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08. Central Force Motion I

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Abstract
Part eight of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc8]

8. Central Force Motion I

Central Force Motion: Two-Body Problem [mln66]

Mechanical system with six degrees of freedom:

Consider two masses m_1, m_2 interacting via a central force.

Central-force potential: $V(\mathbf{r}_1, \mathbf{r}_2) \equiv V(|\mathbf{r}_1 - \mathbf{r}_2|)$.

Lagrangian of two-body problem: $L = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|)$.

Conservation laws inferred from translational and rotational symmetries:

- Energy: $E = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + V(|\mathbf{r}_1 - \mathbf{r}_2|)$.
- Linear momentum: $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2$.
- Angular momentum: $\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$.

Reduction to three degrees of freedom:

Center-of-mass position vector: $\mathbf{R} \doteq \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$.

Distance vector: $\mathbf{r} \doteq \mathbf{r}_2 - \mathbf{r}_1$.

Total mass: $M \doteq m_1 + m_2$.

Reduced mass: $m \doteq \dfrac{m_1 m_2}{m_1 + m_2}$.

Lagrangian (after point transformation):

$$L = L_M(\dot{\mathbf{R}}) + L_m(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}m\dot{\mathbf{r}}^2 - V(|\mathbf{r}|).$$

Center-of-mass motion: $L_M(\dot{\mathbf{R}}) = \frac{1}{2}M\dot{\mathbf{R}}^2$.

- R_x, R_y, R_z are cyclic coordinates.
- Conserved center-of-mass momentum: $\mathbf{P} = M\dot{\mathbf{R}} = \text{const.}$
- Uniform rectilinear center-of-mass motion: $\mathbf{R}(t) = \mathbf{R}_0 + \dfrac{\mathbf{P}}{M}\,t$.

Effective one-body problem: $L_m(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(|\mathbf{r}|)$.

- Three degrees of freedom.
- Particle of mass m moving in a stationary central potential $V(|\mathbf{r}|)$.

Central Force Motion: One-Body Problem [mln67]

Reduction to one degree of freedom:

Consider a particle of mass m moving in a central potential:

Lagrangian: $L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(|\mathbf{r}|)$.

Conservation of angular momentum: $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = \text{const}$.

- Case $\mathbf{L} = 0$: One degree of freedom.
  - Purely radial motion: $\mathbf{r} \parallel \dot{\mathbf{r}} \quad \Rightarrow \; L(r, \dot{r}) = \frac{1}{2}m\dot{r}^2 - V(r)$.
  - Energy conservation: $E(r, \dot{r}) = \frac{1}{2}m\dot{r}^2 + V(r)$.
  - Reduction to quadrature (see [mln4]).

- Case $\mathbf{L} \neq 0$: Two separable degrees of freedom.
  - Motion in plane perpendicular to $\mathbf{L}$.
  - Transformation to polar coordinates: $x = r\cos\vartheta,\; y = r\sin\vartheta$.
  - Lagrangian: $L(r, \dot{r}, \dot{\vartheta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r)$.
  - Cyclic coordinate: ϑ.
  - Conserved angular momentum: $\ell = \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta} = \text{const}$.
  - Routhian: $R(r, \dot{r}; \ell) = L - \ell\dot{\vartheta} = \frac{1}{2}m\dot{r}^2 - \frac{\ell^2}{2mr^2} - V(r)$.
  - Effective potential for radial motion: $\tilde{V}(r; \ell) \doteq V(r) + \frac{\ell^2}{2mr^2}$.
  - Conserved energy: $E(r, \dot{r}; \ell) = \frac{1}{2}m\dot{r}^2 + \tilde{V}(r; \ell)$.
  - Reduction to quadrature (see [mln4]).
  - Integral for angular motion: $\vartheta(t) = \vartheta_0 + \frac{\ell}{m}\int_0^t \frac{dt}{mr^2(t)}$.

Central Force Problem: Formal Solution [mln18]

Lagrangian: $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) - V(r)$.

Lagrange equations (coupled 2$^{\text{nd}}$ order ODEs):

$$m\ddot{r} = mr\dot{\vartheta}^2 - \frac{\partial V}{\partial r}, \qquad \frac{d}{dt}\left(mr^2\dot{\vartheta}\right) = 0.$$

Integrals of the motion (angular momentum and energy):

$$[A] \;\; \ell = mr^2\dot{\vartheta} = \text{const}, \quad [B] \;\; E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) = \text{const}.$$

Motion in time (solution by quadrature):

$$[B] \;\; \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]} \qquad \Rightarrow \; t = \pm\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}}$$

$$\Rightarrow \; r(t) = \ldots$$

$$[A] \;\; \frac{d\vartheta}{dt} = \frac{\ell}{mr^2} \quad \Rightarrow \; \vartheta(t) = \frac{\ell}{m}\int_0^t \frac{dt}{r^2(t)} + \vartheta_0.$$

Integration constants: $E, \; \ell, \; r_0, \; \vartheta_0$.

Orbital integral: eliminate t from $r(t), \vartheta(t)$ to obtain $r(\vartheta)$ or $\vartheta(r)$.

$$\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]} = \frac{dr}{dt} = \frac{dr}{d\vartheta}\frac{d\vartheta}{dt} = \frac{dr}{d\vartheta}\frac{\ell}{mr^2}.$$

$$\Rightarrow \; \int_{r_0}^{r} dr \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} = \int_{\vartheta_0}^{\vartheta} d\vartheta = \vartheta - \vartheta_0 \quad \Rightarrow \; \vartheta(r) = \vartheta_0 + \ldots$$

Orbital integral for power-law potentials $V(r) = -\frac{\kappa}{r^\alpha}$: set $u \doteq 1/r$.

$$\vartheta - \vartheta_0 = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2m\kappa}{\ell^2}\,u^\alpha - u^2}}.$$

For the cases $\alpha = 6, 4, 3, 2, 1, -1, -2, -4, -6$, the orbit can be expressed in terms of elementary functions.

Orbits of Power-Law Potentials [msl21]

$$E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}m\dot{r}^2 + \tilde{V}(r), \quad \tilde{V}(r) = V(r) + \frac{\ell^2}{2mr^2}, \quad E > \tilde{V}(r) > V(r).$$

$$E - V(r) = \frac{1}{2}mv^2, \quad E - \tilde{V}(r) = \frac{1}{2}m\dot{r}^2, \quad \tilde{V}(r) - V(r) = \frac{1}{2}mr^2\dot{\vartheta}^2.$$

particle speed: $v \propto \sqrt{E - V}$.

radial speed: $|\dot{r}| \propto \sqrt{E - \tilde{V}}$.

angular speed: $r|\dot{\vartheta}| \propto \sqrt{\tilde{V} - V}$.

(i) $V(r) = -\dfrac{\kappa}{r^\alpha}, \quad 0 < \alpha < 2$:

$\tilde{V}(r)$ has minimum at $r_0 = (\ell^2/\alpha\kappa m)^{1/(\alpha-2)}$.

$E = E_1$: unbounded orbit, turning point ($\dot{r} = 0$) at $\tilde{V}(r_{min}) = E_1$.

$E = E_3$: bounded orbit, turning points at $\tilde{V}(r_{min}) = \tilde{V}(r_{max}) = E_3$.

$E = E_4$: circular orbit at r_0: $\dot{r} = 0$, $\dot{\vartheta} = \text{const}$.

(ii) $V(r) = -\dfrac{\kappa}{r^\alpha}, \quad \alpha > 2$:

$\tilde{V}(r)$ has maximum at $r_0 = (\alpha\kappa m/\ell^2)^{1/(\alpha-2)}$.

$E < \tilde{V}(r_0)$ and large r: unbounded orbit at $r > r_2$, where $\tilde{V}(r_2) = E$.

$E < \tilde{V}(r_0)$ and small r: bounded orbit at $r < r_1$, where $\tilde{V}(r_1) = E$.

$E > \tilde{V}(r_0)$: Unbounded orbit with particle spiraling through center.

$E = \tilde{V}(r_0)$: Unstable circular orbit exists.

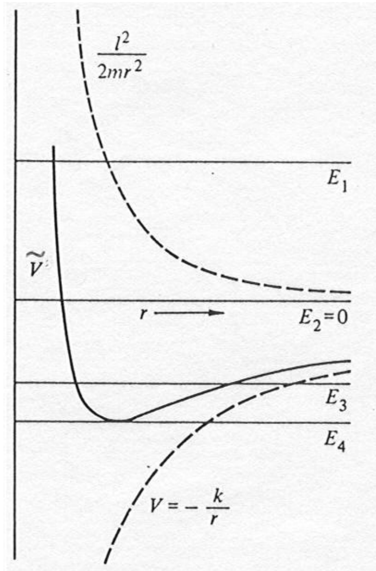
(iii) $V(r) = \kappa' r^{\alpha'}, \quad \kappa' = -\kappa > 0, \quad \alpha' = -\alpha > 0$:

$\tilde{V}(r)$ has minimum at $r_0 = (\ell^2/\alpha'\kappa' m)^{1/(\alpha'+2)}$.

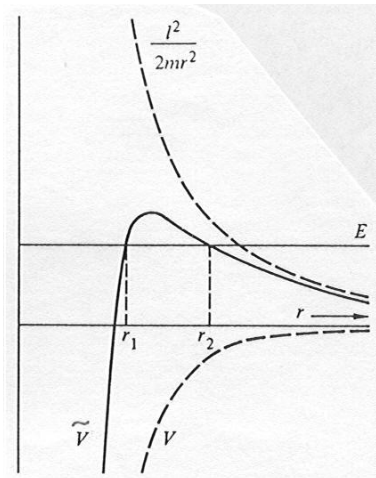
All orbits are bounded: $r_1 < r < r_2$, where $\tilde{V}(r_1) = \tilde{V}(r_2) = E$

$E = \tilde{V}(r_0)$: circular orbit exists.

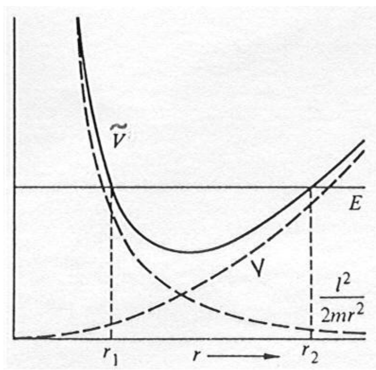
(i) $\alpha = 1$ (gravitation):



(ii) $\alpha = 3$:



(iii) $\alpha' = 2$ (harmonic oscillator):



[Goldstein 1981]

[mex51] Unstable circular orbit

The central force potential $V(r) = -\kappa/r^4$ has an unstable circular orbit of radius R centered at the center of force. (a) Find the angular momentum ℓ, the energy E, and the period τ of this circular orbit. (b) Find a second orbit $r(\vartheta)$ for the same values of E and ℓ which starts at the center of force and approaches the circular orbit of radius R asymptotically.

Solution:

[mex46] Orbit of the inverse-square potential at large angular momentum

Consider the central force potential $V(r) = -\kappa/r^2$. If $\kappa < \ell^2/2m$, all orbits are unbounded and have energies $E > 0$. (a) Show that the orbits can be expressed in the form

$$\frac{1}{r} = \sqrt{\frac{2mE}{\ell^2 - 2m\kappa}} \cos\left(\vartheta\sqrt{1 - \frac{2m\kappa}{\ell^2}}\right).$$

(b) Determine the total angle an orbit describes between the incoming and outgoing asymptotes.

Solution:

[mex47] Orbit of the inverse-square potential at small angular momentum

Consider the central force potential $V(r) = -\kappa/r^2$. If $\kappa > \ell^2/2m$, all orbits at $E > 0$ are unbounded and all orbits at $E < 0$ are bounded. (a) Show that these orbits can be expressed in the form

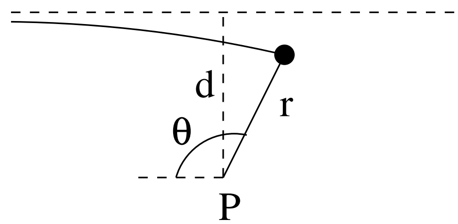
$$E > 0: \ \frac{1}{r} = \sqrt{\frac{2mE}{2m\kappa - \ell^2}} \sinh\left(\vartheta\sqrt{\frac{2m\kappa}{\ell^2} - 1}\right), \quad E < 0: \ \frac{1}{r} = \sqrt{\frac{2m|E|}{2m\kappa - \ell^2}} \cosh\left(\vartheta\sqrt{\frac{2m\kappa}{\ell^2} - 1}\right).$$

(b) Determine the time it takes the particle to move along the bounded orbit from r_{max} to the center of force ($r = 0$).

Solution:

[mex41] In search of some hyperbolic orbit

A particle of unit mass ($m = 1$) moves from infinity along a straight line which, if continued, would allow it to pass a distance $d = b\sqrt{2}$ from a point P. Instead, the particle is attracted toward P by the central force $F(r) = -k/r^5$. If the angular momentum of the particle relative to P is $\ell = \sqrt{k}/b$, show that the orbit is $r(\theta) = b\coth(\theta/\sqrt{2})$.



Solution:

Virial Theorem [mln68]

Consider a system of interacting particles in bounded motion.

Newton's equations of motion: $\dot{\mathbf{p}}_i = m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \quad i = 1, \ldots, N.$

$\mathbf{F}_i$: sum of external and interaction forces acting on particle i.

Definition: $G(t) \doteq \sum_i \mathbf{p}_i \cdot \mathbf{r}_i.$

For bounded motion $G(t)$ is finite.

Time derivative: $\dfrac{dG}{dt} = \sum_i (\mathbf{p}_i \cdot \dot{\mathbf{r}}_i + \dot{\mathbf{p}}_i \cdot \mathbf{r}_i) = \sum_i m_i |\dot{\mathbf{r}}_i|^2 + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i.$

Kinetic energy: $T = \sum_i \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2.$

Time average: $\overline{\dfrac{dG}{dt}} = \dfrac{1}{\tau} \int_0^\tau dt \, \dfrac{dG}{dt} = \dfrac{1}{\tau} [G(\tau) - G(0)] \xrightarrow{\tau \to \infty} 0.$

$$\Rightarrow \; 2\overline{T} + \sum_i \overline{\mathbf{F}_i \cdot \mathbf{r}_i} = 0.$$

Virial: $\overline{T} = -\dfrac{1}{2} \sum_i \overline{\mathbf{F}_i \cdot \mathbf{r}_i}.$

Application to particle in bounded orbit of central-force motion.

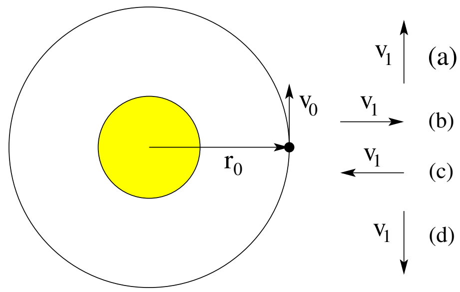
Power-law central force potential: $V(r) = -\dfrac{\kappa}{r^\alpha}.$

$$\overline{T} = -\frac{1}{2} \left(\overline{-r \, \frac{dV}{dr}} \right) = -\frac{1}{2} \alpha \, \overline{V}.$$

- Gravity ($\alpha = 1$): $\overline{T} = -\frac{1}{2} \overline{V}.$
- Harmonic oscillator ($\alpha = -2$): $\overline{T} = \overline{V}.$

[mex163] Changing orbit by brief rocket boost

A satellite orbits the Earth in a circular orbit of radius r_0, traveling with velocity $\mathbf{v}_0$. Then a rocket on the satellite fires such that it acquires an additional velocity $\mathbf{v}_1$ of the same magnitude as $\mathbf{v}_0$ in a very short time. Give a detailed description of the nature of the subsequent orbit of the satellite for the four cases with different directions of $\mathbf{v}_1$ as shown.



Solution:

[mex40] Discounted gravity: 50% off

A particle of mass m moves in a circular orbit of radius r_0 in a central force potential $V(r) = -\kappa/r$. Suddenly the value of κ decreases to half its original value and the particle changes its orbit as a result of the reduced attractive force. Give a detailed description of the new orbit.

Solution:

Bounded Orbits Open or Closed [mln79]

Consider an effective potential $\tilde{V}(r) = V(r) + \ell^2/(2mr^2)$ for the radial part of a central force motion as shown.

The radial coordinate r oscillates between r_P (periapsis) and r_A (apsis).

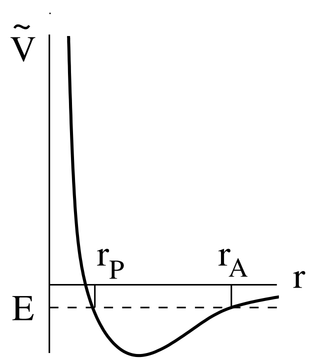
Between successive instances of $r = r_P$ and $r = r_A$ the angular coordinate ϑ always advances the same amount $\Delta\vartheta$.

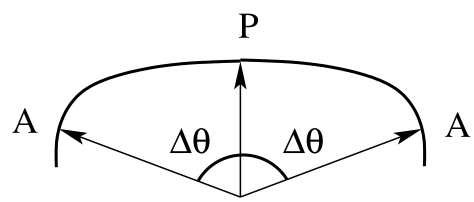
Apsidal vectors: position vectors $\mathbf{r}$ with $|\mathbf{r}| = r_P$ or $|\mathbf{r}| = r_A$.

Orbits are reflection symmetric at apsidal vectors. Hence the complete orbit can be constructed from one segment between successive apsidal vectors.

Apsidal angle: $\Delta\vartheta = \displaystyle\int_{r_P}^{r_A} dr \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}}$.

Condition for closed orbit: $\Delta\vartheta/2\pi$ must be a rational number.





Examples of closed bounded orbits:

- $V(r) = -\dfrac{\kappa}{r} \quad \Rightarrow \quad \vartheta - \vartheta_0 = \arccos \dfrac{\frac{\ell^2}{m\kappa r} - 1}{\sqrt{1 + \frac{2E\ell^2}{m\kappa^2}}} \quad \Rightarrow \quad \Delta\vartheta = \pi.$
- $V(r) = \dfrac{1}{2}kr^2 \quad \Rightarrow \quad \vartheta - \vartheta_0 = \dfrac{1}{2}\arccos \dfrac{\frac{\ell}{mr^2} - \frac{E}{\ell}}{\sqrt{\frac{E^2}{\ell^2} - \frac{k}{m}}} \quad \Rightarrow \quad \Delta\vartheta = \dfrac{\pi}{2}.$

Bertrand's theorem [mln44] proves that only for these two potentials are all bounded orbits closed.

Bertrand's Theorem [mln44]

The only central force potentials $V(r)$ for which all bounded orbits are closed are the following:

- Kepler system: $V(r) = -\dfrac{\kappa}{r}$ (ellipses with $r = 0$ at one focus)
- Harmonic oscillator: $V(r) = \kappa' r^2$ (ellipses with $r = 0$ at center)

J. Bertrand's proof of 1873 is based on a 2^{nd} order perturbation calculation about stable circular orbits. The following derivation follows Arnold [1989] and rests on five lemmas:

1. The central force potential $V(r)$ has a circular orbit at $r = R$ if $V'(R) = \ell^2/mR^3$. This circular orbit is stable if $V''(R) + (3/R)V'(R) > 0$. [mex53] [mex125]
2. For a central force potential $V(r)$ with a circular orbit at $r = R$, the apsidal angle for orbits in the vicinity of this circular orbit is $\Delta\vartheta = \pi\sqrt{V'(R)/[3V'(R) + RV''(R)]}$. [mex126]
3. The only central force potentials for which the apsidal angle of nearly circular orbits is independent of the radius are the power-law potentials $V(r) = -\kappa/r^\alpha, \alpha < 2, \alpha \neq 0$ and the logarithmic potential $V(r) = \kappa \ln r$. The value of the apsidal angle is $\Delta\vartheta = \pi/\sqrt{2-\alpha}$, where the value $\alpha = 0$ pertains to the logarithmic potential. [mex127]
4. For central force potentials with $\lim_{r\to\infty} V(r) = \infty$, the apsidal angle has the property $\lim_{E\to\infty} \Delta\vartheta = \pi/2$. [mex128] [mex129]
5. For power-law central force potentials $V(r) = -\kappa/r^\alpha, 0 \leq \alpha < 2$, the apsidal angle has the property $\lim_{E\to-\infty} \Delta\vartheta = \pi/(2-\alpha)$. [mex130]

Proof of Bertrand's theorem:

- Closed orbits require $\Delta\vartheta = 2\pi(m/n)$ for integer m, n.
- Lemma 3 restricts the class of potentials with no open bounded orbits to potentials (a) $V(r) = \kappa' r^{-\alpha}, \alpha < 0$, (b) $V(r) = -\kappa/r^\alpha, 0 < \alpha < 2$, (c) $V(r) = \kappa \ln r$ (representing $\alpha = 0$).
- For the cases $\alpha < 0$, lemma 4 requires $\pi/\sqrt{2-\alpha} = \pi/2$, which rules out all exponents except $\alpha = -2$ (harmonic oscillator). The apsidal angle is $\Delta\vartheta = \pi/2$ for all orbits of this system.
- For the cases $0 \leq \alpha < 2$, lemma 5 requires $\pi/\sqrt{2-\alpha} = \pi/(2-\alpha)$, which rules out all exponents except $\alpha = 1$ (Kepler system). The apsidal angle is $\Delta\vartheta = \pi$ for all orbits of this system.

[mex53] Stability of circular orbits

Consider a particle of mass m and angular momentum ℓ subject to a central force $F(r) = -V'(r)$.
(a) Show that the condition for the existence of a circular orbit at radius R is $F(R) + \ell^2/mR^3 = 0$.
(b) Show that the stability condition of this circular orbit is $F'(R) + (3/R)F(R) < 0$.

Solution:

[mex125] Small oscillations of radial coordinate about circular orbit

Consider a particle of mass m and angular momentum ℓ subject to a central force $F(r) = -V'(r)$. Under the conditions stated in [mex53] that a stable orbit at radius $r = R$ exists, show that on an orbit starting at radius $r = R + x$ with $|x| \ll R$ next to a stable circular orbit of radius R, the radial coordinate oscillates about R with angular frequency $\omega_0^2 = -3F(R)/mR - F'(R)/m$.

Solution:

[mex126] Angle between apsidal vectors for nearly circular orbits

Consider a particle of mass m and angular momentum ℓ subject to a central force $F(r) = -V'(r)$ and moving in a stable circular orbit of radius $r = R$. Show that nearly circular orbits in the immediate vicinity have an apsidal angle

$$\Delta\vartheta = \pi\sqrt{\frac{V'(R)}{3V'(R) + RV''(R)}}.$$

Solution:

[mex127] Robustness of apsidal angles

(a) Given the result of [mex126], namely that nearly circular orbits at radius $r = R$ of a central force potential $V(r)$ have apsidal angle $\Delta\vartheta = \pi\sqrt{V'(R)/[3V'(R) + RV''(R)]}$, show that the only cases for which this apsidal angle is independent of the radius are the power-law potentials $V(r) = -\kappa/r^\alpha, \alpha < 2, \alpha \neq 0$ and the logarithmic potential $V(r) = \kappa \ln r$. (b) Show that the value of the apsidal angle is $\Delta\vartheta = \pi/\sqrt{2 - \alpha}$, where the value $\alpha = 0$ pertains to the logarithmic potential.

Solution:

[mex128] Apsidal angle reinterpreted

Consider a particle of mass m in a bounded orbit with energy E and angular momentum ℓ of a central force potential $V(r)$. Show that the angle $\Delta\vartheta$ between successive apsidal vectors (between pericenter and apocenter) is related to the period T of the oscillatory motion of a fictitious particle in a 1D potential $W(x)$ as investigated in [mex5]:

$$\Delta\vartheta \equiv \int_{r_{min}}^{r_{max}} dr\, \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} = \frac{T}{2\sqrt{m}}, \quad T = 2\int_{x_{min}}^{x_{max}} \frac{dx}{\sqrt{\frac{2}{m}\left[E - W(x)\right]}}.$$

Find the relation between the variables r and x and determine the function $W(x)$.

Solution:

[mex129] Apsidal angle at very high energies

Use the result of [mex128] to show that for a central force potential with the property $\lim_{r\to\infty} V(r) = \infty$, the apsidal angle of orbits with given angular momentum approaches a universal value at very high energy:

$$\lim_{E\to\infty} \Delta\vartheta = \frac{\pi}{2}.$$

Solution:

[mex130] Apsidal angle at very low energies

Use the result of [mex128] to show that for a power-law central force potential $V(r) = -\kappa/r^\alpha,\ 0 \leq \alpha < 2$ the apsidal angle of orbits with given angular momentum ℓ approaches an ℓ-independent value at very low energy:

$$\lim_{E\to-\infty} \Delta\vartheta = \frac{\pi}{2-\alpha}.$$

Hint: Consider first the case $\alpha = 1$.

Solution: