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07. Applications of Gauss's law to charged conductors at equilibrium

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PHY204 Lecture 7 [rln7]

Charged Conductor Problem (2)

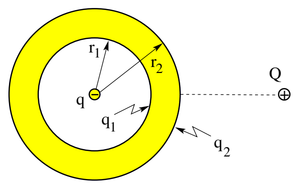
A conducting spherical shell of inner radius $r_1 = 4$cm and outer radius $r_2 = 6$cm carries no net charge. Now we place a point charge $q = -1\mu$C at its center.

(a) Find the surface charge densities σ_1 and σ_2.

(b) Find the electric fields E_1 and E_2 in the immediate vicinity of the shell.

(c) What happens to the electric fields inside and outside the shell when a second point charge $Q = +1\mu$C is placed a distance $d = 20$cm from the center of the shell?

(d) Which objects exert a force on the second point charge?



tsl62

The uncharged conducting spherical by itself does not generate an electric field anywhere. However, as soon as we place the point charge $q = -1\mu$C at the center, the situation changes both inside and outside the shell. Nothing changes in the shell interior between its two surfaces: still no field and no excess charge there.

The (negative) point charge q induces a compensating charge $q_1 = -q$ on the inner surface of the shell. This is mandated by Gauss's law, employed with a Gaussian sphere embedded in the shell: no field there, hence no flux, hence no net charge inside, hence $q + q_1 = 0$, hence $q_1 = -q = +1\mu$C. The net charge on the shell is still zero, therefore we have $q_2 = -q_1 = -1\mu$C on the outer surface, the only other place where charge on the shell can be. The surface charge densities are

$$\sigma_1 = \frac{q_1}{4\pi r_1^2} = 4.97 \times 10^{-5}\mathrm{C/m}^2, \quad \sigma_2 = \frac{q_2}{4\pi r_2^2} = -2.21 \times 10^{-5}\mathrm{C/m}^2.$$

The electric field just outside the surface of a conductor is perpendicular to the surface locally, toward the surface if it is negatively charged and away from it if it is positively charged. In our case this means that the field is radially inward (toward the center) near both surfaces of the shell. The field strengths are

$$E_1 = \frac{|\sigma_1|}{\epsilon_0} = 5.62 \times 10^6\mathrm{N/C}, \quad E_2 = \frac{|\sigma_2|}{\epsilon_0} = 2.50 \times 10^6\mathrm{N/C}.$$

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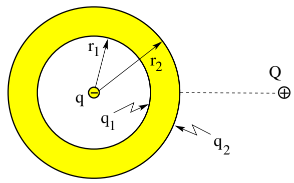


tsl62

Placing a second point charge $Q = +1\mu$C outside the shell throws the spherically symmetric charge distributions on both surfaces of the shell out of equilibrium but quickly restores the one on the inner surface to what it was.

The charge q_2 on the outer surface will find a new distribution such that the field generated by the point charge Q does not penetrate the conducting shell (a condition for equilibrium). The electric field outside the shell will no longer be what it was. The field strength E_2 now varies from point to point on the outer surface, as does the surface charge density σ_2.

There will be an attractive force between the negatively charged outer surface of the shell and the positive point charge Q. How strong is that force?

If the charge density on the outer surface were uniform, the electric field of the shell would be the same as that of the point charge q at its center. In that case the attractive force between shell and point charge Q would be

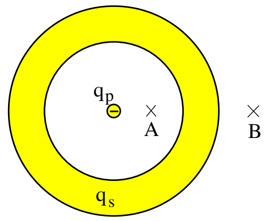
$$F = k\,\frac{|qQ|}{d^2},$$

where d is the distance between the two point charges.

However, on the conducting shell, the equilibrium charge distribution on its outer surface shifts the average distance from the point charge Q to a smaller values, which increases the magnitude of the attractive force.

A point charge $q_p = -7\mu$C is positioned at the center of a conducting spherical shell with a charge $q_s = +4\mu$C on it.

- Find the direction (inward/outward) of the electric field at the points A and B.
- Find the charge q_s^{int} on the inner surface and the charge q_s^{ext} on the outer surface of the shell.



tsl63

This is another situation with spherical symmetry. Only the values of the point charge and the charge on the shell are given, not the inner and outer radius of the shell, nor the radii of the two field points A and B. Nevertheless, we have all we need to answer the questions.

In order to find the direction of the electric field at point A, we use a Gaussian sphere that is concentric with the shell and touches field point A. The electric flux through that Gaussian surface is negative because the net charge inside, $q_p = -7\mu$C, is negative. We conclude that the electric field is directed toward the inside, meaning left at point A.

We determine the direction of the electric field at point B in like manner by using a concentric Gaussian sphere of a larger radius such that it touches field point B. The charge inside is still negative, $q_p + q_s = -3\mu$C, implying that the flux is negative, meaning that the field is inward, i.e. toward the left at point B.

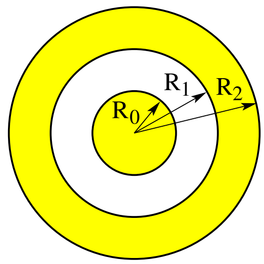
Since the shell is conducting, the charge q_s on it must be distributed between the two surfaces. If we use a Gaussian sphere embedded in the shell, we know that the flux through it is zero because there is no electric field inside the conducting material. Hence the net charge inside that Gaussian sphere must be zero as well, implying that $q_p + q_s^{int} = 0$. We thus conclude that $q_s^{int} = -q_p = +7\mu$C.

The remainder of q_s must be on the outer shell: $q_s^{ext} = q_s - q_s^{int} = -3\mu$C.

Charged Conductor Problem (4)

A long conducting cylinder of radius $R_0 = 3\text{cm}$ carries a charge per unit length $\lambda_c = 5.0\mu\text{C/m}$. It is surrounded by a conducting cylindrical shell of radii $R_1 = 7\text{cm}$ and $R_2 = 11\text{cm}$. The shell carries a charge per unit length $\lambda_s = -8.0\mu\text{C/m}$.

(a) Find the linear charge densities λ_1, λ_2 on the inner and outer surfaces of the shell.
(b) Find the electric fields E_0, E_1, E_3 in the vicinity of the three conducting surfaces.



tsl64

This is an exercise with cylindrical symmetry. We can answer the questions if we use Gaussian surfaces that are cylindrical cans placed coaxially.

Note that the charges on the two conductors are specified as charge per unit length λ rather than as charge per unit area σ even though the charge is spread across the surfaces of both conductors. This is a choice of convenience. We can convert the charge density on the inner cylinder as follows:

$$\sigma_c = \frac{\lambda_c}{2\pi R_0} = \frac{5.0\mu\text{C/m}}{2\pi(0.03\text{m})} = 26.5\mu\text{C/m}^2.$$

We cannot do the same for λ_s because that charge is on surfaces at different radii. Once we have solved part (a) we can convert λ_1 into σ_1 and λ_2 into σ_2.

In order to find λ_1 we use a Gaussian can of radius such that its curved surface is embedded in the shell. Since there is no electric flux through such a can, the net charge inside it must be zero, implying that $(\lambda_c + \lambda_1)L = 0$, where L is the length of the can. We conclude that $\lambda_1 = -\lambda_c = -5.0\mu\text{C/m}$ and infer that $\sigma_1 = \lambda_1/(2\pi R_1) = -11.4\mu\text{C/m}^2$.

The remainder of the charge per unit length on the shell must be on its outer surface: $\lambda_2 = \lambda_s - \lambda_1 = -3.0\mu\text{C/m}$, which we can convert into $\sigma_2 = \lambda_2/(2\pi R_2) = -4.34\mu\text{C/m}^2$.

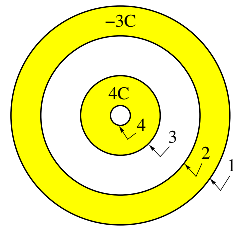
For part (b) the use of σ_c, σ_1, and σ_2 is more convenient because we know the relation, $E = |\sigma|/\epsilon_0$ between the electric field at the surface of a conductor and the local surface charge density. We thus find the results,

$$E_0 = \frac{|\sigma_c|}{\epsilon_0} = 2.99\times10^6\text{N/C}, \quad E_1 = \frac{|\sigma_1|}{\epsilon_0} = 1.29\times10^6\text{N/C}, \quad E_2 = \frac{|\sigma_2|}{\epsilon_0} = 0.49\times10^6\text{N/C}.$$

Given that a positive (negative) σ generates a field away from (toward the surface of the conductor, we conclude that $\vec{E}_0$ and $\vec{E}_1$ are directed radially outward and $\vec{E}_2$ radially inward.

Intermediate Exam I: Problem #4 (Spring '05)

Consider two concentric conducting spherical shells. The total electric charge on the inner shell is 4C and the total electric charge on the outer shell is -3C. Find the electric charges q_1, q_2, q_3, q_4 on each surface of both shells as identified in the figure.



Solution:

Start with the innermost surface.
Note that any excess charge is located at the surface of a conductor.
Note also that the electric field inside a conductor at equilibrium vanishes.

- Gauss's law predicts $q_4 = 0$.
- Charge conservation then predicts $q_3 + q_4 = 4$C. Hence $q_3 = 4$C.
- Gauss's law predicts $q_2 = -(q_3 + q_4) = -4$C.
- Charge conservation then predicts $q_1 + q_2 = -3$C. Hence $q_1 = +1$C.

tsl334

Returning to a situation with spherical symmetry, we consider two concentric conducting shells, the inner one positively charged and the outer one negatively.

The exercise asks how the given charge on each conductor is distributed among its two surfaces.

Since any application of Gauss's law involves the net charge inside a closed surface, it is good practice to design Gaussian surfaces such that it contains just one unknown charge, not several.

Therefore, we choose our first Gaussian surface as a concentric sphere which is embedded in the inner shell, where there is no flux because their is no field. We conclude that the charge on the innermost surface is zero. Hence the entire charge of the inner shell is distributed across its outer surface.

Our second Gaussian surface is again a concentric sphere but now embedded in the outer shell. The flux through that surface vanishes again because there is no field inside the conducting material at equilibrium. Hence the net charge must be zero, which implies that the charge on the inner surface of the outer shell must compensate the charge on the inner shell.

The charge on the outermost surface is the difference between the given total charge on the outer shell and the charge just calculated in its inner surface.

Note that the electric field on the outside of the two concentric shell is directed outward even though the total charge on the outer shell is negative. What matters for the direction of the external electric field is the charge on the outermost surface, which is positive.

Intermediate Exam I: Problem #2 (Spring '06)

Consider a conducting sphere of radius $r_1 = 1\text{m}$ and a conducting spherical shell of inner radius $r_2 = 3\text{m}$ and outer radius $r_3 = 5\text{m}$. The charge on the inner sphere is $Q_1 = -0.6\mu\text{C}$. The net charge on the shell is zero.

(a) Find the charge Q_2 on the inner surface and the charge Q_3 on the outer surface of the shell.

(b) Find magnitude and direction of the electric field at point A between the sphere and the shell.

(c) Find magnitude and direction of the electric field at point B inside the shell.

(d) Find magnitude and direction of the electric field at point C outside the shell.

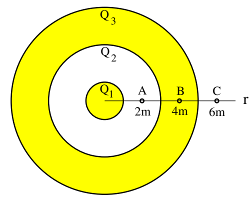
Solution:

(a) Gauss's law implies that $Q_2 = -Q_1 = +0.6\mu\text{C}$.
Given that $Q_2 + Q_3 = 0$ we infer $Q_3 = -0.6\mu\text{C}$.

(b) $E_A = k\dfrac{0.6\mu\text{C}}{(2\text{m})^2} = 1349\text{N/C}$ (inward).

(c) $E_B = 0$ inside conductor.

(d) $E_C = k\dfrac{0.6\mu\text{C}}{(6\text{m})^2} = 150\text{N/C}$ (inward).



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This exercise features a conducting sphere surrounded by a conducting spherical shell. The sphere is negatively charged. The shell is uncharged.

The conducting sphere only has one surface. That is where all the excess charge Q_1 is located. The shell has two surfaces, which will both be charged even though the net charge on the shell is zero.

To find the charge Q_2 on the inner surface of the shell, we use a Gaussian sphere embedded in the shell, where there is no electric flux, hence no charge inside. We conclude that Q_2 must compensate Q_1. We also conclude that Q_3 must compensate Q_2 to make the shell uncharged.

Since the conducting sphere is negatively charged, the field at point A must be directed inward. Its strength can be determined by using a Gaussian sphere with a radius $r_A = 2\text{m}$. Gauss's law then says,

$$E_A(4\pi r_A^2) = \frac{Q_1}{\epsilon_0},$$

from which we infer the answer given on the slide.

The electric field at radius $r_B = 4\text{m}$ must be zero because this is a location inside a conductor at equilibrium.

Finally, we determine the electric field at radius $r_C = 6\text{m}$ by using Gauss's law again with a Gaussian sphere of that radius:

$$E_C(4\pi r_C^2) = \frac{Q_1}{\epsilon_0},$$

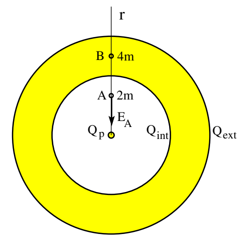
taking into account that the shell is uncharged. The result shown on the slide follows directly.

The direction of the external field is inward, which is consistent with the negative charge on the outermost surface.

Unit Exam I: Problem #2 (Spring '07)

A point charge Q_p is positioned at the center of a conducting spherical shell of inner radius $r_2 = 3.00\text{m}$ and outer radius $r_3 = 5.00\text{m}$. The total charge on the shell $Q_s = +7.00\text{nC}$. The electric field at point A has strength $E_A = 6.75\text{N/C}$ and is pointing radially inward.

(a) Find the value of Q_p (point charge).
(b) Find the charge Q_{int} on the inner surface of the shell.
(c) Find the charge Q_{ext} on the outer surface of the shell.
(d) Find the electric field at point B.



Solution:

(a) Gauss's law implies that $-E_A(4\pi r_A^2) = \dfrac{Q_p}{\epsilon_0}$
$\Rightarrow\ Q_p = -3.00\text{nC}$.
(b) Gauss's law implies that $Q_{int} = -Q_p = +3.00\text{nC}$.
(c) Charge conservation, $Q_{int} + Q_{ext} = Q_s = 7.00\text{nC}$, then implies that $Q_{ext} = +4.00\text{nC}$.
(d) $E_B = 0$ inside conductor.

tsl360

The situation here differs from the one on the previous page in that the inner conductor is replaced by a point charge. Also, the chain of reasoning must be reversed in some instances.

We only know the total charge on the shell, not how it is split between the two surfaces. We do not know the value of the point charge. Instead we know the electric field at point A inside the shell. That is enough to answer all questions.

We begin with a Gaussian sphere of radius 2m. The given field, which is inward, allows us to calculate the flux, which then has to be negative. Gauss's law then predicts that the charge inside, which is the point charge, is negative as well.

The next Gaussian surface to be used has radius 4m. There is no flux because there is no field there because the location is inside the conducting material at equilibrium. Hence the net charge inside is zero, implying that Q_{int} must compensate Q_p.

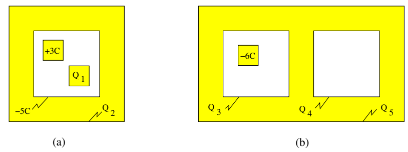
The charge Q_{ext} on the outside surface of the shell must be the difference between the given total charge Q_s on the shell and the charge Q_{int} on the inside surface.

Part (d) we have already answered on the fly.

Unit Exam I: Problem #3 (Spring '09)

(a) Consider a conducting box with no net charge on it. Inside the box are two small charged conducting cubes. For the given charges on the surface of one cube and on the inside surface of the box find the charges Q_1 on the surface of the other cube and Q_2 on the outside surface of the box.

(b) Consider a conducting box with two compartments and no net charge on it. Inside one compartment is a small charged conducting cube. For the given charge on the surface of the cube find the charges Q_3, Q_4, and Q_5 on the three surfaces of the box.



Solution:

(a) Gauss's law implies $Q_1 + 3\mathrm{C} + (-5\mathrm{C}) = 0 \quad \Rightarrow \quad Q_1 = +2\mathrm{C}$.
Net charge on the box: $Q_2 + (-5\mathrm{C}) = 0 \quad \Rightarrow \quad Q_2 = +5\mathrm{C}$.

(b) Gauss's law implies $Q_3 + (-6\mathrm{C}) = 0 \quad \Rightarrow \quad Q_3 = +6\mathrm{C}$.
Gauss's law implies $Q_4 = 0$.
Net charge on box: $Q_3 + Q_4 + Q_5 = 0 \quad \Rightarrow \quad Q_5 = -6\mathrm{C}$.

tsl391

The slide shows two configurations of conductors with rectangular shapes in cross section.

In this problem we have neither spherical nor cylindrical symmetry. The consequence is that the excess charge on conducting surfaces is not uniformly distributed. It is then much harder to calculate the surface charge density σ and the electric field outside those surfaces.

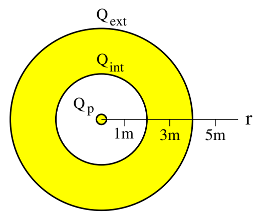
Answering the questions in the problem statement is a simple matter nevertheless. All we need are a few Gaussian surfaces that are embedded in the conducting material and then apply Gauss's law.

The answers given on the slide make it easy to figure out what kind of Gaussian surfaces are being used in each step.

Unit Exam I: Problem #2 (Spring '11)

The charged conducting spherical shell has a 2m inner radius and a 4m outer radius. The charge on the outer surface is $Q_{\rm ext} = 8$nC. There is a point charge $Q_{\rm p} = 3$nC at the center.

(a) Find the charge $Q_{\rm int}$ on the inner surface of the shell.
(b) Find the surface charge density $\sigma_{\rm ext}$ on the outer surface of the shell.
(c) Find the electric flux Φ_E through a Gaussian sphere of radius $r = 5$m.
(d) Find the magnitude of the electric field E at radius $r = 3$m.



Solution:

(a) $Q_{\rm int} = -Q_{\rm p} = -3$nC.
(b) $\sigma_{\rm ext} = \dfrac{Q_{\rm ext}}{4\pi(4{\rm m})^2} = 3.98 \times 10^{-11}{\rm C/m}^2$.
(c) $\Phi_E = \dfrac{Q_{\rm ext}}{\epsilon_0} = 904{\rm Nm}^2/{\rm C}$.
(d) $E = 0$ inside conductor.

tsl402

Back to spherical symmetry as a warm-up to the quiz.

For part (a) we employ a Gaussian sphere embedded in the shell. Since there is no flux through that surface, the net charge inside, which is the sum of the known point charge Q_p and the unknown charge Q_{int} on the inside surface of the shell must vanish.

Part (b) asks for the charge density σ_{ext} on the outside surface. The total charge is given and the area is inferred from the given radius.

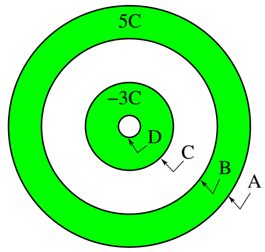
For part (c) we could first calculate the electric field at radius 5m, which is simple enough, and then multiply that by the area of a sphere with that radius. However, it is much simpler to use Gauss's law, which relates the flux sought to the total charge inside, which is $Q_p + Q_{int} + Q_{ext} = Q_{ext}$.

Part (d) is simply trying to remind us that inside the conducting material of the shell at equilibrium the electric field vanishes.

Charged Conductor Problem (5)

Consider two concentric shells with charges on them as indicated.

- Find the charges q_A, q_B, q_C, q_D on each of the four surfaces.



tsl65

This is the quiz for lecture 7. It is a simple variation of the situation discussed on page 5.