\bar{y})^2} \\ \\ \frac{\beta_2}{\bar{y}} & \frac{-(\alpha_2+\beta_2\bar{x})}{\bar{y}^2} \end{pmatrix} = \begin{pmatrix} \frac{-\bar{x}^2}{\alpha_1} & \frac{-\bar{x}^2}{\alpha_1} \\ \\ \frac{\beta_2}{\bar{y}} & \frac{1-\bar{y}}{\bar{y}} \end{pmatrix}.$$

The characteristic equation of the linearized equation of System(12,41) about the equilibrium $(\bar{x}, \bar{y})$ is

$$\lambda^2 + \frac{\bar{x}^2\bar{y} - \alpha_1(1-\bar{y})}{\alpha_1\bar{y}}\lambda + \frac{\bar{x}^2(\bar{y}-1+\beta_2)}{\alpha_1\bar{y}} = 0.$$

By Theorem 3.2.2 we see that both roots are real and lie within the unit disk. Therefore by Theorem 3.2.1, the unique positive equilibrium $(\bar{x}, \bar{y})$ is locally asymptotically stable.

□

3.4 Permanence

We say that System(12,41) is permanent if there exists real numbers l_1, L_1, l_2, and L_2 such that for every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System(12,41), there exists an integer $N \geq 0$, such that

$$l_1 < x_n < L_1 \quad \text{and} \quad l_2 < y_n < L_2 \qquad \text{for every integer n} \geq N.$$

With this in mind, define l_1, L_1, l_2, and L_1 as follows:

1. $l_1 = \dfrac{\alpha_1}{\alpha_1 + \alpha_2 + 1 + \beta_2\alpha_1}$
2. $L_1 = \alpha_1$
3. $l_2 = 1$

4. $L_2 = \alpha_1 + 1 + \beta_2\alpha_1$

Theorem 3.4.1 *System(12,41) is permanent. In particular, let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a positive solution of System(12,41). Then for every integer* $n \geq 4$*, we have*

$$l_1 < x_n < L_1 \quad \text{and} \quad l_2 < y_n < L_2.$$

Proof: Given a non-negative integer $n \geq 0$, note that

$$x_{n+1} = \frac{\alpha_1}{x_n + y_n} \in (0, \infty)$$

and

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + y_n}{y_n} = \frac{\alpha_2 + \beta_2 x_n}{y_n} + 1 \in (1, \infty).$$

Thus $y_n > 1 = l_2$ for $n \geq 1$.

Hence if $n \geq 1$, then

$$0 < x_{n+1} = \frac{\alpha_1}{x_n + y_n} < \frac{\alpha_1}{0 + 1} = \alpha_1$$

and so $x_n < L_1$ for $n \geq 2$.

Hence if $n \geq 2$, then

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + y_n}{y_n} < \frac{\alpha_2 + \beta_2\alpha_1 + 1}{1} = L_2.$$

That is, for every integer $n \geq 3$ we have

$$l_2 < y_n < L_2.$$

If $n \geq 3$, then

$$x_{n+1} = \frac{\alpha_1}{x_n + y_n} > \frac{\alpha_1}{\alpha_1 + \alpha_2 + \beta_2\alpha_1 + 1} = l_1.$$

That is, for every integer $n \geq 4$ we have

$$l_1 < x_n < L_1$$

and the proof is complete. □

3.5 Global Attractivity Analysis

The following theorem gives a sufficient condition for the unique equilibrium of System(12,41) to be globally asymptotically stable.

Theorem 3.5.1 *Suppose that either*

$$0 < \alpha_2 \leq \frac{\alpha_1 \beta_2^2}{1+\beta_2} - 2\sqrt{\frac{\alpha_1 \beta_2^2}{1+\beta_2}}$$

or

$$\frac{\alpha_1 \beta_2^2}{1+\beta_2} \leq \alpha_2.$$

Then the unique equilibrium point $(\bar{x}, \bar{y})$ is globally asymptotically stable.

Proof: The proof will be by Theorem 3.2.3. For $(x, y) \in [0, \infty) \times (0, \infty)$, set

$$f(x,y) = \frac{\alpha_1}{x+y} \qquad \text{and} \qquad g(x,y) = \frac{\alpha_2 + \beta_2 x + y}{y}$$

and let $\mathcal{R} = [a,b] \times [c,d] = [0, \alpha_1] \times [1, \alpha_2 + \beta_2 \alpha_1 + 1]$.

Let $T : [0, \infty) \times (0, \infty) \to (0, \infty) \times (0, \infty)$ be given by $T(x, y) : (f(x, y), g(x, y))$.

We shall first show that $T[\mathcal{R}] \subset \mathcal{R}$. Suppose $(x, y) \in \mathcal{R}$. It suffices to show that

$$f(x,y) \in [a,b] \qquad \text{and} \qquad g(x,y) \in [c,d].$$

1. We shall first show that $a < f(x,y)$.

   Note that
$$a = 0 < \frac{\alpha_1}{x+y} = f(x,y).$$

2. We shall next show that $f(x,y) \leq b$.

   We have
$$f(x,y) = \frac{\alpha_1}{x+y} \leq \frac{\alpha_1}{a+c} = \frac{\alpha_1}{0+1} = \alpha_1 = b.$$

3. We shall next show that $c < g(x,y)$.
$$c = 1 < \frac{\alpha_2 + \beta_2 x}{y} + 1 = \frac{\alpha_2 + \beta_2 x + y}{y} = g(x,y).$$

4. Finally, we shall show that $g(x,y) \leq d$.

   Now
$$g(x,y) = \frac{\alpha_2 + \beta_2 x + y}{y} \leq \frac{\alpha_2 + \beta_2 b + 1}{1} = \alpha_2 + \beta_2 \alpha_1 + 1 = d.$$

Thus $T[\mathcal{R}] \subset \mathcal{R}$.

Clearly f is strictly decreasing in x and strictly decreasing in y, and g is strictly increasing in x and strictly decreasing in y. So to apply Theorem 3.2.3, suppose $(m_1, M_1, m_2, M_2) \in [0, \alpha_1]^2 \times [1, \alpha_2 + 1 + \beta_2\alpha_1]^2$ is a solution of the system of equations
$$\begin{cases} m_1 = \dfrac{\alpha_1}{M_1 + M_2} & , \quad M_1 = \dfrac{\alpha_1}{m_1 + m_2} \\[2ex] m_2 = \dfrac{\alpha_2 + \beta_2 m_1 + M_2}{M_2} & , \quad M_2 = \dfrac{\alpha_2 + \beta_2 M_1 + m_2}{m_2} \end{cases}$$
with
$$0 \leq m_1 \leq M_1 \leq \alpha_1 \qquad \text{and} \qquad 1 \leq m_2 \leq M_2 \leq \alpha_2 + 1 + \beta_2\alpha_1.$$

It suffices to show that

$$m_1 = M_1 \qquad \text{and} \qquad m_2 = M_2.$$

For the sake of contradiction, suppose that this is not the case.

Now

$$m_1 M_1 + m_1 M_2 = \alpha_1 = M_1 m_1 + M_1 m_2$$

and so $m_1 M_2 = M_1 m_2$. Since $m_1 = \dfrac{\alpha_1}{M_1 + M_2}$, we see m_1 is positive, and so as $m_1 M_2 = M_1 m_2$, we have

$$0 < m_1 < M_1 \qquad \text{and} \qquad 1 < m_2 < M_2.$$

Hence

$$M_2 = \frac{m_2}{m_1} M_1.$$

We also have

$$\alpha_2 + \beta_2 m_1 + M_2 = m_2 M_2 = \alpha_2 + \beta_2 M_1 + m_2.$$

Therefore $\beta_2 m_1 + M_2 = \beta_2 M_1 + m_2$, and hence

$$M_2 - m_2 = \beta_2 M_1 - \beta_2 m_1.$$

Thus

$$\beta_2 (M_1 - m_1) = M_2 - m_2 = \frac{m_2}{m_1} M_1 - m_2 = \frac{m_2}{m_1}(M_1 - m_1).$$

So as $M_1 \neq m_1$, we have

$$\beta_2 = \frac{m_2}{m_1} \neq 0.$$

That is,

$$m_2 = \beta_2 m_1 \qquad \text{and} \qquad M_2 = \beta_2 M_1.$$

Recall that

$$m_1 = \frac{\alpha_1}{M_1 + M_2} = \frac{\alpha_1}{M_1 + \beta_2 M_1} = \frac{\alpha_1}{(1 + \beta_2) M_1}$$

and so

$$m_1 M_1 = \frac{\alpha_1}{1+\beta_2}.$$

Thus

1. $M_1 = \frac{\alpha_1}{1+\beta_2} \cdot \frac{1}{m_1}$.

2. $m_2 = \beta_2 m_1$.

3. $M_2 = \beta_2 M_1 = \frac{\alpha_1 \beta_2}{1+\beta_2} \cdot \frac{1}{m_1}$.

In particular, since $m_2 = \beta_2 m_1$, we see that

$$\frac{1}{\beta_2} m_2 M_2 = m_1 M_2 = \frac{\alpha_1 \beta_2}{1+\beta_2}$$

and so

$$m_2 M_2 = \frac{\alpha_1 \beta_2^2}{1+\beta_2}.$$

Thus

$$\begin{aligned} \frac{\alpha_1 \beta_2^2}{1+\beta_2} &= m_2 M_2 \ = \ \alpha_2 + \beta_2 m_1 + M_2 \\ &= \alpha_2 + \beta_2 m_1 + \beta_2 M_1 \\ &= \alpha_2 + \beta_2 m_1 + \frac{\alpha_1 \beta_2}{1+\beta_2} \cdot \frac{1}{m_1} \end{aligned}$$

and so

$$0 = \beta_2 m_1^2 + \left(\alpha_2 - \frac{\alpha_1 \beta_2^2}{1+\beta_2} \right) m_1 + \frac{\alpha_1 \beta_2}{1+\beta_2}.$$

We also have

$$\begin{aligned} \frac{\alpha_1 \beta_2^2}{1+\beta_2} &= m_2 M_2 \ = \ \alpha_2 + \beta_2 M_1 + m_2 \ = \ \alpha_2 + \beta_2 M_1 + \beta_2 m_1 \\ &= \alpha_2 + \beta_2 M_1 + \frac{\alpha_1 \beta_2}{1+\beta_2} \cdot \frac{1}{M_1} \end{aligned}$$

and thus

$$0 = \beta_2 M_1^2 + \left(\alpha_2 - \frac{\alpha_1 \beta_2^2}{1+\beta_2} \right) M_1 + \frac{\alpha_1 \beta_2}{1+\beta_2}.$$

That is, m_1 and M_1 are the two distinct roots of the quadratic equation

$$\beta_2 z^2 + \left(\alpha_2 - \frac{\alpha_1\beta_2^2}{1+\beta_2}\right) z + \frac{\alpha_1\beta_2}{1+\beta_2} = 0.$$

Hence

$$0 < m_1 = \frac{\left(\frac{\alpha_1\beta_2^2}{1+\beta_2} - \alpha_2\right) - \sqrt{\left(\alpha_2 - \frac{\alpha_1\beta_2^2}{1+\beta_2}\right)^2 - \frac{4\alpha_1\beta_2^2}{1+\beta_2}}}{2\beta_2}$$

and

$$m_1 < M_1 = \frac{\left(\frac{\alpha_1\beta_2^2}{1+\beta_2} - \alpha_2\right) + \sqrt{\left(\alpha_2 - \frac{\alpha_1\beta_2^2}{1+\beta_2}\right)^2 - \frac{4\alpha_1\beta_2^2}{1+\beta_2}}}{2\beta_2}.$$

So by our hypothesis this is a contradiction, and the proof of the theorem is complete.

□

Extensive computer simulations lead us to the following conjecture:

Conjecture 3.5.1 *The unique positive equilibrium of System(12,41) is globally asymptotically stable for the entire range of the parameters.*

3.6 References

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Appendix .1

.1 On the Global Behavior of $x_{n+1} = |x_n| + ay_n + b$ and $y_{n+1} = x_n + c|y_n| + d$

To be submitted as a monograph.

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.1.1 General Theorems for the 81 Systems of Piecewise Linear Difference Equations

The following six theorems generalizes the global behavior of 75 of the 81 piecewise systems.

Consider the set of systems

$$\begin{cases} x_{n+1} &= |x_n| - y_n + b \\ \\ y_{n+1} &= x_n + c|y_n| + d \end{cases}, \quad n = 0, 1, \ldots \qquad (I)$$

where $(x_0, y_0) \in R^2$ and

$\{b = -1 \text{ and } c = d = 1\}$

or $\{b = 0 \text{ and } [(c = 1 \text{ and } d \in \{0, 1\}) \text{ or } (c = -1 \text{ and } d \in \{-1, 0\})]\}$

or $\{b = 1 \text{ and } [(c = -1) \text{ or } (c = 1 \text{ and } d \in \{0, 1\})]\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set I with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually a unique equilibrium.

This set of systems are Systems(9-11, 17-21, 26, 27).

....

Consider the set of systems

$$\begin{cases} x_{n+1} &= |x_n| \\ \\ y_{n+1} &= x_n + d \end{cases}, \quad n = 0, 1, \ldots \qquad (Ia)$$

where $(x_0, y_0) \in R^2$ and $d \in \{-1, 0, 1\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set Ia with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually an equilibrium.

This set of systems are Systems(40-42).

....

Consider the set of systems

$$\begin{cases} x_{n+1} &= |x_n| + y_n + b \\ y_{n+1} &= x_n + c|y_n| + d \end{cases}, \quad n = 0, 1, \ldots \qquad (II)$$

where $(x_0, y_0) \in R^2$ and $b, c, d \in \{-1, 0, 1\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set II. Then there exists initial conditions $(x_0, y_0) \in \mathbf{R}^2$ such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness characteristic (U,U). In particular, $\{(x_n, y_n)\}_{n=0}^{\infty}$ increases without bound.

This set of systems are Systems(55-81). Note that of these 27 systems, 19 have one to three equilibrium points and 5 have period-2 solutions.

....

Consider the set of systems

$$\begin{cases} x_{n+1} &= |x_n| + 1 \\ y_{n+1} &= x_n + c|y_n| + d \end{cases}, \quad n = 0, 1, \ldots \qquad (III)$$

where $(x_0, y_0) \in R^2$ and $c, d \in \{-1, 0, 1\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set III with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness characteristic $(\widetilde{U}, \widetilde{U})$. In particular, $\{(x_n, y_n)\}_{n=0}^{\infty}$ increases without bound.

This set of systems are Systems(46-54).

....

Consider the set of systems

$$\left\{ \begin{array}{lcl} x_{n+1} & = & |x_n| - y_n + b \\ \\ y_{n+1} & = & x_n + c|y_n| + d \end{array} \right. , \quad n = 0, 1, \ldots \qquad (IV)$$

where $(x_0, y_0) \in R^2$ and

$\{b = -1$ and $(c = -1$ or $d = -1$ or $c = d = 1)\}$

or $\{b = 0$ and $c + d = 0\}$

or $\{b = 1$ and $[(c = -1)$ or $(c = 1$ and $d = -1)$ or $(c = 0$ and $d = 1)\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set IV with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a unique equilibrium solution or periodic with period greater than two.

This set of systems are Systems(1-4, 7, 8, 12, 14-16, 24, 25).

....

Consider the set of systems

$$\left\{ \begin{array}{lcl} x_{n+1} & = & |x_n| + a \\ \\ y_{n+1} & = & x_n + c|y_n| + d \end{array} \right. , \quad n = 0, 1, \ldots \qquad (V)$$

where $(x_0, y_0) \in R^2$ and

$\{a = -1$ and $[(c = -1$ and $d = 1)$ or $(c = 0)$ or $(c = 1$ and $d \in \{-1, 0\})]\}$

or $\{a = 0$ and $c = -1$ and $d \in \{0, 1\}\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set V with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a unique equilibrium solution or periodic with period-2 or period-4.

This set of systems are Systems(30-35, 38, 39). Note that only Systems(30, 34, 35) exhibit prime period-4 solutions.

....

Consider the set of systems

$$\begin{cases} x_{n+1} = |x_n| + a \\ y_{n+1} = x_n + c|y_n| + d \end{cases}, \quad n = 0, 1, \ldots \qquad (VI)$$

where $(x_0, y_0) \in R^2$ and

$\{a = -1$ and $[(c = -1$ and $d \in \{-1, 0\})$ or $(c = d = 1)\}$

or $\{a = 0$ and $[(c = d = -1)$ or $(c = 1)\}$.

Theorem: Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of a system from set VI with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (B,U).

This set of systems are Systems(28, 29, 36, 37, 43-45). Note: all systems except 44 and 45 exhibit period-2 solutions and Systems(37, 43, 44) have equilibrium points.

.1.2 Systems(10 and 26)

In this section I consider System(10) and System(26). I will begin with System(26):

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ... \tag{26}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers. The theorem that follows gives the global behavior of this system.

Theorem .1.1 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(26) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the unique equilibrium $(\bar{x}, \bar{y}) = (0, 1)$.*

The change of variables, $x_n = -Y_n$ and $y_n = X_n$, reduces the system to

$$\begin{cases} X_{n+1} = |X_n| - Y_n \\ \\ Y_{n+1} = X_n - |Y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{10}$$

which is System(10). The global behavior of System(10) follows.

System(10)

I will now consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{10}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers. This is System(10).

I show that every solution of System(10) is the unique equilibrium $(\bar{x}, \bar{y}) = (1, 0)$.

Global Results

Set

$$
\begin{aligned}
Q_1 &= \{(x,y) : x \geq 0, y \geq 0\} \\
Q_2 &= \{(x,y) : x < 0, y > 0\} \\
Q_3 &= \{(x,y) : x \leq 0, y \leq 0\} \\
Q_4 &= \{(x,y) : x > 0, y < 0\}.
\end{aligned}
$$

Theorem .1.2 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(10) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(\bar{x}, \bar{y}) = (1, 0)$*.*

The proof of Theorem .1.2 is a direct consequence of the following lemmas.

Lemma .1.3 *For a non-negative integer* $N \geq 0$, $(x_{N+1}, y_{N+1}) = (\bar{x}, \bar{y})$ *if and only if* $1 = |x_N| - y_N$ *and* $1 = x_N - |y_N|$.

Proof: We have

$$(x_{N+1}, y_{N+1}) = (\bar{x}, \bar{y})$$

if and only if

$$(x_{N+1}, y_{N+1}) = (1, 0)$$

if and only if

$$1 = |x_N| - y_N \text{ and } 0 = x_N - |y_N| - 1$$

if and only if

$$1 = |x_N| - y_N \text{ and } 1 = x_N - |y_N|.$$

□

Lemma .1.4 *Suppose there exists a non-negative integer* $N \geq 0$*, such that* $(x_N, y_N) \in Q_1$*. Then either* $(x_{N+2}, y_{N+2}) = (\bar{x}, \bar{y})$ *or* $(x_{N+2}, y_{N+2}) \in Q_4$*.*

Proof: We have a non-negative integer $N \geq 0$, such that $(x_N, y_N) \in Q_1$. Then

Case 1: Suppose $x_N \geq y_N$, then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N &&= x_N - y_N &&\geq 0 \\ y_{N+1} &= x_N - |y_N| - 1 &&= x_N - y_N - 1. \end{aligned}$$

Suppose $y_{N+1} = x_N - y_N - 1 \geq 0$, then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} &&= x_N - y_N - (x_N - y_N - 1) &&= 1 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &&= x_N - y_N - (x_N - y_N - 1) - 1 &&= 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) = (\bar{x}, \bar{y})$.

Suppose $y_{N+1} = x_N - y_N - 1 < 0$, then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} &&= x_N - y_N - (x_N - y_N - 1) &&= 1 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &&= 2x_N - 2y_N - 2 &&< 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$.

Case 2: Suppose $x_N < y_N$, then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N &&= x_N - y_N &&< 0 \\ y_{N+1} &= x_N - |y_N| - 1 &&= x_N - y_N - 1 &&< 0 \\ x_{N+2} &= |x_{N+1}| - y_{N+1} &&= -2x_N + 2y_N + 1 &&> 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &&= 2x_N - 2y_N - 2 &&< 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$. The proof is complete. □

Lemma .1.5 *Suppose there exists a non-negative integer* $N \geq 0$, *such that* $(x_N, y_N) \in Q_2$. *Then* $(x_{N+2}, y_{N+2}) \in Q_4$.

Proof: We have a non-negative integer $N \geq 0$, such that $(x_N, y_N) \in Q_2$.

Case 1: Suppose $-x_N \geq y_N$, then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N &&= -x_N - y_N &&\geq 0 \\ y_{N+1} &= x_N - |y_N| - 1 &&= -x_N - y_N - 1 &&< 0 \\ x_{N+2} &= |x_{N+1}| - y_{N+1} &&= -2x_N + 1 &&> 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &&= -2y_N - 2 &&< 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$.

Case 2: Suppose $-x_N < y_N$, then

$$\begin{array}{rclclcl} x_{N+1} & = & |x_N| - y_N & = & -x_N - y_N & < & 0 \\ y_{N+1} & = & x_N - |y_N| - 1 & = & -x_N - y_N - 1 & < & 0 \\ x_{N+2} & = & |x_{N+1}| - y_{N+1} & = & 2y_N + 1 & > & 0 \\ y_{N+2} & = & x_{N+1} - |y_{N+1}| - 1 & = & -2y_N - 2 & < & 0 \end{array}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$. The proof is complete. □

Lemma .1.6 *Suppose there exists a non-negative integer* $N \geq 0$*, such that* $(x_N, y_N) \in Q_3$*. Then* $(x_{N+2}, y_{N+2}) \in Q_4$.

Proof: We have

$$\begin{array}{rclclcl} x_{N+1} & = & |x_N| - y_N & = & -x_N - y_N & \geq & 0 \\ y_{N+1} & = & x_N - |y_N| - 1 & = & x_N + y_N - 1 & < & 0 \\ x_{N+2} & = & |x_{N+1}| - y_{N+1} & = & -2x_N - 2y_N + 1 & > & 0 \\ y_{N+2} & = & x_{N+1} - |y_{N+1}| - 1 & = & -2 & < & 0 \end{array}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$. The proof is complete. □

Lemma .1.7 *Suppose there exists a non-negative integer* $N \geq 0$*, such that* $(x_N, y_N) \in Q_4$*. Then* $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ *is the unique equilibrium* $(\bar{x}, \bar{y})$.

Proof: We have

$$\begin{array}{rclclcl} x_{N+1} & = & |x_N| - y_N & = & x_N - y_N & > & 0 \\ y_{N+1} & = & x_N - |y_N| - 1 & = & x_N + y_N - 1 & & \end{array}$$

Case 1: Suppose $y_{N+1} = x_N + y_N - 1 \geq 0$ then

$$\begin{array}{rcccccc} x_{N+2} & = & |x_{N+1}| - y_{N+1} & = & -2y_N + 1 & \geq & 0 \\ y_{N+2} & = & x_{N+1} - |y_{N+1}| - 1 & = & -2y_N & > & 0 \\ x_{N+3} & = & |x_{N+2}| - y_{N+2} & = & 1 & & \\ y_{N+3} & = & x_{N+2} - |y_{N+2}| - 1 & = & 0 & & \end{array}$$

and so $(x_{N+3}, y_{N+3}) = (\bar{x}, \bar{y})$.

Case 2: Suppose $y_{N+1} = x_N + y_N - 1 < 0$ then

$$\begin{array}{rcccccc} x_{N+2} & = & |x_{N+1}| - y_{N+1} & = & -2y_N + 1 & \geq & 0 \\ y_{N+2} & = & x_{N+1} - |y_{N+1}| - 1 & = & 2x_N - 2. & & \end{array}$$

Subcase 2a: Suppose $y_{N+2} = 2x_N - 2 \geq 0$ then

$$\begin{array}{rcccccc} x_{N+3} & = & |x_{N+1}| - y_{N+1} & = & -2x_N - 2y_N + 3 & > & 0 \\ y_{N+3} & = & x_{N+1} - |y_{N+1}| - 1 & = & -2x_N - 2y_N + 2 & < & 0 \end{array}$$

and so it follows by Lemma .1.3 that $(x_{N+4}, y_{N+4}) = (\bar{x}, \bar{y})$.

Subcase 2b: Suppose $y_{N+2} = 2x_N - 2 < 0$ then

$$\begin{array}{rcccccc} x_{N+3} & = & |x_{N+1}| - y_{N+1} & = & -2x_N - 2y_N + 3 & > & 0 \\ y_{N+3} & = & x_{N+1} - |y_{N+1}| - 1 & = & 2x_N - 2y_N - 2. & & \end{array}$$

Subcase 2bi: Suppose $y_{N+3} = 2x_N - 2y_N - 2 \geq 0$ then

$$\begin{array}{rcccccc} x_{N+4} & = & |x_{N+3}| - y_{N+3} & = & -4x_N + 5 & > & 0 \\ y_{N+4} & = & x_{N+3} - |y_{N+3}| - 1 & = & -4x_N + 4 & > & 0 \end{array}$$

and so it follows by Lemma .1.3 that $(x_{N+5}, y_{N+5}) = (\bar{x}, \bar{y})$.

Subcase 2bii: Suppose $y_{N+3} = 2x_N - 2y_N - 2 < 0$ then

$$\begin{array}{rcccccc} x_{N+4} & = & |x_{N+3}| - y_{N+3} & = & -4x_N + 5 & > & 0 \\ y_{N+4} & = & x_{N+3} - |y_{N+3}| - 1 & = & -4y_N & > & 0 \\ x_{N+5} & = & |x_{N+4}| - y_{N+4} & = & -4x_N + 4y_N + 5 & > & 0 \\ y_{N+5} & = & x_{N+4} - |y_{N+4}| - 1 & = & -4x_N + 4y_N + 4 & > & 0 \end{array}$$

and so it follows by Lemma .1.3 that $(x_{N+6}, y_{N+6}) = (\bar{x}, \bar{y})$.

The proof is complete. $\square$

.1.3 System(12)

I next consider the system of piecewise linear difference equations

$$\left\{\begin{array}{l} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{array}\right. , \qquad n = 0, 1, ... \tag{12}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.
It has the unique equilibrium point $\left(-\frac{1}{5}, \frac{2}{5}\right)$ and the following two prime period-3 solutions:

$$\mathbf{P}_3^1 = \begin{pmatrix} x_0 & = & -1 & , & y_0 & = & 0 \\ \\ x_1 & = & 1 & , & y_1 & = & 0 \\ \\ x_2 & = & 1 & , & y_2 & = & 2 \end{pmatrix} \quad \text{or} \quad \mathbf{P}_3^2 = \begin{pmatrix} x_0 & = & -\frac{1}{3} & , & y_0 & = & 0 \\ \\ x_1 & = & \frac{1}{3} & , & y_1 & = & \frac{2}{3} \\ \\ x_2 & = & -\frac{1}{3} & , & y_2 & = & \frac{2}{3} \end{pmatrix}.$$

Theorem .1.8 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(12) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(12) is either the prime period-3 cycle $\mathbf{P}_3^1$ or the prime period-3 cycle $\mathbf{P}_3^2$.*

The change of variables, $x_n = Y_n$ and $y_n = -X_n$, reduces the system to

$$\left\{\begin{array}{l} X_{n+1} = |X_n| - Y_n - 1 \\ \\ Y_{n+1} = X_n + |Y_n| \end{array}\right. , \qquad n = 0, 1, ... \tag{8}$$

which is System(8). See Theorem 2.3.1.

.1.4 Systems(19 and 27)

In this section I consider System(19) and System(27). I will begin with System(27):

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ... \tag{27}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers. The theorem that follows gives the global behavior.

Theorem .1.9 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(27) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the unique equilibrium $(\bar{x}, \bar{y}) = (-1, 3)$.*

The change of variables, $x_n = -Y_n$ and $y_n = X_n$, reduces the system to

$$\begin{cases} X_{n+1} = |X_n| - Y_n + 1 \\ \\ Y_{n+1} = X_n - |Y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{19}$$

which is System(19). The global behavior of System(19) follows.

System(19)

I will now consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{19}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that every solution of System(19) is the unique equilibrium solution $(\bar{x}, \bar{y}) = (3, 1)$.

Global Results

Set

$$\begin{aligned} Q_1 &= \{(x, y) : x \geq 0, y \geq 0\} \\ Q_2 &= \{(x, y) : x < 0, y > 0\} \\ Q_3 &= \{(x, y) : x \leq 0, y \leq 0\} \\ Q_4 &= \{(x, y) : x > 0, y < 0\}. \end{aligned}$$

Theorem .1.10 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(19) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the unique equilibrium $(\bar{x}, \bar{y}) = (3, 1)$.*

The proof of this theorem is a direct consequence of the following lemmas.

Lemma .1.11 *For $N \geq 0$, $(x_{N+1}, y_{N+1}) = (\bar{x}, \bar{y})$ if and only if $|x_N| - y_N = 2$ and $x_N - |y_N| = 2$.*

Proof: We have

$$(x_{N+1}, y_{N+1}) = (\bar{x}, \bar{y})$$
$$\text{if and only if}$$
$$(x_{N+1}, y_{N+1}) = (3, 1)$$
$$\text{if and only if}$$
$$3 = |x_N| - y_N + 1 \text{ and } 1 = x_N - |y_N| - 1$$
$$\text{if and only if}$$
$$|x_N| - y_N = 2 \text{ and } x_N - |y_N| = 2.$$

In particular, $x_N \geq 0$, $y_N \geq 0$ and $x_N - y_N = 2$. □

Lemma .1.12 *Suppose there exists a non-negative integer $N \geq 0$, such that $(x_N, y_N) \in Q_1$. Then either $(x_{N+4}, y_{N+4}) = (\bar{x}, \bar{y})$ or $(x_{N+2}, y_{N+2}) \in Q_4$.*

Proof: Suppose $(x_N, y_N) \in Q_1$. Then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N + 1 = x_N - y_N + 1 \\ y_{N+1} &= x_N - |y_N| - 1 = x_N - y_N - 1. \end{aligned}$$

Case 1: Suppose $y_{N+1} \geq 0$, then $x_{N+1} > 0$ and so it follows by Lemma .1.11 that $(x_{N+2}, y_{N+2}) = (\bar{x}, \bar{y})$.

Case 2: Suppose $x_{N+1} \leq 0$, then $y_{N+1} < 0$. Hence

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} + 1 = -2x_N + 2y_N + 1 > 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 = 2x_N - 2y_N - 1 < 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$.

Case 3: Suppose $x_{N+1} < 0$ and $y_{N+1} > 0$. Then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} + 1 &=& -2x_N + 2y_N + 1 &>& 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &=& 1 \\ x_{N+3} &= |x_{N+2}| - y_{N+2} + 1 &=& -2x_N + 2y_N + 1 &>& 0 \\ y_{N+3} &= x_{N+2} - |y_{N+2}| - 1 &=& -2x_N + 2y_N - 1 &>& 0 \end{aligned}$$

and so it follows by Lemma .1.11 that $(x_{N+4}, y_{N+4}) = (\bar{x}, \bar{y})$. □

Lemma .1.13 *Suppose there exists a non-negative integer $N \geq 0$, such that $(x_N, y_N) \in Q_2$. Then $(x_{N+2}, y_{N+2}) \in Q_4$.*

Proof: Suppose $(x_N, y_N) \in Q_2$. Then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N + 1 &=& -x_N - y_N + 1 \\ y_{N+1} &= x_N - |y_N| - 1 &=& x_N - y_N - 1 &<& 0. \end{aligned}$$

Case 1: Suppose $x_{N+1} \geq 0$. Then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} + 1 &=& -2x_N + 3 &>& 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &=& -2y_N - 1 &<& 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$.

Case 2: Suppose $x_{N+1} < 0$. Then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} + 1 &=& 2y_N + 1 &>& 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &=& -2y_N - 1 &<& 0 \end{aligned}$$

and so $(x_{N+2}, y_{N+2}) \in Q_4$. □

Lemma .1.14 *Suppose there exists a non-negative integer $N \geq 0$, such that $(x_N, y_N) \in Q_3$. Then $(x_{N+1}, y_{N+1}) \in Q_4$.*

Proof: Suppose $(x_N, y_N) \in Q_3$. Then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N + 1 &= -x_N - y_N + 1 &\geq 0 \\ y_{N+1} &= x_N - |y_N| - 1 &= x_N + y_N - 1 &\leq 0 \end{aligned}$$

and so $(x_{N+1}, y_{N+1}) \in Q_4$. □

Lemma .1.15 *Suppose there exists a non-negative integer $N \geq 0$, such that $(x_N, y_N) \in Q_4$. Then $\{(x_n, y_n)\}_{n=N+5}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$.*

Proof: Suppose $(x_N, y_N) \in Q_4$. Then

$$\begin{aligned} x_{N+1} &= |x_N| - y_N + 1 &= x_N - y_N + 1 &> 0 \\ y_{N+1} &= x_N - |y_N| - 1 &= x_N + y_N - 1. \end{aligned}$$

Case 1: Suppose $y_{N+1} \geq 0$. Then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} + 1 &= -2y_N + 3 &> 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &= -2y_N + 1 &> 0 \end{aligned}$$

and so it follows by Lemma .1.11 that $(x_{N+3}, y_{N+3}) = (\bar{x}, \bar{y})$.

Case 2: Suppose $y_{N+1} \geq 0$. Then

$$\begin{aligned} x_{N+2} &= |x_{N+1}| - y_{N+1} + 1 &= -2y_N + 3 &> 0 \\ y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 &= 2x_N - 1. \end{aligned}$$

Subcase 2a: Suppose $y_{N+2} \geq 0$. Then

$$\begin{aligned} x_{N+3} &= |x_{N+2}| - y_{N+2} + 1 &= -2x_N - 2y_N + 5 &> 0 \\ y_{N+3} &= x_{N+2} - |y_{N+2}| - 1 &= -2x_N - 2y_N + 3 &> 0 \end{aligned}$$

and so it follows by Lemma .1.11 that $(x_{N+4}, y_{N+4}) = (\bar{x}, \bar{y})$.

Subcase 2b: Suppose $y_{N+2} < 0$. Then

$$\begin{aligned} x_{N+3} &= |x_{N+2}| - y_{N+2} + 1 &= -2x_N - 2y_N + 5 &> 0 \\ y_{N+3} &= x_{N+2} - |y_{N+2}| - 1 &= 2x_N - 2y_N + 1 &> 0 \\ x_{N+4} &= |x_{N+3}| - y_{N+3} + 1 &= -4x_N + 5 &> 0 \\ y_{N+4} &= x_{N+3} - |y_{N+3}| - 1 &= -4x_N + 3 &> 0 \end{aligned}$$

and so it follows by Lemma .1.11 that $(x_{N+5}, y_{N+5}) = (\bar{x}, \bar{y})$. □

.1.5 Systems(28 - 33)

All six systems share the same first difference equation, $x_{n+1} = |x_n| - 1$. The following lemma gives the global behavior of $\{x_n\}_{n=0}^{\infty}$ for Systems(28 - 33).

Lemma .1.16 *Every solution of $x_{n+1} = |x_n| - 1$ is eventually periodic with period-2 and there exists prime period-2 solutions. In fact, let $\{x_n\}_{n=0}^{\infty}$ be a real solution of $x_{n+1} = |x_n| - 1$ and write $|x_0| = m + \alpha$ where $m \in \{0, 1, 2, ...\}$, and $0 \leq \alpha < 1$ where $\alpha \in \mathbf{R}$. Then the closed form is*

$$x_j = \begin{cases} x_0 & if \quad j = 0 \\ |x_0| - j & if \quad 0 < j \leq m \\ \alpha - 1 & if \quad j = m + 2n - 1 \quad for \quad n \in \mathbf{N} \\ -\alpha & if \quad j = m + 2n \quad for \quad n \in \mathbf{N} \end{cases}$$

In particular, $-1 \leq x_k \leq 0$ for any natural number k, where $k > m$. The proof is by computations and will be omitted.

System(28)

I first consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{28}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that the boundedness character of System(28) is (B,U).

Global Results

Theorem .1.17 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(28) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (B,U), moreover $\{x_n\}_{n=0}^{\infty}$ is eventually period-2 and $\{y_n\}_{n=0}^{\infty}$ is decreasing without bound.*

Proof: By Lemma .1.16, $\{x_n\}_{n=m+1}^{\infty}$, is period-2 and $-1 \leq x_j \leq 0$ for all $j > m$. Set $k = m + 1$ and suppose $y_k \in \mathbf{R}$. Then for any non-negative integer $n \geq 0$

$$y_{k+n+1} \quad = \quad x_{k+n} - |y_{k+n}| - 1 \quad < \quad y_{k+n}.$$

Therefore $\{y_n\}_{n=m+1}^{\infty}$ is decreasing without bound.

□

System(29)

In this section I next consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ... \tag{29}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that the boundedness character of System(29) is (B,U).

Global Results

Theorem .1.18 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(29) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (B,U), moreover $\{x_n\}_{n=0}^{\infty}$ is eventually period-2 and $\{y_n\}_{n=0}^{\infty}$ is decreasing without bound.*

Proof: By Lemma .1.16, $\{x_n\}_{n=m+1}^{\infty}$, is period-2 and $-1 \leq x_j \leq 0$ for all $j > m$. Set $k = m + 1$ and suppose $y_k \in \mathbf{R}$. Then for any non-negative integer $n \geq 0$

$$y_{k+n+1} \;=\; x_{k+n} - |y_{k+n}| \;<\; y_{k+n}.$$

Therefore $\{y_n\}_{n=m+1}^{\infty}$ is decreasing without bound. □

System(30)

I next consider System(30)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ... \tag{30}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that every solution is either the unique equilibrium $\left(-\frac{1}{2}, \frac{1}{4}\right)$, or periodic with (not necessarily prime) period-4.

Global Results

Theorem .1.19 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(30) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the equilibrium solution $\left(-\frac{1}{2}, \frac{1}{4}\right)$, periodic with prime period-2 or periodic with prime period-4.*

Proof: By Lemma .1.16 we know $\{(x_n)\}_{n=m+1}^{\infty}$ is period-2. Set $M = m+1$ for the remainder of this proof. Also recall that $-1 \leq x_n \leq 0$ for any natural number n, where $n \geq M$.

Lemma .1.20 *Suppose there exists a non-negative natural number N such that $N \geq M$, $x_N = -\frac{1}{2}$ and $y_N \in \mathbf{R}$. Then $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is eventually a period-2 solution.*

Proof: Suppose $x_N = -\frac{1}{2}$. Thus, by Lemma .1.16, $x_{N+k} = -\dfrac{1}{2}$ for all $k \in \mathbf{N}$. Suppose further that $|y_N| \leq \frac{1}{2}$. Then

$$\begin{aligned} y_{N+1} &= x_N - |y_N| + 1 &&= \tfrac{1}{2} - |y_N| \\ y_{N+2} &= x_{N+1} - |y_{N+1}| + 1 &&= -\tfrac{1}{2} - (\tfrac{1}{2} - |y_N|) + 1 = |y_N| \\ y_{N+3} &= x_{N+2} - |y_{N+2}| + 1 &&= \tfrac{1}{2} - |y_N| = y_{N+1} \end{aligned}$$

and so $\{y_n\}_{n=N+1}^{\infty}$ is periodic with period-2.

Now suppose $|y_N| > \frac{1}{2}$. Then for each integer $1 \leq m \leq K-1$, where $K = \lceil 2|y_N| \rceil$, let $P(m)$ be the following statement

$$y_{N+m} = \frac{m}{2} - |y_N| < 0.$$

The proof will be by induction on m. I shall first show that $P(1)$ is true. We have

$$y_{N+1} = x_N - |y_N| + 1 = \frac{1}{2} - |y_N|.$$

Note that

$$\frac{1}{2} - |y_N| \leq \frac{K-1}{2} - |y_N|$$

and

$$\tfrac{K-1}{2} - |y_N| < 0 \quad \text{iff} \quad K < 2|y_N| + 1 \quad \text{iff} \quad \lceil 2|y_N| \rceil < 2|y_N| + 1$$

and so P(1) is true. Thus if $K = 2$, then I have shown that for $1 \leq m \leq K-1$, P(m) is true. It remains to consider the case $K \geq 3$. So assume $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-2$ and suppose $P(m)$ is true. I shall show that $P(m+1)$ is true.

$$\begin{aligned} y_{N+m+1} &= x_{N+m} - |y_{N+m}| + 1 = -\tfrac{1}{2} - (-\tfrac{m}{2} + |y_N|) + 1 \\ &= \tfrac{m+1}{2} - |y_N| \end{aligned}$$

Note that

$$\frac{m+1}{2} - |y_N| \leq \frac{K-1}{2} - |y_N|$$

and

$$\tfrac{K-1}{2} - |y_N| < 0 \quad \text{iff} \quad K < 2|y_N| + 1 \quad \text{iff} \quad \lceil 2|y_N| \rceil < 2|y_N| + 1$$

and so $P(m+1)$ is true. That is $P(m)$ is true for $1 \leq m \leq K-1$. Specifically, $P(K-1)$ is true. Then

$$\begin{aligned} y_{N+K} &= x_{N+K-1} - |y_{N+K-1}| + 1 = -\tfrac{1}{2} - (-\tfrac{K-1}{2} + |y_N|) + 1 \\ &= \tfrac{K}{2} - |y_N|. \end{aligned}$$

In particular,

$$y_{N+K} = \frac{K}{2} - |y_N| \geq \frac{2|y_N|}{2} - |y_N| = 0.$$

Thus

$$\begin{aligned} y_{N+K+1} &= x_{N+K} - |y_{N+K}| + 1 = -\tfrac{1}{2} - (\tfrac{K}{2} - |y_N|) + 1 \\ &= \tfrac{1-K}{2} + |y_N|. \end{aligned}$$

In particular,

$$y_{N+K+1} = \frac{1-K}{2} + |y_N| \geq \frac{1-2|y_N|}{2} + |y_N| = \frac{1}{2}.$$

Then

$$\begin{aligned} y_{N+K+2} &= x_{N+K+1} - |y_{N+K+1}| + 1 = -\tfrac{1}{2} - (\tfrac{1-K}{2} + |y_N|) + 1 \\ &= \tfrac{K}{2} - |y_N| = y_{N+K} \end{aligned}$$

and so the solution is periodic with period-2. Note, if $y_N = \frac{1}{4}$ then we have the equilibrium solution. □

Lemma .1.21 *Suppose there exists a non-negative natural number N such that $N \geq M$ and $|y_N| \geq x_N + 1$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is a period-4 solution.*

Proof: Suppose $-1 \leq x_N \leq 0$ and $|y_N| \geq x_N + 1$. Then

$$
\begin{aligned}
x_{N+1} &= |x_N| - 1 = -x_N - 1 \leq 0 \\
y_{N+1} &= x_N - |y_N| + 1 \leq 0 \\
x_{N+2} &= x_N \\
y_{N+2} &= x_{N+1} - |y_{N+1}| + 1 = -x_N - 1 - (-x_N + |y_N| - 1) + 1 \\
&= |y_N| + 1 \geq 0 \\
x_{N+3} &= x_{N+1} \\
y_{N+3} &= x_{N+2} - |y_{N+2}| + 1 = x_N - (|y_N| + 1) + 1 \\
&= x_N - |y_N| < 0 \\
x_{N+4} &= x_N \\
y_{N+4} &= x_{N+3} - |y_{N+3}| + 1 = -x_N - 1 - (-x_N + |y_N|) + 1 \\
&= -|y_N| \leq 0 \\
x_{N+5} &= x_{N+1} \\
y_{N+5} &= x_{N+4} - |y_{N+4}| + 1 = x_N - |y_N| + 1 \\
&= y_{N+1}.
\end{aligned}
$$

The proof is complete. □

To complete the proof of Theorem .1.19 it remains to consider the region enclosed by $|y_N| < x_N + 1$ and $x_N \in (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, 0)$.

Lemma .1.22 *Suppose there exists a non-negative natural number N such that $N \geq M$ and $|y_N| < x_N + 1$. Then $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually a period-4 solution.*

Set

$$\begin{aligned} R_1 &= \{(x,y) : -1 < x < -\tfrac{1}{2} \text{ and } 0 < y < x+1\} \\ R_2 &= \{(x,y) : -\tfrac{1}{2} < x < 0 \text{ and } 0 < y < x+1\} \\ R_3 &= \{(x,y) : -1 < x < 0 \text{ and } -(x+1) < y \leq 0\}. \end{aligned}$$

First, suppose there exists a non-negative integer N, such that $(x_N, y_N) \in R_3$. Then $x_{N+1} = |x_N| - 1$ and $y_{N+1} = x_N - |y_N| + 1 = x_N + y_N + 1 > 0$. Otherwise $y_{N+1} \geq x_{N+1} + 1$ then by Lemma .1.21, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ is a period-4 solution. If $y_{N+1} < x_{N+1} + 1$ then $(x_{N+1}, y_{N+1}) \in R_1 \cup R_2$.

Next, suppose there exists a non-negative integer N, such that $(x_N, y_N) \in R_2$. Then $x_{N+1} = |x_N| - 1$ and $y_{N+1} = x_N - |y_N| + 1 > 0$. So $(x_{N+1}, y_{N+1}) \in R_1$.

Finally, suppose there exists a non-negative integer N, such that $(x_N, y_N) \in R_1$. For each integer $m \geq 0$, let P(m) be the following statement:

$$x_{N+2m} = x_N$$

$$y_{N+2m} = -2mx_N + y_N - m > 0$$

$$x_{N+2m+1} = x_{N+1}$$

$$y_{N+2m+1} = (2m+1)x_N - y_N + (m+1) > 0.$$

Claim: $P(m)$ is true for $0 \leq m \leq K-1$, where $K = \lceil \frac{y_N - x_N - 1}{2x_N + 1} \rceil$. Note $K \geq 1$. The proof of the Claim will be by induction on m. I shall first show that $P(0)$ is true. Recall that $-1 < x_N < 0$ and $0 < y_N < x_N + 1$. Then

$$x_{N+2(0)} = x_N$$

$$y_{N+2(0)} = y_N > 0$$

$$x_{N+2(0)+1} = x_{N+1}$$

$$y_{N+2(0)+1} = x_N - |y_N| + 1 = x_N - y_N + 1 > 0.$$

Thus if $K = 1$, then I have shown that for $0 \leq m \leq K-1$, $P(m)$ is true. It remains to consider the case $K \geq 2$. So assume $K \geq 2$. Let m be an integer such that $0 \leq m \leq K-2$ and suppose $P(m)$ is true. I shall show that $P(m+1)$ is true. Since $P(m)$ is true I know

$$x_{N+2m} = x_N$$

$$y_{N+2m} = -2mx_N + y_N - m > 0$$

$$x_{N+2m+1} = x_{N+1}$$

$$y_{N+2m+1} = (2m+1)x_N - y_N + (m+1) > 0.$$

Then

$$\begin{aligned} x_{N+2(m+1)} &= |x_{N+2m+1}| - 1 = |x_{N+1}| - 1 \\ &= x_N \\ y_{N+2(m+1)} &= x_{N+2m+1} - |y_{N+2m+1}| + 1 \\ &= x_{N+1} - [(2m+1)x_N - y_N + (m+1)] + 1 \\ &= -x_N - 1 - (2m+1)x_N + y_N - (m+1) + 1 \\ &= -x_N - (2m+1)x_N + y_N - (m+1) \\ &= -2(m+1)x_N + y_N - (m+1). \end{aligned}$$

In particular,

$$\begin{aligned} y_{N+2(m+1)} &= -2(m+1)x_N + y_N - (m+1) \\ &= [-2mx_N + y_N - m] + [-2x_N - 1] > 0 \end{aligned}$$

Then

$$\begin{aligned} x_{N+2(m+1)+1} &= |x_{N+2(m+1)}| - 1 = |x_N| - 1 \\ &= x_{N+1} \\ y_{N+2(m+1)+1} &= x_{N+2(m+1)} - |y_{N+2(m+1)}| + 1 \\ &= x_N - [-2(m+1)x_N + y_N - (m+1)] + 1 \\ &= x_N + 2(m+1)x_N - y_N + (m+1) + 1 \\ &= [2(m+1)+1]x_N - y_N + [(m+1)+1]. \end{aligned}$$

In particular,

$$\begin{aligned} y_{N+2(m+1)+1} &= [2(m+1)+1]x_N - y_N + [(m+1)+1] \\ &= [2(m+1)x_N - y_N + (m+1)] + [x_N + 1] > 0. \end{aligned}$$

and so P(m+1) is true. Thus the proof of the Claim is complete. That is P(m) is true for $0 \leq m \leq K-1$. Specifically, P(K-1) is true, and so

$$\begin{aligned} x_{N+2(K-1)} &= x_N \\ y_{N+2(K-1)} &= -2(K-1)x_N + y_N - (K-1) > 0 \\ x_{N+2(K-1)+1} &= x_{N+1} \\ y_{N+2(K-1)+1} &= [2(K-1)+1]x_N - y_N + [(K-1)+1] > 0 \\ &= (2K-1)x_N - y_N + K > 0. \end{aligned}$$

Then

$$\begin{aligned} x_{N+2(K-1)+2} = x_{N+2K} &= x_N \\ y_{N+2(K-1)+2} = y_{N+2K} &= x_{N+2(K-1)+1} - |y_{N+2(K-1)+1}| + 1 \\ &= x_{N+1} - [(2K-1)x_N - y_N + K] + 1 > 0 \\ &= -x_N - 1 - [(2K-1)x_N - y_N + K] + 1 > 0 \\ &= -2Kx_N + y_N - K. \end{aligned}$$

Recall $K = \lceil \frac{y_N - x_N - 1}{2x_N + 1} \rceil$, thus

$$\begin{aligned} y_{N+2K} &= -2Kx_N + y_N - K \\ &= -2\lceil \tfrac{y_N - x_N - 1}{2x_N + 1} \rceil x_N + y_N - \lceil \tfrac{y_N - x_N - 1}{2x_N + 1} \rceil \\ &= \lfloor 1 + \tfrac{1 - 2y_N}{2x_N + 1} \rfloor x_N + y_N - \lceil -(\tfrac{-x_N - y_N}{2x_N + 1} - 1) \rceil \\ &= \lceil x_N + \tfrac{x_N - 2y_N x_N}{2x_N + 1} \rceil + y_N - \lceil \tfrac{x_N + y_N}{2x_N + 1} + 1 \rceil \\ &= x_N + \lceil \tfrac{x_N - 2y_N x_N}{2x_N + 1} \rceil + y_N + 1 - \lceil \tfrac{x_N + y_N}{2x_N + 1} \rceil \\ &= x_N + 1 + \lceil \tfrac{-y_N(2x_N + 1)}{2x_N + 1} \rceil + y_N \\ &\geq x_N + 1 \quad = \quad x_{N+2k} + 1. \end{aligned}$$

Therefore by Lemma .1.21 $\{(x_n, y_n)\}_{n=N+2K}^{\infty}$ is periodic with period-4.

The proof of Theorem .1.19 is complete.

$\square$

System(31)

I also consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ... \tag{31}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

It has the unique equilibrium point $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

Theorem .1.23 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(31) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually periodic with (not necessarily prime) period-2.*

The change of variables, $x_n = X_n$ and $y_n = Y_n - 1$, reduces the system to

$$\begin{cases} X_{n+1} = |X_n| - 1 \\ \\ Y_{n+1} = X_n \end{cases}, \qquad n = 0, 1, ... \tag{32}$$

which is System(32). See Theorem .1.30.

System(33)

Now consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ... \tag{33}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

It has the unique equilibrium point $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Theorem .1.24 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(33) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually periodic with (not necessarily prime) period-2.*

The change of variables, $x_n = X_n$ and $y_n = Y_n + 1$, reduces the system to

$$\begin{cases} X_{n+1} = |X_n| - 1 \\ \\ Y_{n+1} = X_n \end{cases}, \qquad n = 0, 1, ..., \tag{32}$$

which is System(32). See Theorem .1.30.

System(32)

The next system of piecewise linear difference equations I consider is

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ... \tag{32}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

The unique equilibrium point of this system is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.

Theorem .1.25 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(32) with $(x_0, y_0) \in \mathbf{R}^2$. Every solution of this system is eventually periodic with (not necessarily prime) period-2. In particular, if $|x_0| = m + \alpha$, where $m \in \{0, 1, 2, \ldots\}$, and $\alpha \in \mathbf{R}$ such that $0 \leq \alpha < 1$ then the period-2 solution in $\{x_n\}$ is $\{\alpha - 1, -\alpha\}$.*

The proof is a direct consequence of Lemma .1.16.

.1.6 System(37)

In this section I consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \quad n = 0, 1, ... \tag{37}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

The set of equilibrium points are found on the following line:

$$\{(x,y) : y = \frac{x}{2} - \frac{1}{2}, \text{ if } y \geq 0, \text{ and } x = 1 \text{ if } y < 0\}.$$

The period two cycles are:

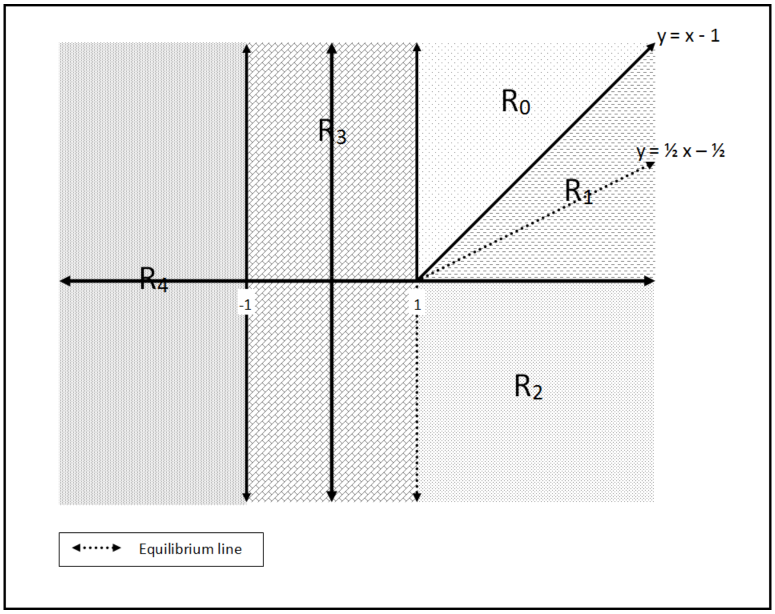
$$\begin{cases} (x_{2m-1}, y_{2m-1}) = (x_0, x_0 - y_0 - 1) \\ \\ (x_{2m}, y_{2m}) = (x_0, y_0) \end{cases}, \quad m = 1, 2, ...$$

I show that every solution of System(37) is either an equilibrium point, is a period-2 solution, or has the boundedness character (B, U).

Global Results

Set

$$\begin{aligned} R_0 &= \{(x,y) : y \leq x - 1, x \geq 1, \text{ and } y \geq 0\} \\ R_1 &= \{(x,y) : y > x - 1, x \geq 1, \text{ and } y \geq 0\} \\ R_2 &= \{(x,y) : x \geq 1, \text{ and } y < 0\} \\ R_3 &= \{(x,y) : |x| < 1\} \\ R_4 &= \{(x,y) : x \leq -1\}. \end{aligned}$$



Theorem .1.26 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(37) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium solution, a period-2 solution, or it has the boundedness character (B,U), that is $\{y_n\}_{n=0}^{\infty}$ is decreasing without bound.*

The proof of Theorem 1.5.1 is a direct consequence of the following lemmas.

Lemma .1.27 *Suppose the initial condition (x_0, y_0) is an element of R_0. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a period-2 solution.*

Proof: Recall that $R_0 = \{(x, y) : y \leq x - 1, x \geq 1, \text{ and } y \geq 0\}$. Suppose the initial condition (x_0, y_0) is an element of R_0. It is clear that $x_n = x_0$ for all $n \in \mathbf{N}$. Then

$$\begin{aligned} y_1 &= x_0 - |y_0| - 1 \geq 0 \\ y_2 &= x_1 - |y_1| - 1 = y_0. \end{aligned}$$

So, for any $m \in \mathbf{N}$, $y_{2m-1} = x_0 - y_0 - 1$ and $y_{2m} = y_0$. The proof is complete. □

Lemma .1.28 *Suppose the initial condition (x_0, y_0) is an element of R_2. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually a period-2 solution.*

Proof: Recall that $R_2 = \{(x,y) : x \geq 1, \text{ and } y < 0\}$. Suppose the initial condition (x_0, y_0) is an element of R_2.

Then

$$\begin{aligned} x_1 &= |x_0| \geq 1 \\ y_1 &= x_0 - |y_0| - 1 = x_0 + y_0 - 1. \end{aligned}$$

It is clear that $x_n = x_0$ for $n \geq 0$.

Case 1 : Suppose $x_0 = 1$. Then $y_1 = y_0$, and so $(x_n, y_n) = (1, y_0)$ for $n \geq 0$.

Case 2 : Suppose $y_1 = x_0 + y_0 - 1 \geq 0$. Recall $R_0 = \{(x,y) : y \leq x-1, x \geq 1, \text{ and } y \geq 0\}$, so one more condition must be satisfied to utilize Lemma .1.27.

Note that, $y_1 = x_0 + y_0 - 1 = x_1 + y_0 - 1 \geq x_1 - 1$

and so by Lemma .1.27, the solution $\{(x_n, y_n)\}_{n=1}^{\infty}$ period-2.

Case 3 : Suppose $y_1 = x_0 + y_0 - 1 < 0$. For each integer $m \geq 1$, let $P(m)$ be the following statement

$$y_m = (m)(x_0 - 1) + y_0 < 0.$$

Claim: $P(m)$ is true for $1 \leq m \leq K-1$, where $K = \lceil \frac{-y_0}{x_0 - 1} \rceil$. Note $K \geq 2$. The proof of the Claim will be by induction on m. It is cleat that $P(1)$ is true. Thus if $K = 1$, then we know that for $1 \leq m \leq K-1$, $P(m)$ is true. It remains to consider the case $K \geq 3$. So assume $K \geq 3$. Let m be an integer such that $1 \leq m \leq K-4$ and suppose $P(m)$ is true. I shall show that $P(m+1)$ is true.

Since $P(m)$ is true I know

$$y_m = (m)(x_0 - 1) + y_0 < 0$$

and so

$$\begin{aligned} y_{m+1} &= x_m - |y_m| - 1 \\ &= x_0 - [-(m)(x_0 - 1) - y_0] - 1 \\ &= (m+1)(x_0 - 1) + y_0 < 0. \end{aligned}$$

In particular,

$$\begin{aligned} y_{m+1} &= (m+1)(x_0-1)+y_0 \\ &\leq (K+1)(x_0-1)+y_0 \\ &= (\lceil \tfrac{-y_0}{x_0-1} \rceil - 1 + 1)(x_0-1)+y_0 \\ &< 0. \end{aligned}$$

and so $P(m+1)$ is true. Thus the proof of the Claim is complete. Specifically, $P(K-1)$ is true and so

$$\begin{aligned} y_{K-1} &= (K-1)(x_0-1)+y_0 < 0 \\ y_K &= x_K - |y_K| - 1 \\ &= x_0 - 1 + (K-1)(x_0-1)+y_0 \\ &= (K)(x_0-1)+y_0. \end{aligned}$$

Recall $K = \lceil \frac{-y_0}{x_0-1} \rceil$, then

$$y_K \;=\; (\lceil \tfrac{-y_0}{x_0-1} \rceil)(x_0-1)+y_0 \;\geq\; 0$$

and so $(x_K, y_K) \in R_0$, and by Lemma .1.27, the solution $\{(x_n, y_n)\}_{n=1}^{\infty}$ is eventually period-2.

□

Lemma .1.29 *Suppose the initial condition* (x_0, y_0) *is an element of* $R_1 \cup R_4$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually a period-2 solution.*

Proof: Recall that $R_1 = \{(x,y) : y > x-1, \text{ and } x \geq 1, y \geq 0\}$ and $R_4 = \{(x,y) : x \leq -1\}$. Suppose the initial condition (x_0, y_0) is an element of R_1.

Then

$$\begin{aligned} x_1 &= |x_0| = x_0 > 0 \\ y_1 &= x_0 - |y_0| - 1 = x_0 - y_0 - 1 < 0 \end{aligned}$$

and so $(x_1, y_1) \in R_2$ and by Lemma .1.28 the solution $\{(x_n, y_n)\}_{n=1}^{\infty}$ is eventually period-2.

Suppose the initial condition (x_0, y_0) is an element of R_4.

Then

$$\begin{aligned} x_1 &= |x_0| > 0 \\ y_1 &= x_0 - |y_0| - 1 < 0 \end{aligned}$$

and so $(x_1, y_1) \in R_2$ and by Lemma .1.28 the solution $\{(x_n, y_n)\}_{n=1}^{\infty}$ is eventually period-2. □

Lemma .1.30 *Suppose the initial condition* (x_0, y_0) *is an element of* R_3*. Then the boundedness character is (B, U).*

Proof: Recall that $R_3 = \{(x, y) : |x| < 1\}$. Suppose the initial condition (x_0, y_0) is an element of R_3.

First note that, $x_n = |x_0| < 1$ for all $n = 1, 2, 3, \ldots$.

For each integer $m \geq 1$, let $P(m)$ be the following statement

$$y_m = y_1 - (m-1)(1 - |x_0|) < 0.$$

Claim: $P(m)$ is true for $m \geq 2$ The proof of the Claim will be by induction on m. I shall first show that $P(2)$ is true. Then

$$\begin{aligned} y_1 &= x_0 - |y_0| - 1 < 0 \\ y_2 &= x_1 - |y_1| - 1 \\ &= |x_0| + (x_0 - |y_0| - 1) - 1 \\ &= (x_0 - |y_0| - 1) - (1 - |x_0|) \\ &= y_1 - (2-1)(1 - |x_0|) < 0 \end{aligned}$$

Let m be an integer such that $m \geq 2$ and suppose $P(m)$ is true. I shall show that $P(m+1)$ is true.

Since $P(m)$ is true I know

$$\begin{aligned} y_m &= y_1 - (m-1)(1-|x_0|) \quad < \quad 0 \\ &\text{and so} \\ y_{m+1} &= x_m - |y_m| - 1 \\ &= |x_0| - (-y_1 + (m-1)(1-|x_0|)) - 1 \\ &= y_1 - (m-1)(1-|x_0|) - (1-|x_0|) \\ &= y_1 - m(1-|x_0|) \quad < \quad 0 \end{aligned}$$

and so $P(m+1)$ is true. The proof of the claim is complete. Clearly, $\{y_n\}_{n=2}^{\infty}$ is decreasing at a constant rate with no lower bound. The proof is complete. □

.1.7 System(38)

In this section I consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ... \tag{38}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

The set of equilibrium points are found on the following line:

$$\{(x, y) : y = \frac{x}{2}, \text{ if } y > 0, \text{ and } x = 0 \text{ if } y \leq 0\}.$$

The period two cycles are:

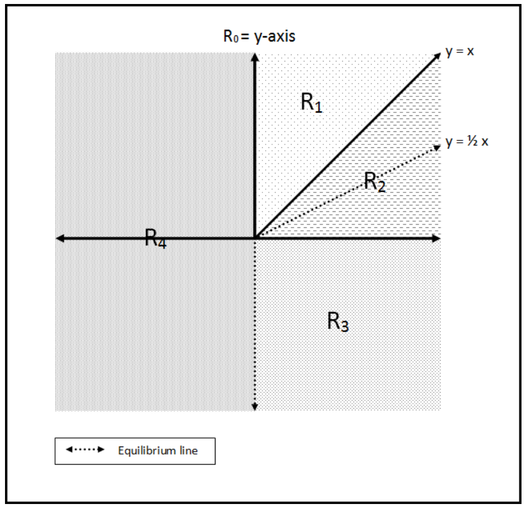
$$\begin{cases} (x_{2m-1}, y_{2m-1}) = (x_0, x_0 - y_0) \\ \\ (x_{2m}, y_{2m}) = (x_0, y_0) \end{cases}, \qquad m = 1, 2, ...$$

I show that every solution of System(38) is either an equilibrium point or a period-2 solution.

Global Results

Set

$$\begin{aligned} R_0 &= \{(x, y) : x = 0 \text{ and } y \in \mathbf{R}\} \\ R_1 &= \{(x, y) : x < y \text{ and } x > 0\} \\ R_2 &= \{(x, y) : x \geq y \text{ and } y > 0\} \\ R_3 &= \{(x, y) : x > 0 \text{ and } y < 0\} \\ R_4 &= \{(x, y) : x < 0\}. \end{aligned}$$



Theorem .1.31 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(38) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium solution or a period-2 solution.*

The proof of Theorem 1.6.1 is a direct consequence of the following lemmas.

Lemma .1.32 *Suppose the initial condition (x_0, y_0) is an element of R_0. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a equilibrium solution.*

Proof: Suppose the initial condition (x_0, y_0) is an element of $R_0 = \{(x, y) : x = 0 \text{ and } y \in \mathbf{R}\}$. It is clear that $x_n = 0$ for $n = 0, 1, 2, ...$

Then

$$\begin{aligned} y_1 &= x_0 - |y_0| = -|y_0| \\ y_2 &= x_1 - |y_1| = 0 - |-|y_0|| = -|y_0| \end{aligned}$$

and so $(x_n, y_n) = (0, -|y_0|)$ for $n = 1, 2, ...$ The proof is complete. □

Lemma .1.33 *Suppose the initial condition (x_0, y_0) is an element of R_2. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a period two solution.*

Proof: Suppose the initial condition (x_0, y_0) is an element of $R_2 = \{(x, y) : x \geq y \text{ and } y > 0\}$. First note that, $x_n = |x_0| = x_0$ for $n = 1, 2, 3, \ldots$.

Then

$$\begin{aligned} y_1 &= x_0 - |y_0| = x_0 - y_0 \geq 0 \\ y_2 &= x_1 - |y_1| = x_0 - |x_0 - y_0| = y_0. \end{aligned}$$

So, for any $m \in \mathbf{N}$, $(x_{2m-1}, y_{2m-1}) = (x_0, x_0 - y_0)$ and $(x_{2m}, y_{2m}) = (x_0, y_0)$. The proof is complete. $\square$

Lemma .1.34 *Suppose the initial condition* (x_0, y_0) *is an element of* R_3. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually a period two solution.*

Proof: Suppose the initial condition (x_0, y_0) is an element of $R_3 = \{(x, y) : x > 0 \text{ and } y < 0\}$. It is clear that $x_n = x_0 > 0$ for $n = 0, 1, 2, ...$

Then

$$y_1 = x_0 - |y_0| = x_0 + y_0.$$

Case 1 : Suppose $y_1 = x_0 + y_0 \geq 0$. Recall $R_2 = \{(x, y) : x \geq y \text{ and } y > 0\}$, so an additional condition must be satisfied to utilize Lemma .1.33. Note that $y_1 = x_0 + y_0 = x_1 + y_0 < x_1$,
and so by Lemma .1.33, the solution $\{(x_n, y_n)\}_{n=1}^{\infty}$ period-2.

Case 2 : Suppose $y_1 = x_0 + y_0 < 0$. For each integer $m \geq 1$, let $P(m)$ be the following statement

$$y_m = mx_0 + y_0 < 0$$

Claim: $P(m)$ is true for $1 \leq m \leq K - 1$, where $K = \lceil \frac{-y_0}{x_0} \rceil$. Note $K \geq 2$. The proof of the Claim will be by induction on m. $P(1)$ is clearly true. Thus if $K = 2$, then for $1 \leq m \leq K - 1$, $P(m)$ is true. It remains to consider the case $K \geq 3$. So assume $K \geq 3$. Let m be an integer such that $1 \leq m \leq K - 2$ and suppose $P(m)$ is true. I shall show that $P(m + 1)$ is true.
Since $P(m)$ is true I know

$$y_m = mx_0 + y_0 < 0$$

and so

$$\begin{aligned} y_{m+1} &= x_{m+1} - |y_{m+1}| = x_0 + mx_0 + y_0 \\ &= (m + 1)x_0 + y_0. \end{aligned}$$

In particular,

$$y_{m+1} = (m+1)x_0 + y_0 \le (K-1)x_0 + y_0 = Kx_0 + y_0 - x_0 \le 0.$$

The proof of the Claim is complete. So $P(m)$ is true for $1 \le m \le K-1$. Specifically, $P(K-1)$ is true. Hence

$$\begin{aligned} y_{K-1} &= (K-1)x_0 + y_0 < 0 \\ y_K &= x_{K-1} - |y_{K-1}| = x_0 + (K-1)x_0 + y_0 \\ &= Kx_0 + y_0 = \lceil \tfrac{-y_0}{x_0} \rceil x_0 + y_0 \ge 0, \end{aligned}$$

and so $(x_K, y_K) \in R_2$, and by Lemma .1.33, the solution $\{(x_n, y_n)\}_{n=1}^{\infty}$ is eventually period-2.

□

Lemma .1.35 *Suppose the initial condition* (x_0, y_0) *is an element of* $R_1 = \{(x,y) : x < y \text{ and } x > 0\}$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually a period two solution.*

Proof: We have

$$\begin{aligned} x_1 &= |x_0| = x_0 > 0 \\ y_1 &= x_0 - |y_0| = x_0 - y_0 < 0 \end{aligned}$$

and so $(x_1, y_1) \in R_3$ and by Lemmas .1.34 and .1.33 the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually a period two. □

Lemma .1.36 *Suppose the initial condition* (x_0, y_0) *is an element of* $R_4 = \{(x,y) : x < 0\}$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually a period two solution.*

Proof: We have

$$\begin{aligned} x_1 &= |x_0| = -x_0 > 0 \\ y_1 &= x_0 - |y_0| < 0 \end{aligned}$$

and so $(x_1, y_1) \in R_3$ and by Lemmas .1.34 and .1.33 the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually a period two. □

.1.8 System(43)

In this section I consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{43}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

The set of equilibrium points are found on the following line:

$$\{(x, y) : y = \frac{x}{2} - \frac{1}{2} < 0 \text{ if } x \geq 0, \text{ and } x = 1 \text{ if } y \geq 0\}.$$

I show that every solution of System(43) is either an equilibrium point, a period-2 solution or has the boundedness characteristic (B, U).

Global Results

Set

$$\begin{aligned} R_0 &= \{(x, y) : |x| = 1 \text{ and } y = \tfrac{x}{2} - \tfrac{1}{2} < 0 \text{ if } x \geq 0\} \\ R_1 &= \{(x, y) : |x| > 1\} \\ R_2 &= \{(x, y) : |x| < 1\} \end{aligned}$$

Theorem .1.37 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(43) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium, a period-2 solution or has the boundedness character (B,U). In particular, $\{(x_n)\}_{n=1}^{\infty} = |x_0|$ and $\{(y_n)\}_{n=1}^{\infty}$ is increasing without bound.*

The proof of Theorem .1.37 is a direct consequence of the following lemmas.

Lemma .1.38 *Suppose the initial condition (x_0, y_0) is an element of R_0. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is an equilibrium solution.*

Proof: Recall $R_0 = \{(x, y) : |x| = 1 \text{ and } y = \frac{x}{2} - \frac{1}{2} < 0 \text{ if } x \geq 0\}$. Clearly, $\{(x_n)\}_{n=1}^{\infty} = |x_0|$.

Case 1 : Suppose $x_0 = 1$. Then

$$\begin{aligned} y_1 &= x_0 + |y_0| - 1 &= |y_0| &> 0 \\ y_2 &= x_1 + |y_1| - 1 &= |y_1| &= y_1 \end{aligned}$$

and so $(\bar{x}, \bar{y}) = (1, |y_0|)$.

Case 2 : Now suppose $x_0 = -1$ and $|y_0| \leq 2$. Then

$$\begin{aligned} y_1 &= x_0 + |y_0| - 1 &= |y_0| - 2 & &\leq 0 \\ y_2 &= x_1 + |y_1| - 1 &= 1 + |y_1| - 1 &= -|y_0| + 2 &\geq 0 \\ y_3 &= x_2 + |y_2| - 1 &= 1 + |y_2| - 1 &= y_2 & \end{aligned}$$

and so $(\bar{x}, \bar{y}) = (1, -|y_0| + 2)$.

It remains to consider $|y_0| > 2$. Then

$$\begin{aligned} y_1 &= x_0 + |y_0| - 1 &= |y_0| - 2 &> 0 \\ y_2 &= x_1 + |y_1| - 1 &= 1 + |y_1| - 1 &= y_1 \end{aligned}$$

and so $(\bar{x}, \bar{y}) = (1, |y_0| - 2)$.

Case 3 : Finally, suppose $x_0 \geq 0$ and $y_0 = \frac{x_0}{2} - \frac{1}{2} < 0$. Then

$$y_1 = x_0 + |y_0| - 1 = (x_0 - 1) + (\tfrac{x_0}{2} - \tfrac{1}{2}) = \tfrac{x_0}{2} - \tfrac{1}{2} = y_0$$

and so $(\bar{x}, \bar{y}) = (x_0, y_0)$.

□

Lemma .1.39 *Suppose the initial condition (x_0, y_0) is an element of R_1. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (B,U). In particular, $\{y_n\}_{n=0}^{\infty}$ is increasing without bound.*

Proof: Recall $R_1 = \{(x, y) : |x| > 1\}$. Clearly, $\{(x_n)\}_{n=1}^{\infty} = x_0$.

Case 1 : Suppose $x_0 > 1$ and $y_0 \in \mathbf{R}$. For each integer $m \geq 1$, let $P(m)$ be the following statement:

$$y_m = m(x_0 - 1) + |y_0| > 0$$

Claim: $P(m)$ is true for $m \geq 1$. The proof of the claim will be by induction on m. I shall first show that $P(1)$ is true.

$$y_1 = x_0 + |y_0| - 1 = 1(x_0 - 1) + |y_0| > 0$$

Suppose $P(m)$ is true. I shall show that $P(m+1)$ is true.

$$\begin{aligned} y_{m+1} &= x_m + |y_m| - 1 \\ &= x_0 + [m(x_0 - 1) + |y_0|] - 1 \\ &= (m+1)(x_0 - 1) + |y_0| \quad > \quad 0 \end{aligned}$$

and so $P(m+1)$ is true. Thus the proof of the claim is complete. So $\{(y_n)\}_{n=1}^{\infty}$ is increasing at a constant rate, therefore the boundedness character is (B,U).

Case 2 : Suppose $x_0 < -1$ and $y_0 \in \mathbf{R}$. Then

$$x_1 \;=\; |x_0| \;=\; -x_0 \;>\; 1$$

$$y_1 \;=\; x_0 + |y_0| - 1 \;=\; -x_0 + |y_0| - 1 \;\in\; \mathbf{R}$$

and so, by Case 1, the solution $\{(x_n, y_n)\}_{n=N+1}^{\infty}$ has the boundedness character (B,U).

□

Lemma .1.40 *Suppose the initial condition* (x_0, y_0) *is an element of* R_2. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually periodic with period-2.*

Proof: Recall $R_2 = \{(x, y) : |x| < 1\}$. Clearly, $x_n = |x_0|$ for all $n \geq 0$.

Case 1 : Suppose $0 \leq x_0 < 1$ and $|y_0| \leq 1 - x_0$. Then

$$\begin{aligned} y_1 &= x_0 + |y_0| - 1 \leq 0 \\ y_2 &= x_1 + |y_1| - 1 = x_1 + (-x_0 - |y_0| + 1) - 1 = -|y_0| \\ y_3 &= x_2 + |y_2| - 1 = x_0 + |y_0| - 1 = y_1. \end{aligned}$$

So for $n \geq 1$ the periodic solution is

$$\begin{aligned} (x_{2n}, y_{2n}) &= (x_0, -|y_0|) \\ (x_{2n+1}, y_{2n+1}) &= (x_0, x_0 + |y_0| - 1). \end{aligned}$$

Case 2 : Suppose $0 \leq x_0 < 1$ and $|y_0| > 1 - x_0$. Then for each integer $1 \leq m \leq K - 1$, where $K = \lceil \frac{-|y_0|}{x_0 - 1} \rceil$, note $K \geq 2$, let $P(m)$ be the following statement:

$$y_m = m(x_0 - 1) + |y_0| > 0$$

Claim: $P(m)$ is true for $1 \leq m \leq K - 1$. The proof of the Claim will be by induction on m. It is clear that $P(1)$ is true because $y_1 = x_0 + |y_0| - 1 > 0$. So if $K = 2$ then I have shown that for $1 \leq m \leq K - 1$, the claim is true. So assume $K \geq 3$. Let m be an integer such that $1 \leq m \leq K - 2$ and suppose $P(m)$ is true. I shall show that $P(m + 1)$ is true. So

$$\begin{aligned} y_{m+1} &= x_m + |y_m| - 1 \\ &= x_0 + [m(x_0 - 1) + |y_0|] - 1 \\ &= (m + 1)(x_0 - 1) + |y_0| > 0. \end{aligned}$$

The proof of the Claim is complete. So $P(m)$ is true for $1 \leq m \leq K - 1$. In particular $P(K - 1)$ is true. Then

$$\begin{aligned} y_{K-1} &= (K - 1)(x_0 - 1) + y_0 > 0 \\ y_K &= x_{K-1} + |y_{K-1}| - 1 \\ &= x_0 + [(K - 1)(x_0 - 1) + |y_0|] - 1 \end{aligned}$$

$$
\begin{aligned}
&= K(x_0 - 1) + |y_0| \\
&= (\lceil \tfrac{|y_0|}{x_0-1} \rceil)(x_0 - 1) + y_0 \quad \leq \quad 0
\end{aligned}
$$

and so

$$
\begin{aligned}
y_{K+1} &= x_K + |y_K| - 1 \\
&= x_0 + [-K(x_0 - 1) - y_0] - 1 \\
&= x_0 - K(x_0 - 1) - y_0 - 1 \\
&= (-K + 1)(x_0 - 1) - y_0 \\
&= -[(K - 1)(x_0 - 1) + y_0] \\
&= -y_{K-1} \quad < \quad 0 \\
y_{K+2} &= x_{K+1} + |y_{K+1}| - 1 \\
&= x_0 + y_{K-1} - 1 \\
&= x_0 + [(K - 1)(x_0 - 1) + y_0] - 1 \\
&= K(x_0 - 1) - y_0 \quad = \quad y_K.
\end{aligned}
$$

So for $n \geq 1$ the periodic solution is

$$
\begin{aligned}
(x_{K+2n}, y_{K+2n}) &= (x_0, K(x_0 - 1) + y_0) \\
(x_{K+2n+1}, y_{K+2n+1}) &= (x_0, -[(K - 1)(x_0 - 1) + y_0])
\end{aligned}
$$

and this completes the proof of Case 2.

Case 3 : Suppose $-1 < x_0 < 0$ and $y_0 \in \mathbf{R}$. Then

$$
\begin{aligned}
x_1 &= |x_0| = -x_0 > 1 \\
y_1 &= x_0 + |y_0| - 1 = -x_0 + |y_0| - 1 \in \mathbf{R}
\end{aligned}
$$

and so, by Cases 1 and 2, the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ is eventually periodic with period-2.

$\square$

.1.9 Systems(44 and 45)

In this section I first consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ... \tag{44}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

The set of equilibrium points are found on the following line: $\{(x, y) : x = 0 \text{ and } y \geq 0\}$.

Global Results

Theorem .1.41 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(44) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium, or has the boundedness character (B,U). In particular, if $|x_0| = 0$ then the solution is $(\bar{x}, \bar{y}) = (0, |y_0|)$, and if $|x_0| \neq 0$ then $\{(x_n)\}_{n=1}^{\infty} = |x_0|$ and $\{(y_n)\}_{n=1}^{\infty}$ is increasing without bound.*

The proof is by computations and will be omitted.

System(45)

Next, I consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ... \tag{45}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

Global Results

Theorem .1.42 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(45) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (B,U). In particular, $\{(x_n)\}_{n=1}^{\infty} = |x_0|$ and $\{(y_n)\}_{n=1}^{\infty}$ is increasing without bound.*

The proof is by computations and will be omitted.

.1.10 Systems(46 - 48, 52, 53)

All five systems share the same first difference equation, $x_{n+1} = |x_n| + 1$. It is clear that $x_n = |x_0| + n$ for $n \geq 1$ and $\{x_n\}_{n=0}^{\infty}$ is increasing without bound.

System(46)

I first consider the system of piecewise linear difference equations

$$\left\{\begin{array}{l} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{array}\right. , \qquad n = 0, 1, ... \tag{46}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that the boundedness characteristic of every solution of System(46) is (U,U).

Global Results

Theorem .1.43 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(46) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U). In particular, $\{x_n\}_{n=N}^{\infty}$ and $\{x_n\}_{n=N}^{\infty}$ are both increasing without bound.*

Recall $\{x_n\}_{n=0}^{\infty}$ is increasing without bound.

Lemma .1.44 *Suppose there exists a non-negative natural number N such that $|y_N| \leq x_N - 1$ and $x_N \geq 0$. Then the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ has the boundedness character (U,U).*

Proof: Suppose $|y_N| \leq x_N - 1$ and $x_N \geq 0$. For each integer $m \geq 1$, let $P(m)$ be the following statement:

$$y_{N+2m} = |y_N| + m > 0$$

$$y_{N+2m+1} = x_N - |y_N| + (m-1) > 0.$$

Claim: $P(m)$ is true for $m \geq 1$. The proof of the claim will be by induction on m.

I shall first show that $P(1)$ is true. Hence

$$
\begin{aligned}
y_{N+1} &= x_N - |y_N| - 1 \quad \geq \quad 0 \\
y_{N+2(1)} &= x_{N+1} - |y_{N+1}| - 1 \\
&= |x_0| + N + 1 - [x_N - |y_N| - 1] - 1 \\
&= |x_0| + N + 1 - [|x_0| + N - |y_N| - 1] - 1 \\
&= |y_N| + 1 \quad > \quad 0 \\
y_{N+2(1)+1} &= x_{N+2} - |y_{N+2}| - 1 \\
&= |x_0| + N + 2 - [|y_N| + 1] - 1 \\
&= |x_0| + N - |y_N| \\
&= x_N - |y_N| \quad > \quad 0.
\end{aligned}
$$

So $P(1)$ is true. Suppose $P(m)$ is true. I shall show that $P(m+1)$ is true. Hence

$$
\begin{aligned}
y_{N+2(m+1)} &= x_{N+2m+1} - |y_{N+2m+1}| - 1 \\
&= |x_0| + N + 2m + 1 - [x_N - |y_N| + (m-1)] - 1 \\
&= x_N + 2m - x_N + |y_N| - (m-1) \\
&= |y_N| + (m+1) \quad > \quad 0 \\
y_{N+2(m+1)+1} &= x_{N+2m+2} - |y_{N+2m+2}| - 1 \\
&= |x_0| + N + 2m + 2 - [|y_N| + (m+1)] - 1 \\
&= x_N - |y_N| + [(m+1) - 1] \quad > \quad 0
\end{aligned}
$$

and so $P(m+1)$ is true and the proof of the Claim is complete. So $\{y_{N+2m}\}_{m=1}^{\infty}$ and $\{y_{N+2m+1}\}_{m=1}^{\infty}$ are increasing without bound. □

Lemma .1.45 *Suppose there exists a non-negative natural number N such that $|y_N| > x_N - 1$ and $x_N \geq 0$. Then the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ has the boundedness character (U,U).*

Proof: It suffices to show that there exists an integer $M \geq 0$ such that $\{y_n\}_{n=N+M}^{\infty}$ is increasing without bound (that is, $y_{N+M} \geq 0$).

For the sake of contradiction assume that it is false that there exists and integer $M \geq 0$ such that $y_{N+M} \geq 0$ and $\{y_n\}_{n=N+M}^{\infty}$ is increasing without bound.

Suppose $|y_N| > x_N - 1$ and $x_N \geq 0$. Then $y_{N+1} = x_N - |y_N| - 1 < 0$. For each integer $m \geq 2$, let $P(m)$ be the following statement

$$y_{N+m} = y_{N+m-1} + x_{N+m-2}.$$

Recall by assumption that $y_{N+m} < 0$ for every integer $m \geq 2$. Claim: $P(m)$ is true for $m \geq 2$. The proof of the claim will be by induction on m. I shall first show that $P(2)$ is true. Hence

$$\begin{aligned} y_{N+2} &= x_{N+1} - |y_{N+1}| - 1 \\ &= |x_0| + N + 1 - [-x_N + |y_N| + 1] - 1 \\ &= |x_0| + N + x_N - |y_N| - 1 \\ &= (x_N - |y_N| - 1) + x_N \\ &= y_{N+1} + x_N. \end{aligned}$$

So $P(2)$ is true. Suppose $P(m)$ is true. I shall show that $P(m+1)$ is true. Hence

$$\begin{aligned} y_{N+(m+1)} &= x_{N+m} - |y_{N+m}| - 1 \\ &= |x_0| + N + m - (-y_{N+m-1} - x_{N+m-2}) - 1 \\ &= |x_0| + N + m + y_{N+m-1} + |x_0| + N + m - 2 - 1 \\ &= y_{N+m-1} + |x_0| + N + m - 1 + |x_0| + N + m - 2 \\ &= y_{N+m-1} + x_{N+m-1} + x_{N+m-2} \\ &= y_{N+m} - x_{N+m-2} + x_{N+m-1} + x_{N+m-2} \\ &= y_{N+m} + x_{N+m-1} \end{aligned}$$

and so P(m+1) is true. Clearly, $\{y_n\}_{n=N+m}^{\infty}$ is increasing at a growing rate. This leads to a contradiction. So there exists a smallest M such that $y_{N+M} \geq 0$. Then, by Lemma .1.46, the solution $\{y_n\}_{n=N+M}^{\infty}$ is increasing without bound. □

System(47)

I next consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ... \tag{47}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that the boundedness characteristic of every solution of System(47) is (U,U).

Global Results

Theorem .1.46 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(47) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U). In particular, $\{x_n\}_{n=N}^{\infty}$ and $\{x_n\}_{n=N}^{\infty}$ are both increasing without bound.*

Recall $\{x_n\}_{n=0}^{\infty}$ is increasing without bound. So there exists a smallest non-negative integer N such that $x_N \geq 0$. Suppose $n \geq N$ then the change of variables, $x_n = X_n - 1$ and $Y_n = y_n$ reduces the system to

$$\begin{cases} X_{n+1} = |X_n| + 1 \\ \\ Y_{n+1} = X_n - |Y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{46}$$

which is System(46). See Theorem .1.43.

System(48)

I next consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ... \tag{48}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that the boundedness characteristic of every solution of System(48) is (U,U).

Global Results

Theorem .1.47 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(48) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U). In particular, $\{x_n\}_{n=N}^{\infty}$ and $\{x_n\}_{n=N}^{\infty}$ are both increasing without bound.*

Recall $\{x_n\}_{n=0}^{\infty}$ is increasing without bound. So there exists a smallest non-negative integer N such that $x_N \geq 0$. Suppose $n \geq N$ then the change of variables, $x_n = X_n - 2$ and $Y_n = y_n$ reduces the system to

$$\begin{cases} X_{n+1} = |X_n| + 1 \\ \\ Y_{n+1} = X_n - |Y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{46}$$

which is System(46). See Theorem .1.43.

Systems(52 and 53)

I next consider the system of piecewise linear difference equations

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ... \tag{52}$$

where the initial conditions x_0 and y_0 are arbitrary real numbers.

I show that the boundedness characteristic of every solution of System(52) is (U,U).

Global Results

Theorem .1.48 *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(52) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U). In particular, $\{x_n\}_{n=N}^{\infty}$ and $\{x_n\}_{n=N}^{\infty}$ are both increasing without bound.*

Recall $\{x_n\}_{n=0}^{\infty}$ is increasing without bound. So there exists a smallest non-negative integer N such that $x_N \geq 0$. Suppose $n \geq N$ then the change of variables, $x_n = X_n + 1$ and $Y_n = y_n$ reduces the system to

$$\begin{cases} X_{n+1} = |X_n| + 1 \\ \\ Y_{n+1} = X_n + |Y_n| \end{cases}, \qquad n = 0, 1, ... \tag{53}$$

which is System(53). By inspection, it is clear that $\{y_n\}_{n=0}^{\infty}$ of System(53) is increasing without bound.

Appendix .2

.2 Summary of Results for the 81 Systems of Piecewise Linear Difference Equations

System(1)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1,-1), and the following two prime period-6 solutions:

$$\mathbf{P}_6^1 = \begin{pmatrix} x_0 & = & 3 & , & y_0 & = & -3 \\ x_1 & = & 5 & , & y_1 & = & -1 \\ x_2 & = & 5 & , & y_2 & = & 3 \\ x_3 & = & 1 & , & y_3 & = & 1 \\ x_4 & = & -1 & , & y_4 & = & -1 \\ x_5 & = & 1 & , & y_5 & = & -3 \end{pmatrix} \text{ or } \mathbf{P}_6^2 = \begin{pmatrix} x_0 & = & \frac{7}{5} & , & y_0 & = & -3 \\ x_1 & = & \frac{17}{5} & , & y_1 & = & -\frac{13}{5} \\ x_2 & = & 5 & , & y_2 & = & -\frac{1}{5} \\ x_3 & = & \frac{21}{5} & , & y_3 & = & -\frac{19}{5} \\ x_4 & = & -\frac{3}{5} & , & y_4 & = & -\frac{3}{5} \\ x_5 & = & \frac{1}{5} & , & y_5 & = & -\frac{11}{5} \end{pmatrix}.$$

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(1) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(1) is either the prime period-6 cycle $\mathbf{P}_6^1$ or the prime period-6 cycle $\mathbf{P}_6^2$.*

See [5].

....

System(2)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (0, -1) and the following two prime period-5 solutions:

$$\mathbf{P}_5^1 = \begin{pmatrix} x_0 &=& -2 &,& y_0 &=& -1 \\ x_1 &=& 2 &,& y_1 &=& -3 \\ x_2 &=& 3 &,& y_2 &=& -1 \\ x_3 &=& 4 &,& y_3 &=& 3 \\ x_4 &=& 0 &,& y_4 &=& 1 \end{pmatrix} \text{ or } \mathbf{P}_5^2 = \begin{pmatrix} x_0 &=& 0 &,& y_0 &=& \frac{1}{7} \\ x_1 &=& -\frac{8}{7} &,& y_1 &=& -\frac{1}{7} \\ x_2 &=& \frac{2}{7} &,& y_2 &=& -\frac{9}{7} \\ x_3 &=& \frac{4}{7} &,& y_3 &=& -1 \\ x_4 &=& \frac{4}{7} &,& y_4 &=& -\frac{3}{7} \end{pmatrix}.$$

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(2) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(2) is either the prime period-5 cycle $\mathbf{P}_5^1$ or the prime period-5 cycle $\mathbf{P}_5^2$.*

See [4]. Also see Manuscript 1.

....

System(3)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point $\left(-\frac{3}{5}, \frac{1}{5}\right)$ and the following two prime period-3 and prime period-4 solutions:

$$\mathbf{P}_3^1 = \begin{pmatrix} x_0 & = & -1 & , & y_0 & = & \frac{1}{3} \\ \\ x_1 & = & -\frac{1}{3} & , & y_1 & = & -\frac{1}{3} \\ \\ x_2 & = & -\frac{1}{3} & , & y_2 & = & \frac{1}{3} \end{pmatrix} \text{ or } \mathbf{P}_3^2 = \begin{pmatrix} x_0 & = & -\frac{1}{5} & , & y_0 & = & -\frac{3}{5} \\ \\ x_1 & = & -\frac{1}{5} & , & y_1 & = & \frac{3}{5} \\ \\ x_2 & = & -\frac{7}{5} & , & y_2 & = & \frac{1}{3} \end{pmatrix}.$$

$$\mathbf{P}_4^1 = \begin{pmatrix} x_0 & = & -1 & , & y_0 & = & 1 \\ \\ x_1 & = & -1 & , & y_1 & = & -1 \\ \\ x_2 & = & 1 & , & y_2 & = & -1 \\ \\ x_3 & = & 1 & , & y_3 & = & 1 \end{pmatrix} \text{ or } \mathbf{P}_4^2 = \begin{pmatrix} x_0 & = & -3 & , & y_0 & = & -1 \\ \\ x_1 & = & 3 & , & y_1 & = & -3 \\ \\ x_2 & = & 5 & , & y_2 & = & 1 \\ \\ x_2 & = & 3 & , & y_2 & = & 5 \end{pmatrix}.$$

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(3) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then either* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is the unique equilibrium* $(\bar{x}, \bar{y})$, *or else there exists a non-negative integer* $N \geq 0$ *such that the solution* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *of System(3) is either the prime period-3 cycle* $\mathbf{P}_3^1$ *or the prime period-3 cycle* $\mathbf{P}_3^2$, *or the prime period-4 cycle* $\mathbf{P}_4^1$ *or the prime period-4 cycle* $\mathbf{P}_4^2$.

See [5].

....

System(4)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n - 1 \end{cases} , \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is (0, -1).

We can see that the system can be reduced to the second order difference equation

$$x_{n+1} = |x_n| - (x_{n-1} - 1) - 1 = |x_n| - x_{n-1}.$$

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(4) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then every solution is periodic with prime period-9.*

See [2].

....

System(5)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Open problem.

....

System(6)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{2}{3}, \frac{1}{3}\right)$.

Open problem.

....

System(7)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has the unique equilibrium point $\left(-\frac{1}{5}, -\frac{3}{5}\right)$ and the following two prime period-3 and prime period-4 solutions:

$$\mathbf{P}_3^1 = \begin{pmatrix} x_0 & = & -\frac{1}{3} & , & y_0 & = & -1 \\ x_1 & = & \frac{1}{3} & , & y_1 & = & -\frac{1}{3} \\ x_2 & = & -\frac{1}{3} & , & y_2 & = & -\frac{1}{3} \end{pmatrix} \text{ or } \mathbf{P}_3^2 = \begin{pmatrix} x_0 & = & \frac{3}{5} & , & y_0 & = & \frac{1}{5} \\ x_1 & = & -\frac{3}{5} & , & y_1 & = & -\frac{1}{5} \\ x_2 & = & -\frac{1}{5} & , & y_2 & = & -\frac{7}{5} \end{pmatrix}.$$

$$\mathbf{P}_4^1 = \begin{pmatrix} x_0 & = & -1 & , & y_0 & = & -1 \\ x_1 & = & 1 & , & y_1 & = & -1 \\ x_2 & = & 1 & , & y_2 & = & 1 \\ x_3 & = & -1 & , & y_3 & = & 1 \end{pmatrix} \text{ or } \mathbf{P}_4^2 = \begin{pmatrix} x_0 & = & 1 & , & y_0 & = & -3 \\ x_1 & = & 3 & , & y_1 & = & 3 \\ x_2 & = & -1 & , & y_2 & = & 5 \\ x_3 & = & -5 & , & y_3 & = & 3 \end{pmatrix}.$$

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(3) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(3) is either the prime period-3 cycle $\mathbf{P}_3^1$ or the prime period-3 cycle $\mathbf{P}_3^2$, or the prime period-4 cycle $\mathbf{P}_4^1$ or the prime period-4 cycle $\mathbf{P}_4^2$.*

See [5].

....

System(8)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n + |y_n| \end{cases} , \qquad n = 0, 1, ...$$

This system possesses the unique equilibrium point $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ and the following two prime period-3 solutions:

$$\mathbf{P}_3^1 = \begin{pmatrix} x_0 & = & 0 & , & y_0 & = & -1 \\ x_1 & = & 0 & , & y_1 & = & 1 \\ x_2 & = & -2 & , & y_2 & = & 1 \end{pmatrix} \text{ or } \mathbf{P}_3^2 = \begin{pmatrix} x_0 & = & 0 & , & y_0 & = & -\frac{1}{3} \\ x_1 & = & -\frac{2}{3} & , & y_1 & = & \frac{1}{3} \\ x_2 & = & -\frac{2}{3} & , & y_2 & = & -\frac{1}{3} \end{pmatrix}.$$

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(8) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(8) is either the prime period-3 cycle $\mathbf{P}_3^1$ or the prime period-3 cycle $\mathbf{P}_3^2$.*

See [3]. Also see Manuscript 2.

....

System(9)

$$\begin{cases} x_{n+1} = |x_n| - y_n - 1 \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (-1, 1).

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(10) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the unique equilibrium $(\bar{x}, \bar{y})$.*

See [5].

....

System(10)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 0).

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(10) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=6}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$.*

See Theorem .1.2.

....

System(11)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n - |y_n| \end{cases} , \qquad n = 0, 1, ...$$

It has the unique equilibrium point (0, 0).

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(11) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the unique equilibrium $(\bar{x}, \bar{y})$.*

See [5].

....

System(12)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases} , \qquad n = 0, 1, ...$$

It has the unique equilibrium point $\left(-\frac{1}{5}, \frac{2}{5}\right)$ and the following two prime period-3 solutions:

$$\mathbf{P}_3^1 = \begin{pmatrix} x_0 & = & -1 & , & y_0 & = & 0 \\ \\ x_1 & = & 1 & , & y_1 & = & 0 \\ \\ x_2 & = & 1 & , & y_2 & = & 2 \end{pmatrix} \quad \text{or} \quad \mathbf{P}_3^2 = \begin{pmatrix} x_0 & = & -\frac{1}{3} & , & y_0 & = & 0 \\ \\ x_1 & = & \frac{1}{3} & , & y_1 & = & \frac{2}{3} \\ \\ x_2 & = & -\frac{1}{3} & , & y_2 & = & \frac{2}{3} \end{pmatrix}.$$

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(12) with $(x_0, y_0) \in \mathbf{R}^2$. Then either $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the unique equilibrium $(\bar{x}, \bar{y})$, or else there exists a non-negative integer $N \geq 0$ such that the solution $\{(x_n, y_n)\}_{n=N}^{\infty}$ of System(12) is either the prime period-3 cycle $\mathbf{P}_3^1$ or the prime period-3 cycle $\mathbf{P}_3^2$.*

See [3]. Also see Theorem .1.8.

....

System(13)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 0).

Open problem. Note: The change of variables, $y_n = Y_n - 1$, reduces it to System(23), the Gingerbread man map.

See [1].

....

System(14)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 0).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(14) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is a prime period-9 cycle* $\mathbf{P}_3^2$.

See [2].

....

System(15)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point $\left(-\frac{1}{3}, \frac{2}{3}\right)$.

Open problem.

Note: The change of variables, $y_n = Y_n + 1$, reduces it to System(5).

....

System(16)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases} , \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 0) and the following two prime period-5 solutions:

$$\mathbf{P}_5^1 = \begin{pmatrix} x_0 & = & -1 & , & y_0 & = & 0 \\ x_1 & = & 1 & , & y_1 & = & -2 \\ x_2 & = & 3 & , & y_2 & = & 2 \\ x_3 & = & 1 & , & y_3 & = & 4 \\ x_4 & = & -3 & , & y_4 & = & 4 \end{pmatrix} \text{ or } \mathbf{P}_5^2 = \begin{pmatrix} x_0 & = & 1 & , & y_0 & = & \frac{4}{7} \\ x_1 & = & \frac{3}{7} & , & y_1 & = & -\frac{4}{7} \\ x_2 & = & -\frac{1}{7} & , & y_2 & = & 0 \\ x_3 & = & \frac{1}{7} & , & y_3 & = & -\frac{8}{7} \\ x_4 & = & \frac{9}{7} & , & y_4 & = & \frac{2}{7} \end{pmatrix}.$$

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(16) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then either* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is the unique equilibrium* $(\bar{x}, \bar{y})$*, or else there exists a non-negative integer* $N \geq 0$ *such that the solution* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *of System(16) is either the prime period-5 cycle* $\mathbf{P}_5^1$ *or the prime period-5 cycle* $\mathbf{P}_5^2$.

Note: The change of variables, $x_n = -Y_n$ and $y_n = X_n$, reduces it to System(2).

See [4]. Also see Manuscript 1.

....

System(17)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ \\ y_{n+1} = x_n + |y_n| \end{cases} , \qquad n = 0, 1, ...$$

It has the unique equilibrium point (0, 0).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(17) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(0, 0)$.

See [5].

....

System(18)

$$\begin{cases} x_{n+1} = |x_n| - y_n \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (-1, 2).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(18) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(-1, 2)$*.*

See [5].

....

System(19)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (3, 1).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(19) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(3, 1)$*.*

See Theorem .1.10.

....

System(20)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (2, 1).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(20) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(2, 1)$*.*

See [5].

....

System(21)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 1).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(21) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(1, 1)$.

See [5].

....

System(22)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ...$$

It possesses the unique equilibrium point (2, 1).

Open problem.

....

System(23)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ...$$

It possesses the unique equilibrium point (1, 1).

This system is the Gingerbread man map.

Open problem.

....

System(24)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 0).

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(24) with $(x_0, y_0) \in \mathbf{R}^2$. Then every solution is periodic with prime period-9.*

Note: The change of variables, $y_n = Y_n + 2$, reduces it to System(4).

See [2].

....

System(25)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

It has the unique equilibrium point (1, 1), and the following two prime period-6 solutions:

$$\mathbf{P}_6^1 = \begin{pmatrix} x_0 & = & 3 & , & y_0 & = & 3 \\ x_1 & = & 1 & , & y_1 & = & 5 \\ x_2 & = & -3 & , & y_2 & = & 5 \\ x_3 & = & -1 & , & y_3 & = & 1 \\ x_4 & = & 1 & , & y_4 & = & -1 \\ x_5 & = & 3 & , & y_5 & = & 1 \end{pmatrix} \text{ or } \mathbf{P}_6^2 = \begin{pmatrix} x_0 & = & -3 & , & y_0 & = & \frac{7}{5} \\ x_1 & = & \frac{13}{5} & , & y_1 & = & \frac{17}{5} \\ x_2 & = & \frac{1}{5} & , & y_2 & = & 5 \\ x_3 & = & -\frac{19}{5} & , & y_3 & = & \frac{21}{5} \\ x_4 & = & \frac{3}{5} & , & y_4 & = & -\frac{3}{5} \\ x_5 & = & -\frac{11}{5} & , & y_5 & = & \frac{1}{5} \end{pmatrix}.$$

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(25) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either the unique equilibrium solution, the prime period-6 cycle $\mathbf{P}_6^1$*

or the prime period-6 cycle $\mathbf{P}_6^2$.

See [5].

....

System(26)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| \end{cases} , \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is (0, 1).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(26) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(0, 1)$.

See Theorem .1.1.

....

System(27)

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases} , \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is (-1, 3).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(27) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the unique equilibrium* $(-1, 3)$.

See Theorem .1.9.

....

System(28)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The boundedness characteristic of this system is (B, U).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(28) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then every solution is eventually period-2 in* $\{x_n\}$. *More precisely, if* $|x_0| < 1$ *then the period-2 solution is* $\{-|x_0|, |x_0| - 1\}$. *Every solution is unbounded in* $\{y_n\}$.

See Theorem .1.17.

....

System(29)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The boundedness characteristic of this system is (B, U).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(29) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then every solution is eventually period-2 in* $\{x_n\}$ *and every solution is unbounded in* $\{y_n\}$.

See Theorem .1.18.

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System(30)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{2}, \frac{1}{3}\right)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(30) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the equilibrium solution* $\left(-\frac{1}{2}, \frac{1}{4}\right)$, *periodic with prime period-2 or periodic with prime period-4.* See Theorem .1.19

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System(31)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(31) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually periodic with period-2.*

See Theorem .1.23.

....

System(32)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n \end{cases} , \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(32) with* $(x_0, y_0) \in \mathbf{R}^2$*. Every solution of this system is eventually periodic with period-2. In particular, if* $|x_0| = m + \alpha$*, where* $m \in \{0, 1, 2, \ldots\}$*, and* $\alpha \in \mathbf{R}$ *such that* $0 \leq \alpha < 1$ *then the period-2 solution in* $\{x_n\}$ *is* $\{\alpha - 1, -\alpha\}$*.*

See Theorem .1.25.

....

System(33)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n + 1 \end{cases} , \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(33) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually periodic with period-2.*

See Theorem .1.24.

....

System(34)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases} , \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{2}, -\frac{3}{4}\right)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(34) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually the equilibrium solution* $\left(-\frac{1}{2}, \frac{1}{4}\right)$*, periodic with prime period-2 or periodic with prime period-4.* See [5].

....

System(35)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ...$$

The unique equilibrium point of this system is $\left(-\frac{1}{2}, -\frac{1}{4}\right)$.

Conjecture: *The boundedness characteristic is (B, B) and there exist prime period-4 solutions.*

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System(36)

$$\begin{cases} x_{n+1} = |x_n| - 1 \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The boundedness characteristic of this system is (B, U).

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(36) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then every solution is eventually (not necessarily prime) period-2 in* $\{x_n\}$ *and every solution is unbounded in* $\{y_n\}$. *In particular, if* $|x_0| = m + \alpha$, *where* $m \in \{-1, 0, 1, 2, \ldots\}$,*and* $\alpha \in \mathbf{R}$ *such that* $0 < \alpha \leq 1$ *then the period-2 solution in* $\{x_n\}$ *is* $\{\alpha - 1, -\alpha\}$.

See [5].

....

System(37)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium lines are:

$$\text{if } y \geq 0, \text{ then } y = \frac{1}{2}x - \frac{1}{2}, \text{ and if } y < 0, \text{ then } x = 1.$$

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(37) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, a period-2 solution, or it has the boundedness character (B,U).*

See Theorem .1.26.

....

System(38)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium lines are:

$$\text{if } y \geq 0, \text{ then } y = \frac{1}{2}x, \text{ and} \qquad \text{if } y < 0, \text{ then } x = 0.$$

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(38) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium solution or a period-2 solution.*

See Theorem .1.31.

....

System(39)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium lines are:

$$\text{if } y \geq 0, \text{ then } y = \frac{1}{2}x + \frac{1}{2}, \text{ and} \qquad \text{if } y < 0, \text{ then } x = -1.$$

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(39) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium solution or a period-2 solution. In particular, if the initial condition is* (x_0, y_0) *then the prime period-2 solution in* $\{y_n\}$ *is* $\{x_0 - y_0 + 1, y_0\}$.

See [5].

....

System(40)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ...$$

If the initial condition is (x_0, y_0) then the equilibrium point is $(|x_0|, |x_0| - 1)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(40) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually an equilibrium solution.*

Note: The change of variables: $y_n = Y_n - 1$, reduces it to System(41).

....

System(41)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ...$$

If the initial condition is (x_0, y_0) then the equilibrium point is $(|x_0|, |x_0|)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(41) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually an equilibrium solution.*

Clearly, $x_n = y_n = |x_0|$ for $n \geq 2$.

....

System(42)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ...$$

If the initial condition is (x_0, y_0) then the equilibrium point is $(|x_0|, |x_0| + 1)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(42) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is eventually an equilibrium solution.*

Note: The change of variables: $y_n = Y_n + 1$, reduces it to System(41).

....

System(43)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium lines are:

if $y \geq 0$, then $x = 1$, and if $y < 0$, then $y = \frac{1}{2}x - \frac{1}{2}$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(43) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, a period-2 solution, or it has the boundedness character (B,U).*

See Theorem .1.37.

....

System(44)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium line is the positive y-axis.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(44) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (B,U).*

See Theorem .1.41.

....

System(45)

$$\begin{cases} x_{n+1} = |x_n| \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(45) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (B,U).*

See Theorem .1.42.

....

System(46)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(46) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

See Theorem .1.43.

....

System(47)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(47) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

See Theorem .1.46.

....

System(48)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(48) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U).*

See Theorem .1.47.

....

System(49)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n - 1 \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(49) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U).*

Note: The change of variables: $y_n = Y_n - 1$, reduces it to System(50).

....

System(50)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(50) with $(x_0, y_0) \in \mathbf{R}^2$. Then $\{(x_n, y_n)\}_{n=0}^{\infty}$ has the boundedness character (U,U).*

....

System(51)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(51) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

Note: The change of variables: $y_n = Y_n + 1$, reduces it to System(50).

....

System(52)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(52) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

See Theorem .1.48.

....

System(53)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(53) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(54)

$$\begin{cases} x_{n+1} = |x_n| + 1 \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(54) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(55)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(3, 1)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(55) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, or periodic with period-2, or it has the boundedness character (U,U).*

....

System(56)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(2, 1)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(56) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, or periodic with period-2, or it has the boundedness character (U,U).*

....

System(57)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium points of this system are $(1, 1), (-1, -1)$, and $(-\frac{1}{3}, \frac{1}{3})$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(57) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, or periodic with period-2, or it has the boundedness character (U,U).*

....

System(58)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium points of this system are $(2, 1)$, and $(-2, -3)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(58) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, or periodic with period-2, or it has the boundedness character (U,U).*

....

System(59)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium points of this system are $(1, 1)$, and $(-1, -1)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(59) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

....

System(60)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n + 1 \end{cases} , \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(0, 1)$.

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(60) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium point or it has the boundedness character (U,U).*

Note: The change of variables: $y_n = Y_n + 2$, reduces it to System(76).

....

System(61)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases} , \qquad n = 0, 1, ...$$

The equilibrium points of this system are $(1, 1)$, and $(-1, -1)$.

Conjecture: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(61) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium point or it has the boundedness character (U,U).*

....

System(62)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n + |y_n| \end{cases} , \qquad n = 0, 1, ...$$

The equilibrium points of this system are $(0, 1)$, and $\left(-\frac{2}{3}, -\frac{1}{3}\right)$.

Conjecture: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(62) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium point or it has the boundedness character (U,U).*

....

System(63)

$$\begin{cases} x_{n+1} = |x_n| + y_n - 1 \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system does not possess an equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(63) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(64)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has the equilibrium point $(1, 0)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(64) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

Note: The change of variables, $x_n = -Y_n$ and $y_n = -X_n$, reduces it to System(74).

....

System(65)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n - |y_n| \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(0, 0)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(65) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

....

System(66)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has the equilibrium point $(-1, -2)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(66) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point, or periodic with (not necessarily prime) period-2, or it has the boundedness character (U,U).*

Note: The change of variables, $x_n = -Y_n$ and $y_n = -X_n$, reduces it to System(56).

....

System(67)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n - 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has the equilibrium points $(1, 0)$ and $(-1, -2)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(67) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

Note: The change of variables, $y_n = Y_n - 1$, reduces it to System(59).

....

System(68)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(0, 0)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(68) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then* *the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

Note: The change of variables, $y_n = Y_n - 1$, reduces it to System(60).

....

System(69)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(69) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

Note: The change of variables: $y_n = Y_n + 1$, reduces it to System(77).

....

System(70)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium points of this system are $(1, 0)$, and $\left(-\frac{1}{3}, -\frac{2}{3}\right)$.

Conjecture: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(70) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

Note: The change of variables, $x_n = Y_n$ and $y_n = X_n$, reduces it to System(62).

....

System(71)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(0, 0)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(71) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

....

System(72)

$$\begin{cases} x_{n+1} = |x_n| + y_n \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(72) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(73)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n - |y_n| - 1 \end{cases}, \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(1, -1)$.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(73) with* $(x_0, y_0) \in \mathbf{R}^2$*. Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is either an equilibrium point or it has the boundedness character (U,U).*

See [5].

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System(74)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n - |y_n| \end{cases} , \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(0, -1)$.

Theorem: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(74) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium point or it has the boundedness character (U,U).*

....

System(75)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n - |y_n| + 1 \end{cases} , \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(-1, -3)$.

Conjecture: *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(75) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either an equilibrium point, or periodic with period-2, or it has the boundedness character (U,U).*

....

System(76)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n - 1 \end{cases} , \qquad n = 0, 1, ...$$

The equilibrium point of this system is $(0, -1)$.

Theorem:*Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(76) with $(x_0, y_0) \in \mathbf{R}^2$. Then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is either the equilibrium point or it has the boundedness character (U,U).*

Note: The change of variables, $y_n = Y_n - 2$, reduces it to System(60).

....

System(77)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The Theorem:*Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(77) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

Note: The change of variables, $y_n = Y_n - 1$, reduces it to System(69).

....

System(78)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n + 1 \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The Theorem:*Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(78) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(79)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| - 1 \end{cases} , \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The Theorem:*Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(79) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(80)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

The Theorem:*Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(80) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

....

System(81)

$$\begin{cases} x_{n+1} = |x_n| + y_n + 1 \\ \\ y_{n+1} = x_n + |y_n| + 1 \end{cases}, \qquad n = 0, 1, ...$$

This system has no equilibrium point.

Theorem: *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System(81) with* $(x_0, y_0) \in \mathbf{R}^2$. *Then the solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *has the boundedness character (U,U).*

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Appendix .3

.3 On the Global Behavior of $x_{n+1} = \dfrac{\alpha_1}{x_n + y_n}$ and $y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + y_n}{B_2 x_n + y_n}$

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.3.1 Abstract

We investigate the system of rational difference equations in the title, where the parameters and initial conditions are positive real values. We show that the system is permanent. We also find sufficient conditions to insure that every positive solution of the system converges.

.3.2 Introduction

We show that the system of rational difference equations

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{x_n + y_n} \\ \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + y_n}{B_2 x_n + y_n} \end{cases}, \quad n = 0, 1, \ldots \tag{.5}$$

is permanent, where the parameters $\alpha_1, \alpha_2, \beta_2, B_2$ and the initial conditions x_0, y_0 of the system are positive real numbers. We actually show that there exist positive real numbers l_1, l_2, L_1, L_2 such that for every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (.5), we have

$$l_1 < x_n < L_1 \qquad \text{and} \qquad l_2 < y_n < L_2 \qquad \text{for n} \geq 4.$$

We also find sufficient conditions to insure that every positive solution of system (.5) converges.

During the last four years we have been interested in the boundedness character and the global behavior of systems of rational difference equations. This paper is part of a general project which involves the system of rational difference equations

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\ \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{cases}, \quad n = 0, 1, \ldots \tag{.6}$$

which includes 2401 special cases. In the numbering system which was introduced by Camouzis, Kulenović, Ladas, and Merino in ([6]), system (2) is referred to as System(12,48). Related work has recently been given in ([1]-[10]) and ([13]-[15]).

This theorem gives a sufficient condition to insure that a system of k continuous functions have a unique positive equilibrium, and it is a global attractor.

Theorem .3.1 *([11]) Let k be a positive integer. For $i \in \{1, 2, \dots, k\}$, assume $[a_i, b_i]$ is a closed and bounded interval of real numbers, and let $F^i : [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k] \to [a_i, b_i]$ be a continuous function. For each $i, j \in \{1, 2, \dots, k\}$, let $M_{i,j} : [a_i, b_i] \to [a_i, b_i]$ and $m_{i,j} : [a_i, b_i] \to [a_i, b_i]$ be defined as follows: given $m_i, M_i \in [a_i, b_i]$*

set

$$M_{i,j}(m_i, M_i) = \begin{cases} M_i, & \text{if } \mathrm{F^j} \text{ is increasing in } z_i \\ m_i, & \text{if } \mathrm{F^j} \text{ is non} - \text{increasing in } z_i \end{cases}$$

and

$$m_{i,j}(m_i, M_i) = M_{i,j}(M_i, m_i).$$

Assume that for each $i \in \{1, 2, \dots, k\}$, that the function F^i, satisfies the following conditions:

1. *$F^i(z_1, z_2, \dots, z_k)$ is weakly monotonic in each of its arguments.*

2. *If $M_1, M_2, \dots, M_k, m_1, m_2, \dots, m_k$, where $m_i \leq M_i$ for each $i \in \{1, 2, \dots, k\}$, is a solution of the system of 2k equations:*

$$\begin{cases} M_i = F^i(M_{1,i}(m_1, M_1), M_{2,i}(m_2, M_2), \dots, M_{k,i}(m_k, M_k)) \\ m_i = F^i(m_{1,i}(m_1, M_1), m_{2,i}(m_2, M_2), \dots, m_{k,i}(m_k, M_k)) \end{cases}$$

   then

$$M_i = m_i, \text{for all } i \in \{1, 2, \dots, k\}.$$

Then the system of k difference equations:

$$\begin{cases} x^1_{n+1} = F^1(x^1_n, x^2_n, \dots, x^k_n) \\ x^2_{n+1} = F^2(x^1_n, x^2_n, \dots, x^k_n) \\ \vdots \\ x^k_{n+1} = F^k(x^1_n, x^2_n, \dots, x^k_n) \end{cases}, \quad n = 0, 1, \dots$$

with initial condition $(x_0^1, x_0^2, \ldots, x_0^k) \in [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_k, b_k]$, *has exactly one equilibrium point* $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^k)$, *and it is a global attractor.*

.3.3 Permanence

Recall System(12,48)

$$\begin{cases} x_{n+1} &= \dfrac{\alpha_1}{x_n + y_n} \\ & \\ y_{n+1} &= \dfrac{\alpha_2 + \beta_2 x_n + y_n}{B_2 x_n + y_n} \end{cases}, \quad n = 0, 1, \ldots \tag{2}$$

where the parameters $\alpha_1, \alpha_2, \beta_2, B_2$ and the initial conditions x_0, y_0 of the system are positive real numbers.

System(12,48) is permanent if there exist positive real numbers l_1, L_1, l_2, L_2 such that for every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System(12,48), there exists an integer $N \geq 0$ (possibly depending upon the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System(12,48) such that

$$l_1 < x_n < L_1 \qquad \text{and} \qquad l_2 < y_n < L_2$$

for every integer $n \geq N$.

In view of the above, set

$$U = \frac{\alpha_1}{\alpha_2} \max\{B_2, 1\}$$

and define l_1, L_1, l_2, L_2 as follows:

1. $l_1 = \dfrac{\alpha_1}{(U+1)[\alpha_2(B_2 U + 1) + (\beta_2 U + 1)]}$
2. $L_1 = U[\alpha_2(B_2 U + 1) + (\beta_2 U + 1)]$
3. $l_2 = \dfrac{1}{B_2 U + 1}$
4. $L_2 = \alpha_2(B_2 U + 1) + (\beta_2 U + 1).$

In particular, note that

$$l_1 = \frac{\alpha_1}{L_1 + L_2} \qquad \text{and} \qquad L_1 = UL_2.$$

Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a positive solution of System(12,48).

Given a non-negative integer $n \geq 0$, note that

$$\begin{aligned}
\frac{x_{n+1}}{y_{n+1}} &= \frac{\alpha_1}{x_n + y_n} \cdot \frac{B_2 x_n + y_n}{\alpha_2 + \beta_2 x_n + y_n} = \frac{\alpha_1}{\alpha_2 + \beta_2 x_n + y_n} \cdot \frac{B_2 x_n + y_n}{x_n + y_n} \\
&< \frac{\alpha_1}{\alpha_2} \cdot \frac{\max\{B_2, 1\}(x_n + y_n)}{x_n + y_n} = \frac{\alpha_1}{\alpha_2} \cdot \max\{B_2, 1\} \\
&= U.
\end{aligned}$$

Thus

$$x_n < U y_n \qquad \text{for all} \qquad n \geq 1.$$

Hence if $n \geq 1$ is an integer, then

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + y_n}{B_2 x_n + y_n} > \frac{y_n}{B_2 x_n + y_n} > \frac{y_n}{B_2 U y_n + y_n} = \frac{1}{B_2 U + 1} = l_2$$

and so

$$y_n > l_2 \qquad \text{for all} \qquad n \geq 2.$$

Hence if $n \geq 2$ is an integer, then

$$\begin{aligned}
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + y_n}{B_2 x_n + y_n} < \frac{\alpha_2 + \beta_2 x_n + y_n}{y_n} < \frac{\alpha_2 + \beta_2 U y_n + y_n}{y_n} \\
&= \frac{\alpha_2}{y_n} + (\beta_2 U + 1) < \frac{\alpha_2}{l_2} + (\beta_2 U + 1) = L_2.
\end{aligned}$$

That is, for every integer $n \geq 3$ we have

$$l_2 < y_n < L_2.$$

Now if $n \geq 3$ is an integer, then

$$x_n < U y_n < UL_2 = L_1.$$

Hence for every integer $n \geq 3$, we also have

$$x_{n+1} = \frac{\alpha_1}{x_n + y_n} > \frac{\alpha_1}{L_1 + L_2} = l_1.$$

In conclusion we see that the following theorem is true.

Theorem .3.2 *System(12,48) is permanent. In particular, let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a positive solution of System(12,48). Then for every integer* $n \geq 4$, *we have*

$$l_1 < x_n < L_1$$

and

$$l_2 < y_n < L_2.$$

.3.4 Global Attractivity Analysis

In this section we give the result in the case $\alpha_1, \alpha_2, B_2, \beta_2 \in (0, \infty)$. The case $\alpha_1, \alpha_2, B_2 \in (0, \infty)$ and $\beta_2 = 0$ was given in [13].

The following theorem gives a sufficient condition for the unique equilibrium of System(12,48) to be a global attractor.

Theorem .3.3 *Suppose that* $B_2 \geq \beta_2$ *and*

$$(\alpha_2 B_2 + \beta_2)(B_2 - \beta_2)\alpha_1^2(\max\{B_2, 1\})^2 + (\alpha_2 + 1)(B_2 - \beta_2)\alpha_1\alpha_2 \max\{B_2, 1\} \leq \alpha_2^3.$$

Then System(12,48) has a unique positive equilibrium point $(\bar{x}, \bar{y})$, *and every positive solution of System(12,48) converges to* $(\bar{x}, \bar{y})$.

Proof: For $(x, y) \in (0, \infty) \times (0, \infty)$, set

$$f(x, y) = \frac{\alpha_1}{x + y} \qquad \text{and} \qquad g(x, y) = \frac{\alpha_2 + \beta_2 x + y}{B_2 x + y}$$

and let $\mathcal{R} = [l_1, L_1] \times [l_2, L_2]$, where l_1, L_1, l_2, and L_2 are as defined in section 3.2.

We have the following lemma.

Lemma .3.4 *For $(x, y) \in \mathcal{R}$, the following statements are true.*

1. $\dfrac{\partial f}{\partial x}(x, y) < 0$
2. $\dfrac{\partial f}{\partial y}(x, y) < 0$
3. $\dfrac{\partial g}{\partial x}(x, y) < 0$
4. $\dfrac{\partial g}{\partial y}(x, y) \leq 0.$

Proof: The proofs of Statements (1), (2) are trivial and will be omitted. For the proofs of Statements (3) and (4), let $(x, y) \in \mathcal{R}$.

We shall first show that Statement (3) is true. Note that as $B_2 \geq \beta_2$, we have

$$\frac{\partial g}{\partial x}(x, y) = \frac{(B_2 x + y)\beta_2 - (\alpha_2 + \beta_2 x + y)B_2}{(B_2 x + y)^2} = \frac{-\alpha_2 B_2 - (B_2 - \beta_2)y}{(B_2 x + y)^2} < 0$$

and so Statement (3) is true.

Finally, we shall show that Statement (4) is true. Note that the proof that Statement (4) is true depends upon the fact that $(x, y) \in \mathcal{R}$. We have

$$\begin{aligned}
\frac{\partial g}{\partial y}(x, y) &= \frac{(B_2 - \beta_2)x - \alpha_2}{(B_2 x + y)^2} \leq \frac{(B_2 - \beta_2)L_1 - \alpha_2}{(B_2 x + y)^2} \\
&= \frac{(B_2 - \beta_2)UL_2 - \alpha_2}{(B_2 x + y)^2} \\
&= \frac{(B_2 - \beta_2)U[\alpha_2(B_2 U + 1) + (\beta_2 U + 1)] - \alpha_2}{(B_2 x + y)^2} \\
&= \frac{(B_2 - \beta_2)U[\alpha_2 B_2 U + \alpha_2 + \beta_2 U + 1] - \alpha_2}{(B_2 x + y)^2} \\
&= \frac{(B_2 - \beta_2)U[(\alpha_2 B_2 + \beta_2)U + (\alpha_2 + 1)] - \alpha_2}{(B_2 x + y)^2} \\
&= \frac{(\alpha_2 B_2 + \beta_2)(B_2 - \beta_2)U^2 + (\alpha_2 + 1)(B_2 - \beta_2)U - \alpha_2}{(B_2 x + y)^2} \\
&= \frac{(\alpha_2 B_2 + \beta_2)(B_2 - \beta_2)\alpha_1^2(\max\{B_2, 1\})^2}{\alpha_2^2(B_2 x + y)^2} \\
&\quad + \frac{(\alpha_2 + 1)(B_2 - \beta_2)\alpha_1\alpha_2 \max\{B_2, 1\} - \alpha_2^3}{\alpha_2^2(B_2 x + y)^2} \\
&\leq 0
\end{aligned}$$

and the proof of the lemma is complete. $\square$

Let $T : (0,\infty) \times (0,\infty) \to (0,\infty) \times (0,\infty)$ be given by

$$T(x,y) = (f(x,y), g(x,y)).$$

Lemma .3.5 $T[\mathcal{R}] \subset \mathcal{R}$.

Proof: Let $(x,y) \in \mathcal{R}$. It suffices to show that

$$f(x,y) \in [l_1, L_1] \qquad \text{and} \qquad g(x,y) \in [l_2, L_2].$$

1. We shall first show that $l_1 \leq f(x,y)$.

   Note that

   $$l_1 = \frac{\alpha_1}{L_1 + L_2} \leq \frac{\alpha_1}{x+y} = f(x,y)$$

   as was to be shown.

2. We shall next show that $f(x,y) \leq L_1$.

   We have

   $$f(x,y) = \frac{\alpha_1}{x+y} \leq \frac{\alpha_1}{l_1 + l_2}$$

   and so it suffices to show that

   $$\frac{\alpha_1}{l_1 + l_2} \leq L_1.$$

   That is, we must show that

   $$\alpha_1 \leq L_1(l_1 + l_2).$$

   Now

   $$L_1(l_1+l_2) = UL_2(l_1+l_2) = UL_2\left(\frac{\alpha_1}{L_1+L_2} + \frac{1}{B_2U+1}\right) = \frac{UL_2\alpha_1}{L_1+L_2} + \frac{UL_2}{B_2U+1}$$

   and so

   $$\alpha_1 \leq L_1(l_1 + l_2)$$

if and only if

$$\alpha_1 \leq \frac{UL_2\alpha_1}{L_1 + L_2} + \frac{UL_2}{B_2U + 1}$$

if and only if

$$\alpha_1(L_1 + L_2)(B_2U + 1) \leq UL_2\alpha_1(B_2U + 1) + UL_2(L_1 + L_2)$$

if and only if

$$\alpha_1(B_2U + 1)L_2 \leq UL_2(L_1 + L_2)$$

if and only if

$$\alpha_1 B_2 U + \alpha_1 \leq UL_1 + U(B_2U\alpha_2 + \alpha_2 + \beta_2 U + 1)$$

if and only if

$$\alpha_1 B_2 U + \alpha_1 \leq UL_1 + B_2U\alpha_2\frac{\alpha_1}{\alpha_2}\max\{B_2, 1\} + \alpha_2 U + \beta_2 U^2 + U$$

if and only if

$$\alpha_1 \leq UL_1 + \alpha_1 B_2 U(\max\{B_2, 1\} - 1) + \alpha_2 U + \beta_2 U^2 + U$$

if and only if

$$\alpha_1 \leq UL_1 + B_2U\alpha_1(\max\{B_2, 1\} - 1) + \alpha_2\frac{\alpha_1}{\alpha_2}\max\{B_2, 1\} + \beta_2 U^2 + U$$

if and only if

$$0 \leq UL_1 + B_2U\alpha_1(\max\{B_2, 1\} - 1) + \alpha_1(\max\{B_2, 1\} - 1) + \beta_2 U^2 + U$$

which is true, and so $f(x, y) \leq L_1$ as was to be shown.

3. We shall next show that $l_2 \leq g(x, y)$.

   Recall that $l_2 = \dfrac{1}{B_2U + 1}$ and by Lemma 3.2, $g(x, y)$ is decreasing in both arguments.

   So,

$$g(x, y) \geq g(L_1, L_2) = \frac{\alpha_2 + \beta_2 L_1 + L_2}{B_2 L_1 + L_2}$$

   and so it suffices to show that

$$\frac{1}{B_2U + 1} \leq \frac{\alpha_2 + \beta_2 L_1 + L_2}{B_2 L_1 + L_2}.$$

   Now

$$\frac{1}{B_2U + 1} \leq \frac{\alpha_2 + \beta_2 L_1 + L_2}{B_2 L_1 + L_2}$$

   if and only if

$$B_2 L_1 + L_2 \leq (B_2U + 1)(\alpha_2 + \beta_2 L_1 + L_2)$$

   if and only if

$$B_2 U L_2 + L_2 \leq B_2 U \alpha_2 + B_2 U \beta_2 L_1 + B_2 U L_2 + \alpha_2 + \beta_2 L_1 + L_2$$

   if and only if

$$0 \leq B_2 U \alpha_2 + B_2 U \beta_2 L_1 + \alpha_2 + \beta_2 L_1$$

   which is true. Hence $l_2 \leq g(x, y)$.

4. Finally, we shall show that $g(x, y) \leq L_2$.

   Recall that

$$L_2 = B_2 U \alpha_2 + \alpha_2 + \beta_2 U + 1 = \alpha_2(B_2U + 1) + \beta_2 U + 1 = \frac{\alpha_2}{l_2} + \beta_2 U + 1.$$

By Lemma 3.2, $g(x, y) \leq g(l_1, l_2) = \dfrac{\alpha_2 + \beta_2 l_1 + l_2}{B_2 l_1 + l_2}$, and so it suffices to show that

$$\frac{\alpha_2 + \beta_2 l_1 + l_2}{B_2 l_1 + l_2} \leq B_2 U \alpha_2 + \alpha_2 + \beta_2 U + 1.$$

Now

$$\frac{\alpha_2 + \beta_2 l_1 + l_2}{B_2 l_1 + l_2} \leq B_2 U \alpha_2 + \alpha_2 + \beta_2 U + 1$$

if and only if

$$\alpha_2 + \beta_2 l_1 + l_2 \leq (B_2 U \alpha_2 + \alpha_2 + \beta_2 U + 1)(B_2 l_1 + l_2)$$

if and only if

$$\alpha_2 + \beta_2 l_1 + l_2 \leq B_2 U \alpha_2 B_2 l_1 + B_2 U \alpha_2 l_2 + \alpha_2 B_2 l_1 + \alpha_2 l_2 + \beta_2 U B_2 l_1 + \beta_2 U l_2 + B_2 l_1 + l_2$$

if and only if

$$\alpha_2 + \beta_2 l_1 \leq \alpha_2 B_2^2 U l_1 + \alpha_2 B_2 l_1 + B_2 l_1 + (B_2 U + 1)\alpha_2 l_2 + \beta_2 U B_2 l_1 + \beta_2 U l_2$$

if and only if

$$\alpha_2 + \beta_2 l_1 \leq \alpha_2 B_2^2 U l_1 + \alpha_2 B_2 l_1 + B_2 l_1 + \frac{1}{l_2}\alpha_2 l_2 + \beta_2 U B_2 l_1 + \beta_2 U l_2$$

if and only if

$$\beta_2 l_1 \leq \alpha_2 B_2^2 U l_1 + \alpha_2 B_2 l_1 + B_2 l_1 + \beta_2 l_1 B_2 U + \beta_2 U l_2$$

if and only if

$$0 \leq \left(\alpha_2 B_2^2 U l_1 + \alpha_2 B_2 l_1 + B_2 l_1\right) + \left[\beta_2 l_1 (B_2 U - 1) + \beta_2 U l_2\right].$$

It suffices to show that $\beta_2 l_1 (B_2 U - 1) + \beta_2 U l_2 \geq 0$.

Now

$$\begin{aligned}
\beta_2 l_1(B_2U-1)+\beta_2 U l_2 &= \beta_2\frac{\alpha_1}{L_1+L_2}(B_2U-1)+\beta_2 U\left(\frac{1}{B_2U+1}\right)\\
&= \frac{\alpha_1\beta_2}{(U+1)L_2}(B_2U-1)+\frac{\beta_2 U}{B_2U+1}\\
&= \frac{\alpha_1\beta_2}{U+1}\left(\frac{1}{\frac{\alpha_2}{l_2}+\beta_2U+1}\right)(B_2U-1)+\frac{\beta_2U}{B_2U+1}\\
&= \frac{\alpha_1\beta_2}{U+1}\cdot\frac{1}{\alpha_2(B_2U+1)+(\beta_2U+1)}\cdot(B_2U-1)\\
&\quad +\frac{\beta_2U}{B_2U+1}
\end{aligned}$$

Note that

$$0 \leq \alpha_1\beta_2\cdot\frac{1}{\alpha_2(B_2U+1)+(\beta_2U+1)}\cdot(B_2U-1)+\frac{\beta_2U}{B_2U+1}$$

if and only if

$$0 \leq \alpha_1\beta_2(B_2U-1)(B_2U+1)+\beta_2U\left[\alpha_2(B_2U+1)+(\beta_2U+1)\right]$$

if and only if

$$0 \leq \alpha_1\beta_2(B_2^2U^2-1)+\alpha_2\beta_2(B_2U+1)U+\beta_2U(\beta_2U+1)$$

if and only if

$$\begin{aligned}
\alpha_1\beta_2 &\leq \alpha_1\beta_2B_2^2U^2+\alpha_2\beta_2B_2U^2+\alpha_2\beta_2U+\beta_2^2U^2+\beta_2U\\
&= \alpha_1\beta_2B_2^2U^2+\alpha_2\beta_2B_2U^2+\alpha_2\beta_2\frac{\alpha_1}{\alpha_2}\max\{B_2,1\}+\beta_2^2U^2+\beta_2U
\end{aligned}$$

which is true because

$$\alpha_1\beta_2 \leq \alpha_2\beta_2\frac{\alpha_1}{\alpha_2}\max\{B_2,1\}.$$

Therefore it follows that

$$0 \le \left(\alpha_2 B_2^2 U l_1 + \alpha_2 B_2 l_1 + B_2 l_1\right) + \left[\beta_2 l_1 (B_2 U - 1) + \beta_2 U l_2\right].$$

□

Let $((m_1, M_1), (m_2, M_2)) \in [l_1, L_1]^2 \times [l_2, L_2]^2$ be a solution of the system of equations

$$m_1 = \frac{\alpha_1}{M_1 + M_2} \quad , \quad M_1 = \frac{\alpha_1}{m_1 + m_2} \tag{.7}$$

$$m_2 = \frac{\alpha_2 + \beta_2 M_1 + M_2}{B_2 M_1 + M_2} \quad , \quad M_2 = \frac{\alpha_2 + \beta_2 m_1 + m_2}{B_2 m_1 + m_2}. \tag{.8}$$

Then it follows by Theorem .3.1 and Theorem .3.2 that the proof of Theorem .3.3 will be completed by showing

$$m_1 = M_1 \qquad \text{and} \qquad m_2 = M_2. \tag{.9}$$

By (.7), we see that

$$m_1(M_1 + M_2) = \alpha_1 = M_1(m_1 + m_2)$$

and hence that

$$m_1 M_2 = M_1 m_2. \tag{.10}$$

By (.8), we see similarly that

$$B_2 M_1 m_2 + m_2 M_2 = \alpha_2 + \beta_2 M_1 + M_2$$

$$B_2 m_1 M_2 + m_2 M_2 = \alpha_2 + \beta_2 m_1 + m_2$$

and so

$$B_2(M_1m_2 - m_1M_2) = \beta_2(M_1 - m_1) + (M_2 - m_2). \quad (.11)$$

Thus by (.10) and (.11) we see that

$$0 = \beta_2(M_1 - m_1) + (M_2 - m_2)$$

and so as $\beta_2 > 0$, we must have

$$m_1 = M_1 \qquad \text{and} \qquad m_2 = M_2 \quad (.12)$$

and so the proof of Theorem .3.3 is complete.

□

Extensive computer simulations lead us to the following conjecture:

Conjecture .3.1 *The unique positive equilibrium of System(12,48) is a global attractor for the entire range of the parameters.*

.3.5 References

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