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Unboundedness Results for Rational Difference Equations

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UNBOUNDEDNESS RESULTS FOR RATIONAL DIFFERENCE EQUATIONS

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ABSTRACT

We present a collection of techniques for demonstrating the existence of unbounded solutions. We then use these techniques to determine the boundedness character of rational difference equations and systems of rational difference equations.

We study the rational difference equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that for nonnegative α and C, whenever $C\alpha = 0$ and $C+\alpha > 0$, unbounded solutions exist for some choice of nonnegative initial conditions. Moreover, we study the rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0, 1]$, unbounded solutions exist for some choice of nonnegative initial conditions.

Following these two results, we then present some new results regarding the boundedness character of the k^{th} order rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

When applied to the general fourth order rational difference equation, these results prove the existence of unbounded solutions for 49 special cases of the fourth order rational difference equation, where the boundedness character has not been established yet. This resolves 49 conjectures posed by E. Camouzis and G. Ladas.

Finally, we study k^{th} order systems of two rational difference equations

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N}.$$

In particular, we assume non-negative parameters and non-negative initial conditions. We develop several approaches, which allow us to prove that unbounded solutions exist for certain initial conditions in a range of the parameters.

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DEDICATION

I dedicate this to the poor.

PREFACE

This dissertation is prepared in accordance with the "University of Rhode Island Manuscript Plan" option. The main body of the dissertation consists of three manuscripts which have been written in a form that is suitable for publication in peer reviewed international mathematical journals.

Manuscript 1, Unboundedness for some classes of rational difference equations, has been published in the *International Journal of Difference Equations* **15**(2009), 253-260.

Manuscript 2, Unboundedness results for fourth order rational difference equations, has been accepted in the *Journal of Difference Equations and Applications.*

Manuscript 3, Unboundedness results for systems, has been published in the *Central European Journal of Mathematics* **7**(2009), 741-756.

The references for each individual manuscript are located at the end of the manuscript before the start of the next manuscript.

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MANUSCRIPT 1

Unboundedness for some classes of rational difference equations

by

Gabriel Lugo and Frank J. Palladino

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Abstract.

We study the rational difference equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that for nonnegative α and C, whenever $C\alpha = 0$ and $C+\alpha > 0$, unbounded solutions exist for some choice of nonnegative initial conditions. Moreover, we study the rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0,1]$, unbounded solutions exist for some choice of nonnegative initial conditions.

1.1 Introduction

In Ref. [1], Camouzis and Ladas devote a chapter to the study of unbounded solutions for the k^{th} order rational difference equation with nonnegative parameters and nonnegative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

In the introduction of said chapter, the authors of Ref. [1] pose five conjectures regarding the boundedness character of five different special cases of the third order rational difference equation. Particularly, we are referring to the special cases #28, #44, #56, #70, and #120. These are the only remaining cases of third order for which the boundedness character has not been established.

First, we study special cases #56 and #120

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}.$$

Using a standard induction technique, we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0,1]$, unbounded solutions exist for some choice of nonnegative initial conditions.

We then study special cases #44 and #28

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

We show that for nonnegative α and C, whenever $C\alpha = 0$ and $C + \alpha > 0$, unbounded solutions exist for some choice of nonnegative initial conditions. The proof is presented in two special cases. The case where $\alpha > 0$ and the case where $C > 0$.

1.2 Todd's Equation

Consider the third order rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}. \tag{1.2.1}$$

There have been significant results concerning the case where $\beta = 1$. In this case, the equation is generally referred to by the cognomen "Todd's equation" and possesses the invariant:

$$(\alpha + x_n + x_{n-1} + x_{n-2}) \left(1 + \frac{1}{x_n}\right) \left(1 + \frac{1}{x_{n-1}}\right) \left(1 + \frac{1}{x_{n-2}}\right) = constant.$$

For more information regarding Todd's equation see Refs. [5-7]. In the following theorem we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0, 1]$, the difference equation (1.2.1) has unbounded solutions for some choice of nonnegative initial conditions.

Theorem 1. *Suppose* $0 < \beta < \frac{1}{3}$ *and* $\alpha \in [0, 1]$, *then the difference equation* (1.2.1) *has unbounded solutions for some choice of nonnegative initial conditions.*

Proof. Choose initial conditions so that

$$\min(x_0, x_{-2}) > \max\left(\frac{1}{\beta}, \frac{x_{-1}}{\beta}\right).$$

We shall first prove by induction that for all $j \in \mathbb{N}$,

$$\min\left(x_{2j}, x_{2j-2}\right) > \max\left(\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right). \tag{1.2.2}$$

The initial conditions provide the base case. Assume the following holds for some $j \in \mathbb{N}$,

$$\min\left(x_{2j-2}, x_{2j-4}\right) > \max\left(\frac{1}{\beta}, \frac{x_{2j-3}}{\beta}\right).$$

Since $\beta x_{2j-2} > x_{2j-3}$, $x_{2j-2} > \frac{1}{\beta}$, $\beta x_{2j-2} > 1 \geq \alpha$, and $x_{2j-4} > \frac{1}{\beta} > 3$, we see that

$$x_{2j-1} = \frac{\alpha + \beta x_{2j-2} + x_{2j-3}}{x_{2j-4}} < \frac{3\beta x_{2j-2}}{x_{2j-4}} < \beta x_{2j-2}.$$

Thus we have shown

$$x_{2j-2} > \max\left(\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right).$$

Since $\beta x_{2j-4} > x_{2j-3}$ and $0 < \beta < \frac{1}{3}$, we have

$$x_{2j} = \frac{\alpha + \beta x_{2j-1} + x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-4}} > \frac{3x_{2j-2}}{x_{2j-4}} = \frac{3\beta x_{2j-2}}{\beta x_{2j-4}} > \frac{x_{2j-1}}{\beta}.$$

Also

$$x_{2j} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-2}} = \frac{1}{\beta}.$$

Thus

$$\min\left(x_{2j}, x_{2j-2}\right) > \max\left(\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right).$$

This completes the induction.

Using Equation (1.2.2) we now prove that $x_{8\eta} > \frac{x_{8\eta-8}}{9\beta^2}$ for all $\eta \in \mathbb{N}$:

$$x_{8\eta} = \frac{\alpha + \beta x_{8\eta-1} + x_{8\eta-2}}{x_{8\eta-3}} > \frac{x_{8\eta-2}}{x_{8\eta-3}} =$$

$$\left(\frac{x_{8\eta-6}}{\alpha + \beta x_{8\eta-4} + x_{8\eta-5}}\right)\left(\frac{\alpha + \beta x_{8\eta-3} + x_{8\eta-4}}{x_{8\eta-5}}\right) > \left(\frac{x_{8\eta-6}}{3\beta x_{8\eta-4}}\right)\left(\frac{x_{8\eta-4}}{x_{8\eta-5}}\right) =$$

$$\frac{x_{8\eta-6}}{3\beta x_{8\eta-5}} = \frac{x_{8\eta-6}x_{8\eta-8}}{3\beta\left(\alpha + \beta x_{8\eta-6} + x_{8\eta-7}\right)} > \frac{x_{8\eta-6}x_{8\eta-8}}{9\beta^2 x_{8\eta-6}} = \frac{x_{8\eta-8}}{9\beta^2}.$$

Since $0 < \beta < \frac{1}{3}$, $9\beta^2 < 1$. Thus, we have a subsequence of our solution which diverges to ∞. Hence, the solution is unbounded. ∎

1.3 Special Case #44

We now study special case #44

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}}, \quad n \in \mathbb{N}. \tag{1.3.1}$$

Particularly, we show that whenever $\alpha > 0$, Equation (1.3.1) has unbounded solutions for some initial conditions. The following lemma provides a useful technique for constructing divergent subsequences of solutions for rational difference equations.

Lemma 1. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $[0, \infty)$. Suppose that there exists $D > 1$ and hypotheses $H_1, \dots, H_k$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \mathbb{N}$ so that the following holds. Whenever x_{n-i} satisfies H_i for all $i \in \{1, \dots, k\}$, then x_{n+p_n-i} satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{n+p_n-1} \geq Dx_{n-1}$. Further assume that for some $N \in \mathbb{N}$, x_{N-i} satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{N-1} > 0$. Then $\{x_n\}_{n=1}^{\infty}$ is unbounded. Particularly, $\{x_{z_m-1}\}_{m=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ which diverges to ∞, where $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$.*

Proof. Let $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$. Using induction, we prove that given $m \in \mathbb{N}$ the following holds. $x_{z_m-1} \geq D^m x_{N-1}$ and x_{z_m-i} satisfies H_i for all $i \in \{1, \dots, k\}$. By assumption, x_{N-i} satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{N-1} \geq D^0 x_{N-1}$. This provides the base case. Assume $x_{z_{m-1}-i}$ satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{z_{m-1}-1} \geq D^{m-1} x_{N-1}$. Using our earlier assumption, this implies that there exists $p_{z_{m-1}}$ so that $x_{z_{m-1}+p_{z_{m-1}}-i}$ satisfies H_i for all $i \in \{1, \dots, k\}$ and $x_{z_{m-1}+p_{z_{m-1}}-1} \geq D x_{z_{m-1}-1} \geq (D) D^{m-1} x_{N-1} = D^m x_{N-1}$. So we have shown that $x_{z_m-1} \geq D^m x_{N-1}$ for all $m \in \mathbb{N}$. Hence, the subsequence $\{x_{z_m-1}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ clearly diverges to ∞, since $D > 1$. ∎

The above argument merely simplifies the following arguments by removing a somewhat onerous construction.

Theorem 2. *If* $\alpha > 0$*, then Equation* (1.3.1) *has unbounded solutions for some initial conditions.*

Proof. We choose initial conditions so that

$$x_0 > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \quad x_{-1} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \quad x_{-2} > \frac{\alpha}{2}.$$

We show that there exists $D = \frac{4}{3}$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \{7, 8\}$ so that the following holds. Whenever

$$x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \quad x_{n-2} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \quad x_{n-3} > \frac{\alpha}{2}.$$

Then we have

$$x_{n+p_n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \quad x_{n+p_n-2} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right),$$

$$x_{n+p_n-3} > \frac{\alpha}{2}, \quad x_{n+p_n-1} \geq \left(\frac{4}{3}\right) x_{n-1}.$$

First assume

$$x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right), \quad x_{n-2} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right), \quad x_{n-3} > \frac{\alpha}{2}.$$

Since $x_{n-1}, x_{n-2}, x_{n-3} > 0$, we may write $\eta = log_2(x_{n-1})$, $\ell = log_2(x_{n-2})$, and $\rho = log_2(x_{n-3})$. Hence $2^\eta = x_{n-1}$, $2^\ell = x_{n-2}$, and $2^\rho = x_{n-3}$. We use such representations for ease of computations. First we see that

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}} = \frac{\alpha}{x_{n-3}} + \frac{x_{n-1}}{x_{n-3}} = \frac{\alpha}{2^\rho} + 2^{\eta-\rho}; \tag{1.3.2}$$

$$x_{n+1} = \frac{\alpha}{x_{n-2}} + \frac{x_n}{x_{n-2}} = \frac{\alpha}{2^\ell} + \left(\frac{\alpha}{2^\rho} + 2^{\eta-\rho}\right)\frac{1}{2^\ell} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}; \tag{1.3.3}$$

$$\left.\begin{aligned} x_{n+2} = \frac{\alpha}{x_{n-1}} + \frac{x_{n+1}}{x_{n-1}} &= \frac{\alpha}{2^\eta} + \left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right)\left(\frac{1}{2^\eta}\right) \\ &= \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}}; \end{aligned}\right\} \tag{1.3.4}$$

$$x_{n+3} = \frac{\alpha}{x_n} + \frac{x_{n+2}}{x_n} = \frac{\alpha 2^\rho}{\alpha + 2^\eta} + \left(\frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}}\right)\left(\frac{2^\rho}{\alpha + 2^\eta}\right). \quad (1.3.5)$$

We will make use of these identities later. We prove the result in two cases. Let us first assume $\ell + \rho \geq \eta$. We show that if this inequality is satisfied for some $n \in \mathbb{N}$, then $p_n = 7$. First, we prove that $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$. Notice that

$$x_{n+4} = \frac{\alpha + x_{n+3}}{x_{n+1}} > \frac{\alpha}{x_{n+1}}.$$

From Equation (1.3.3) we see that

$$\frac{\alpha}{x_{n+1}} = \frac{\alpha}{\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}} = \frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}}.$$

We now use the assumption $\ell + \rho \geq \eta$. This assumption implies that $2^{\eta-\ell-\rho} \leq 2^0 = 1$. Earlier we assumed that $2^{-\rho} < \frac{2}{\alpha}$. Moreover, from our assumptions,

$$2^\ell > (\alpha+1)^2 2^{11} = \left(\alpha^2 + 2\alpha + 1\right) 2^{11}$$

so $2^{-\ell} < \frac{1}{\alpha 2^{12}}$ and $2^{-\ell} < 2^{-11}$. Using these inequalities we obtain the following.

$$\frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}} > \frac{\alpha}{\alpha \frac{1}{\alpha 2^{12}} + \alpha 2^{-11}\frac{2}{\alpha} + 1} = \frac{\alpha}{2^{-12} + 2^{-10} + 1} > \frac{\alpha}{2}.$$

So we have shown $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$.

We now prove that $x_{n+p_n-2} = x_{n+5} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right)$. Notice that

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{x_{n+4}}{x_{n+2}} > \frac{\alpha}{2x_{n+2}}.$$

Since $\ell+\rho \geq \eta$, $2^{\ell+\rho} \geq 2^\eta$. Moreover, as we have recently shown, $2^\ell > 2^{11}$, similarly $2^\eta > 2^{15}$. So $\ell > 11 > 0$ and $\eta > 15 > 0$. So from Equation (1.3.4),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{1}{2^\eta} = \frac{3\alpha+1}{2^\eta}. \quad (1.3.6)$$

Hence,

$$x_{n+5} > \frac{\alpha}{2x_{n+2}} > \frac{\alpha}{2\frac{3\alpha+1}{2^\eta}} = \frac{\alpha}{3\alpha+1}\left(2^{\eta-1}\right) > \left(\frac{1}{3\alpha+1}\right)\max\left(\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right).$$

So,

$$x_{n+5} > \left(\frac{1}{3\alpha+1}\right)\max\left(\frac{2^{14}}{\alpha^2},(\alpha+1)^4 2^{14}\right) \geq \frac{(\alpha+1)^4 2^{14}}{3\alpha+1} > \frac{(\alpha+1)^4 2^{14}}{3\alpha+3}$$
$$= \frac{(\alpha+1)^3 2^{14}}{3} > (\alpha+1)^2 2^{11}.$$

When $\alpha \geq 1$, $\frac{1}{\alpha^2} \leq 1 < (\alpha+1)^2$ so

$$(\alpha+1)^2 2^{11} = \max\left(\frac{2^{11}}{\alpha^2},(\alpha+1)^2 2^{11}\right).$$

Thus, the only remaining case is when $\alpha < 1$. In this case we have the following:

$$x_{n+5} > \left(\frac{1}{3\alpha+1}\right)\max\left(\frac{2^{14}}{\alpha^2},(\alpha+1)^4 2^{14}\right) \geq \frac{2^{14}}{(3\alpha+1)\alpha^2} > \frac{2^{14}}{4\alpha^2} > \frac{2^{11}}{\alpha^2}.$$

So we have shown $x_{n+p_n-2} = x_{n+5} > \max\left(\frac{2^{11}}{\alpha^2},(\alpha+1)^2 2^{11}\right)$.

We now prove that

$$x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right) x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3},\frac{(\alpha+1)^4 2^{15}}{\alpha}\right).$$

First assume,

$$\max\left((\alpha+1)2^5,\frac{(\alpha+1)2^5}{\alpha}\right) \geq 2^{\eta-\rho}. \tag{1.3.7}$$

Notice that

$$x_{n+6} = \frac{\alpha+x_{n+5}}{x_{n+3}} > \frac{x_{n+5}}{x_{n+3}} = \frac{\alpha+x_{n+4}}{x_{n+2}x_{n+3}} > \frac{x_{n+4}}{x_{n+2}x_{n+3}} = \frac{\alpha+x_{n+3}}{x_{n+2}x_{n+3}x_{n+1}} > \frac{1}{x_{n+2}x_{n+1}}.$$

We use Equation (1.3.4), our induction assumption, our assumption (1.3.7), and the fact that $2^{-\rho} < \frac{2}{\alpha}$ to obtain

$$x_{n+1} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{2^{\eta-\rho}}{2^\ell} < \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{\max\left((\alpha+1)2^5,\frac{(\alpha+1)2^5}{\alpha}\right)}{\max\left(\frac{2^{11}}{\alpha^2},(\alpha+1)^2 2^{11}\right)}.$$

Notice that if $\alpha \geq 1$,

$$\frac{\max\left((\alpha+1)2^5,\frac{(\alpha+1)2^5}{\alpha}\right)}{\max\left(\frac{2^{11}}{\alpha^2},(\alpha+1)^2 2^{11}\right)} \leq \frac{(\alpha+1)2^5}{(\alpha+1)^2 2^{11}} = \frac{1}{(\alpha+1)2^6} < \frac{1}{(\alpha+1)2^3}.$$

Also if $\alpha < 1$,

$$\frac{\max\left((\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right)}{\max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right)} \leq \frac{(\alpha+1)2^5}{\alpha\left(\frac{2^{11}}{\alpha^2}\right)} = \frac{(\alpha+1)\alpha}{2^6} < \frac{(\alpha+1)^2}{2^6} < \frac{1}{(\alpha+1)2^3}.$$

So,

$$x_{n+1} < \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{1}{(\alpha+1)2^3} < \frac{\alpha}{(\alpha+1)^2 2^{11}} + \frac{1}{(\alpha+1)^2 2^{10}} + \frac{1}{(\alpha+1)2^3}$$

$$< \frac{1}{(\alpha+1)2^{11}} + \frac{1}{(\alpha+1)2^{10}} + \frac{1}{(\alpha+1)2^3} = \frac{1}{\alpha+1}\left(2^{-11} + 2^{-10} + 2^{-3}\right) < \frac{1}{4\alpha+4}.$$

Now, using the inequality we have just shown and Equation (1.3.6), we have

$$x_{n+6} > \frac{1}{x_{n+1}x_{n+2}} > \frac{4\alpha+4}{3\alpha+1}(2^\eta) > \frac{4\alpha+4}{3\alpha+3}(2^\eta) = \left(\frac{4}{3}\right)x_{n-1}.$$

Thus we have shown

$$x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right)$$

when (1.3.7) holds. Now assume the opposite inequality in (1.3.7). Using Equation (1.3.5) and Equation (1.3.6), we have the following:

$$x_{n+3} = \frac{\alpha 2^\rho}{\alpha+2^\eta} + (x_{n+2})\left(\frac{2^\rho}{\alpha+2^\eta}\right)$$

$$< \frac{\alpha 2^\rho}{\alpha+2^\eta} + \left(\frac{3\alpha+1}{2^\eta}\right)\left(\frac{2^\rho}{\alpha+2^\eta}\right) < 2^{\rho-\eta}\left(\alpha + \frac{3\alpha+1}{2^\eta}\right).$$

So,

$$x_{n+6} > \frac{x_{n+5}}{x_{n+3}} > \frac{\alpha + x_{n+4}}{x_{n+3}x_{n+2}} > \frac{x_{n+4}}{x_{n+3}x_{n+2}}$$

$$> \frac{\alpha}{x_{n+3}x_{n+2}x_{n+1}} > \frac{\alpha}{2^{\rho-\eta}\left(\alpha + \frac{3\alpha+1}{2^\eta}\right)x_{n+2}x_{n+1}}.$$

Since

$$2^\eta > \frac{(\alpha+1)^4 2^{15}}{\alpha} > (\alpha+1)^3 2^{15} > 3\alpha+1,$$

we see that $\frac{3\alpha+1}{2^\eta} < 1$ and using Equation (1.3.3), we get

$$x_{n+6} > \frac{\alpha}{2^{\rho-\eta}(\alpha+1)x_{n+2}x_{n+1}} = \frac{\alpha}{2^{\rho-\eta}(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right)}.$$

Distributing the $2^{\rho-\eta}$, we have

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{1}{2^{\ell}}\right)}. \tag{1.3.8}$$

Now let us assume $\alpha \geq 1$. Then we have

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{\alpha}{2^{\ell}}\right)} = \frac{2^{\ell}}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}} + 1\right)}.$$

In this case

$$2^{\ell} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right) \geq (\alpha+1)^2 2^{11},$$

so

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)}.$$

Since we assumed the reversed inequality in (1.3.7), we have that $2^{\eta-\rho} > 2^5 > 1$. Furthermore, we know from earlier that $2^{\eta} > 2^{15} > 1$. Using this information, we obtain

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)} > \frac{2^{11}(\alpha+1)}{3x_{n+2}}.$$

Now we use Equation (1.3.6) and we obtain

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{3(3\alpha+1)}(x_{n-1}) > \frac{2^{11}(\alpha+1)}{3(3\alpha+3)}(x_{n-1}) = \left(\frac{2^{11}}{9}\right)x_{n-1} > \left(\frac{4}{3}\right)x_{n-1}.$$

We now prove the case when $\alpha < 1$. Here we continue from Equation (1.3.8) with the following:

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\ell+\eta-\rho}} + \frac{1}{2^{\ell+\eta}} + \frac{1}{2^{\ell}}\right)} = \frac{\alpha 2^{\ell}}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}} + 1\right)}.$$

In this case

$$2^{\ell} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right) \geq \frac{2^{11}}{\alpha},$$

so we have

$$x_{n+6} > \frac{2^{11}}{(\alpha+1)x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)}.$$

Since we assumed the reverse inequality in (1.3.7), we have that $2^{\eta-\rho} > 2^5 > 1$. Furthermore, we know from earlier that $2^\eta > 2^{15} > 1$. Using this information, we obtain

$$x_{n+6} > \frac{2^{11}}{(\alpha+1)x_{n+2}\left(1+\frac{1}{2^{\eta-\rho}}+\frac{1}{2^\eta}\right)} > \frac{2^{11}}{3(\alpha+1)x_{n+2}}.$$

Now we use Equation (1.3.6) and the assumption $\alpha < 1$ to obtain

$$x_{n+6} > \frac{2^{11}}{3(3\alpha+1)(\alpha+1)}(x_{n-1}) > \frac{2^{11}}{24}(x_{n-1}) > \left(\frac{4}{3}\right)x_{n-1}.$$

Thus, we have shown

$$x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right)x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right)$$

when the opposite inequality of (1.3.7) holds. Therefore, we have finished the case where $\ell+\rho \geq \eta$.

We now consider the case $\ell+\rho < \eta$. We show that if this inequality is satisfied for some $n \in \mathbb{N}$, then $p_n = 8$. First we prove that $x_{n+p_n-3} = x_{n+5} > \frac{\alpha}{2}$. Notice that since our assumptions have changed, Equations (1.3.6) and (1.3.8) no longer hold. We will now make a new analogue for Equation (1.3.6), namely the forthcoming Equation (1.3.9). Since $\ell+\rho < \eta$ we have $2^{\ell+\rho} < 2^\eta$. Moreover, since $2^\ell > 2^{11}$ and $2^\eta > 2^{15}$ we have $\ell > 0$ and $\eta > 0$. So from Equation (6),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho}} = \frac{3\alpha+1}{2^{\ell+\rho}}. \quad (1.3.9)$$

So we have

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{\alpha}{x_{n+2}} > \frac{\alpha 2^{\ell+\rho}}{3\alpha+1}.$$

Notice that

$$\frac{\alpha 2^\ell}{3\alpha+1} > \max\left(\frac{\alpha 2^{11}}{(3\alpha+1)\alpha^2}, \frac{\alpha(\alpha+1)^2 2^{11}}{(3\alpha+1)}\right) \geq \max\left(\frac{2^{11}}{(3\alpha+1)\alpha}, \frac{\alpha(\alpha+1)^2 2^{11}}{(3\alpha+3)}\right)$$

$$= \max\left(\frac{2^{11}}{(3\alpha+1)\alpha}, \frac{\alpha(\alpha+1)2^{11}}{3}\right) > 1.$$

So $x_{n+5} > 2^{\rho} > \frac{\alpha}{2}$.

We now prove that

$$x_{n+p_n-2} = x_{n+6} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right).$$

Notice that from Equation (1.3.5), we have

$$x_{n+3} = \frac{\alpha 2^{\rho}}{\alpha + 2^{\eta}} + (x_{n+2})\left(\frac{2^{\rho}}{\alpha + 2^{\eta}}\right) < 2^{\rho-\eta}(\alpha + x_{n+2}). \tag{1.3.10}$$

Since $x_{n+5} > 2^{\rho}$, we get

$$x_{n+6} > \frac{x_{n+5}}{x_{n+3}} > 2^{\eta-\rho}\frac{x_{n+5}}{\alpha + x_{n+2}} > \frac{2^{\eta}}{\alpha + x_{n+2}}. \tag{1.3.11}$$

We assume $\frac{1}{\alpha} \leq \alpha + 1$ and we use Equation (1.3.9). We know that $2^{\rho} > \frac{\alpha}{2}$ and $2^{\ell} > (\alpha+1)^2 2^{11} \geq \frac{(\alpha+1)2^{11}}{\alpha}$. So,

$$x_{n+2} < \frac{3\alpha+1}{2^{\ell+\rho}} < \frac{3\alpha+1}{\frac{(\alpha+1)2^{11}}{\alpha}\left(\frac{\alpha}{2}\right)} = \frac{3\alpha+1}{(\alpha+1)2^{10}} < \frac{3\alpha+3}{(\alpha+1)2^{10}} < 2^{-8}.$$

So, since $\frac{1}{\alpha} \leq \alpha + 1$,

$$x_{n+6} > \frac{2^{\eta}}{\alpha + x_{n+2}} > \frac{2^{\eta}}{\alpha + 2^{-8}} > \max\left(\frac{2^{15}}{(\alpha+2^{-8})\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{(\alpha+2^{-8})\alpha}\right)$$

$$\geq \frac{(\alpha+1)^4 2^{15}}{(\alpha+2^{-8})\alpha} > \frac{(\alpha+1)^4 2^{15}}{(\alpha+1)^2} = (\alpha+1)^2 2^{15} > (\alpha+1)^2 2^{11} \geq \frac{2^{11}}{\alpha^2}.$$

We now assume $\frac{1}{\alpha} > \alpha + 1$ and we use Equation (1.3.9). We know that $2^{\rho} > \frac{\alpha}{2}$ and $2^{\ell} > \frac{2^{11}}{\alpha^2} \geq \frac{(\alpha+1)2^{11}}{\alpha}$. So,

$$x_{n+2} < \frac{3\alpha+1}{2^{\ell+\rho}} < \frac{3\alpha+1}{\frac{(\alpha+1)2^{11}}{\alpha}\left(\frac{\alpha}{2}\right)} = \frac{3\alpha+1}{(\alpha+1)2^{10}} < \frac{3\alpha+3}{(\alpha+1)2^{10}} < 2^{-8}.$$

We now use Equation (1.3.11) and our assumption $\frac{1}{\alpha} > \alpha + 1$ to obtain the following:

$$x_{n+6} > \frac{2^{\eta}}{\alpha + x_{n+2}} > \frac{2^{\eta}}{\alpha + 2^{-8}} > \max\left(\frac{2^{15}}{(\alpha+2^{-8})\,\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{(\alpha+2^{-8})\,\alpha}\right)$$

$$\geq \frac{2^{15}}{(\alpha+2^{-8})\,\alpha^3} > \frac{(\alpha+1)2^{15}}{(\alpha+1)\alpha^2} = \frac{2^{15}}{\alpha^2} > \frac{2^{11}}{\alpha^2} > (\alpha+1)^2 2^{11}.$$

Thus we have shown that

$$x_{n+p_n-2} = x_{n+6} > \max\left(\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right).$$

Now we prove

$$x_{n+p_n-1} = x_{n+7} \geq \left(\frac{4}{3}\right) x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right).$$

Notice that

$$x_{n+7} = \frac{\alpha + x_{n+6}}{x_{n+4}} > \frac{x_{n+6}}{x_{n+4}} = \frac{\alpha + x_{n+5}}{x_{n+4}x_{n+3}} > \frac{x_{n+5}}{x_{n+4}x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}x_{n+4}} > \frac{1}{x_{n+2}x_{n+3}}.$$

Using Equations (1.3.9) and (1.3.10), we have

$$x_{n+7} > \frac{1}{x_{n+2}x_{n+3}} > \frac{2^{\ell+\rho}}{(3\alpha+1)x_{n+3}} > \frac{2^{\ell+\eta}}{(3\alpha+1)(\alpha+x_{n+2})}.$$

Earlier we demonstrated that $x_{n+2} < 2^{-8}$. Furthermore, we have assumed that $2^{\ell} > (\alpha+1)^2 2^{11}$. Thus,

$$x_{n+7} > \frac{2^{\ell+\eta}}{(3\alpha+1)(\alpha+x_{n+2})} > \frac{2^{\ell+\eta}}{(3\alpha+3)(\alpha+1)} > \frac{(\alpha+1)^2 2^{11} 2^{\eta}}{(3\alpha+3)(\alpha+1)}$$

$$= \left(\frac{2^{11}}{3}\right) x_{n-1} > \left(\frac{4}{3}\right) x_{n-1}.$$

Hence $x_{n+p_n-1} = x_{n+7} \geq \left(\frac{4}{3}\right) x_{n-1} > \max\left(\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right)$. Now we apply Lemma 1, and then the proof is done. ∎

1.4 Special Case #28

We now study special case #28

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}. \tag{1.4.1}$$

Particularly, we show that whenever $C > 0$, the difference equation (1.4.1) has unbounded solutions for some initial conditions.

Theorem 3. *If $C > 0$, then the difference equation* (1.4.1) *has unbounded solutions for some initial conditions.*

Proof. We choose initial conditions so that

$$x_0 > \max\left(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right),$$

$$x_{-1} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right).$$

We show that there exists $D = 2$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \{7, 8\}$ so that the following holds: Whenever

$$x_{n-1} > \max\left(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right),$$

$$x_{n-2} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right),$$

then we have

$$x_{n+p_n-1} > \max\left(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right),$$

$$x_{n+p_n-2} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right), \quad x_{n+p_n-1} \geq 2x_{n-1}.$$

First assume

$$x_{n-1} > \max\left(1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right),$$

$$x_{n-2} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right).$$

Using algebra, we immediately obtain the following:

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10}; \quad x_{n+1} = \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{x_{n-2}} < \frac{x_n}{10};$$

$$x_{n+1} = \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{Cx_{n-1}} < \frac{x_n}{1000C};$$

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10C};$$

$$x_{n+2} = \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{100C}; \quad x_{n+2} = \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{1000}.$$

So we get the following inequalities:

$$100000x_{n+2} < 100x_{n+1} < 10x_n < x_{n-1}; \tag{1.4.2}$$

$$1000000C^3x_{n+2} < 10000C^2x_{n+1} < 10Cx_n < x_{n-1}. \tag{1.4.3}$$

Using Equation (1.4.3) we get

$$\frac{x_{n+2}}{x_{n+3}} = Cx_{n+1} + x_n < 2x_n. \tag{1.4.4}$$

We use Equations (1.4.2) and (1.4.3) to get

$$\begin{aligned} Cx_{n+3} + x_{n+2} &= x_{n+2}\left(1 + \frac{C}{Cx_{n+1} + x_n}\right) \\ &< x_{n+2}\left(1 + \frac{1}{x_{n+1}}\right) < x_{n+2}\left(1 + \frac{1}{1000x_{n+2}}\right) \\ &= x_{n+2} + \frac{1}{1000} = \frac{x_n}{(Cx_n + x_{n-1})(Cx_{n-1} + x_{n-2})} + \frac{1}{1000} \\ &< \frac{1}{Cx_{n-2}} + \frac{1}{1000} < \frac{1}{10} + \frac{1}{1000} < 1. \end{aligned}$$

In short,

$$Cx_{n+3} + x_{n+2} < 1. \tag{1.4.5}$$

Using Equations (1.4.3) and (1.4.4), we have

$$x_{n+5} = \frac{x_{n+4}}{Cx_{n+3} + x_{n+2}} = \frac{1}{Cx_{n+2} + x_{n+1}}\left(\frac{x_{n+3}}{Cx_{n+3} + x_{n+2}}\right).$$

$$x_{n+5} = \frac{1}{Cx_{n+2} + x_{n+1}}\left(\frac{1}{C + \frac{x_{n+2}}{x_{n+3}}}\right) > \frac{1}{2x_{n+1}}\left(\frac{1}{C + 2x_n}\right). \tag{1.4.6}$$

Furthermore we have

$$x_{n+4} = \frac{x_{n+3}}{Cx_{n+2} + x_{n+1}} < \frac{x_{n+3}}{x_{n+1}} = \frac{1}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} < \frac{1}{x_{n-1}x_n}. \tag{1.4.7}$$

Using Equations (1.4.6) and (1.4.7), we obtain

$$x_{n+6} = \frac{x_{n+5}}{Cx_{n+4} + x_{n+3}} > \left(\frac{1}{2x_{n+1}}\right)\left(\frac{1}{C + 2x_n}\right)\left(\frac{1}{\frac{C}{x_{n-1}x_n} + x_{n+3}}\right)$$

$$= \left(\frac{x_{n-1}}{2x_{n+1}}\right)\left(\frac{1}{C+2x_n}\right)\left(\frac{1}{\frac{C}{x_n}+x_{n-1}x_{n+3}}\right)$$

$$= \left(\frac{Cx_{n-1}+x_{n-2}}{2C+4x_n}\right)\left(\frac{Cx_{n-2}+x_{n-3}}{\frac{C}{x_n}+x_{n-1}x_{n+3}}\right)$$

$$= \left(\frac{(Cx_{n-1}+x_{n-2})^2}{2C+4x_n}\right)\left(\frac{Cx_{n-2}+x_{n-3}}{\frac{C}{x_{n+1}}+x_{n-1}x_{n+3}(Cx_{n-1}+x_{n-2})}\right).$$

Using Equations (1.4.2) and (1.4.3), we see

$$x_{n-1}x_{n+3}(Cx_{n-1}+x_{n-2}) = \frac{x_{n-1}x_{n+2}(Cx_{n-1}+x_{n-2})}{Cx_{n+1}+x_n} = \frac{x_{n-1}x_{n+1}(Cx_{n-1}+x_{n-2})}{(Cx_{n+1}+x_n)(Cx_n+x_{n-1})}$$

$$= \frac{x_{n-1}x_n}{(Cx_{n+1}+x_n)(Cx_n+x_{n-1})} < 1$$

$$< C^3x_{n-2} < \frac{C(Cx_{n-1}+x_{n-2})(Cx_{n-2}+x_{n-3})}{x_{n-1}} = \frac{C}{x_{n+1}}.$$

Using this in the prior inequality, we get

$$x_{n+6} > \frac{x_{n-1}(Cx_{n-1}+x_{n-2})}{4C^2+8Cx_n}. \tag{1.4.8}$$

Using this fact, we have

$$x_{n+7} = \frac{x_{n+6}}{Cx_{n+5}+x_{n+4}} = \frac{x_{n+6}}{\left(\frac{Cx_{n+4}}{Cx_{n+3}+x_{n+2}}\right)+x_{n+4}} = \left(\frac{x_{n+6}}{x_{n+4}}\right)\left(\frac{1}{1+\frac{C}{Cx_{n+3}+x_{n+2}}}\right)$$

$$= \left(\frac{x_{n+4}}{Cx_{n+3}+x_{n+2}}\right)\left(\frac{1}{Cx_{n+4}+x_{n+3}}\right)\left(\frac{1}{x_{n+4}})(\frac{1}{1+\frac{C}{Cx_{n+3}+x_{n+2}}}\right)$$

$$= \left(\frac{1}{C+Cx_{n+3}+x_{n+2}}\right)\left(\frac{1}{Cx_{n+4}+x_{n+3}}\right) > \left(\frac{1}{Cx_{n+4}+x_{n+3}}\right)\left(\frac{1}{C+1}\right).$$

We now use Equations (1.4.2) and (1.4.3) to show

$$x_{n+3} = \frac{x_{n+2}}{Cx_{n+1}+x_n} = \frac{x_{n+1}}{(Cx_{n+1}+x_n)(Cx_n+x_{n-1})}$$

$$< \frac{x_{n+1}}{(Cx_{n+1}+500Cx_{n+1}+5x_{n+1})(Cx_n+x_{n-1})}$$

$$< \frac{1}{(501C+5)x_{n-1}}$$

$$< \frac{1}{(4C+4)x_{n-1}}.$$

We now use this fact and Equation (1.4.7) to obtain

$$x_{n+7} > \left(\frac{1}{\frac{C}{x_n x_{n-1}} + \frac{1}{x_{n-1}(4C+4)}}\right)\left(\frac{1}{C+1}\right) = \left(\frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right). \quad (1.4.9)$$

Suppose $x_n \leq 4C(C+1)$ we will show that in this case $p_n = 7$. Using Equation (1.4.6), we have

$$x_{n+p_n-2} = x_{n+5} > \frac{1}{2x_{n+1}}\left(\frac{1}{C+2x_n}\right) =$$

$$\frac{Cx_{n-1}+x_{n-2}}{2x_n}\left(\frac{1}{C+2x_n}\right) > \frac{Cx_{n-1}}{2x_n(C+2x_n)} \geq \frac{Cx_{n-1}}{8C(C+1)(C+8C(C+1))}$$

$$= \frac{x_{n-1}}{8(C+1)(C+8C(C+1))} > \frac{x_{n-1}}{100(C+1)^3} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right).$$

Also using Equation (1.4.8), we have

$$x_{n+p_n-1} = x_{n+6} > \frac{x_{n-1}(Cx_{n-1}+x_{n-2})}{4C^2+8Cx_n} > \frac{Cx_{n-1}^2}{4C^2+8C(4C(C+1))}$$

$$= \frac{x_{n-1}^2}{4C+8(4C(C+1))} > \frac{x_{n-1}^2}{50(C^2+C)} > 2x_{n-1}.$$

Now suppose $x_n > 4C(C+1)$. We will show that in this case $p_n = 8$. Using Equation (1.4.8), we have

$$x_{n+p_n-2} = x_{n+6} > \frac{x_{n-1}(Cx_{n-1}+x_{n-2})}{4C^2+8Cx_n} > \frac{x_{n-1}(Cx_{n-1}+x_{n-2})}{9Cx_n}$$

$$= \frac{(Cx_{n-1}+x_{n-2})(Cx_{n-2}+x_{n-3})}{9C} > x_{n-2} > \max\left(10, \frac{10}{C}, \frac{1}{C^3}\right).$$

Also using Equation (1.4.9), we have

$$x_{n+p_n-1} = x_{n+7} > \left(\frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right)$$

$$> \left(\frac{x_{n-1}}{\frac{C}{4C(C+1)} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right) = 2x_{n-1}.$$

Hence, after application of Lemma 1, the proof is complete. ■

1.5 Conclusion

Theorem 1 establishes the boundedness character of special cases #56 and #120 in a range of their parameters. Theorems 2 and 3 establish the boundedness character of the special cases #44 and #28 respectively. Further work should focus on expanding the range for which boundedness character of special cases #56 and #120 is known and resolving Conjecture 3.0.1 in Ref. [1]. There remains only one special case for which Conjecture 3.0.1 has not yet been established. This is special case #70. It is worthwhile to note that special case #70 is part of the period-six trichotomy conjecture. The resolution of the period-six trichotomy conjecture will immediately resolve Conjecture 3.0.1 in Ref. [1]. See Ref. [2] for more details regarding the period-six trichotomy conjecture. We restate, for the convenience of the reader, the period-six trichotomy conjecture.

Conjecture 1. *Assume that* $\alpha, C \in [0,\infty)$. *Then the following period-six trichotomy result is true for the rational equation*

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}. \tag{1.5.1}$$

(a) Every solution of Equation (1.5.1) *converges to its positive equilibrium if and only if* $\alpha C^2 > 1$.

(b) Every solution of Equation (1.5.1) *converges to a not necessarily prime period-six solution of Equation* (1.5.1) *if and only if* $\alpha C^2 = 1$.

(c) Equation (1.5.1) *has unbounded solutions if and only if* $\alpha C^2 < 1$.

For more on boundedness character see Refs. [1-4].

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MANUSCRIPT 2

Unboundedness results for fourth order rational difference equations

by

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Abstract.

We present some new results regarding the boundedness character of the k^{th} order rational difference equation

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

When applied to the general fourth order rational difference equation, these results prove the existence of unbounded solutions for 49 special cases of the fourth order rational difference equation, where the boundedness character has not been established yet. This resolves 49 conjectures posed by E. Camouzis and G. Ladas.

2.1 Introduction

In Ref. [1], E. Camouzis and G. Ladas posed numerous conjectures regarding the boundedness character of third and fourth order rational difference equations. Later on, the same authors published [2], where they proved that there exist unbounded solutions for 60 additional special cases, whose boundedness character had not been established at that time. After this, there remained 149 special cases of third and fourth order, for which E. Camouzis and G. Ladas had conjectured that there exist unbounded solutions and the conjecture remained open. These cases were listed in Ref. [2] in Appendix A.

Later on, [5] was published. In Ref. [5], the authors resolved the conjectures 28, 44, 56, and 120, listed in Appendix A of Ref. [2]. Thus, there remains only one special case of third order for which the boundedness character has not been established yet. This is special case #70, which is the following equation,

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \; n \in \mathbb{N}.$$

In the process of proving a general periodic trichotomy result, the author of Ref.

[8] resolved the conjectures 296, 578, 586, 608, 610, 616, and 618. Furthermore, the conjectures 584, 609, 611, 617, and 619 were resolved in Ref. [4]. Thus, there remain $149 - 16 = 133$ special cases of third and fourth order, for which E. Camouzis and G. Ladas have conjectured that there exist unbounded solutions and the conjecture has not been established yet.

When applied to fourth order rational difference equations, the results in this manuscript establish 49 out of the 133 remaining conjectures from Appendix A of Ref. [2]. Theorem 4 establishes the conjectures 620, 621, 622, 623, 624, 625, 632, 633, 634, 635, 636, 637, 638, 639, 864, 865, 872, 873, 874, 875, 876, 877, 878, 879, 880, 881, 888, 889, 890, 891, 892, 893, 894, and 895. Theorem 5 establishes the conjectures 614, 615, 626, 627, 630, 631, 866, 867, 870, 871, 882, 883, 886, and 887. Theorem 6 establishes conjecture 585. So, there now remain $133 - 49 = 84$ special cases of third and fourth order, for which E. Camouzis and G. Ladas have conjectured that there exist unbounded solutions and the conjecture has not been established yet. We include a list of the remaining cases in Appendix A. The results in this paper can be considered a generalization of the results in Refs. [2-5,7-8]. To be more specific, the idea for Theorem 4 came from attempts to combine the methods used in Ref. [2] with the modulo class techniques used in Ref. [8]. The idea for Theorem 5 came from attempts to combine the methods used in Ref. [2] with the technique of iteration developed in Refs. [3] and [7]. Furthermore, Theorem 6 is a direct generalization of methods used in Refs. [4] and [5].

2.2 Some General Unboundedness Results

In Ref. [8], techniques for proving unboundedness involving modular arithmetic on the indices are introduced. These techniques are expanded upon in Ref. [4] and partially extended to systems in Ref. [6]. Here, we extend these ideas further in order to solve some conjectures in Ref. [2].

We first introduce a condition which allows us to construct unbounded solutions, namely Condition 1. Before doing so, let us first introduce some notation. Let us define the following sets of indices :

$$I_\beta = \{i \in \{1, 2, \ldots, k\} | \beta_i > 0\} \;\; and \;\; I_B = \{j \in \{1, 2, \ldots, k\} | B_j > 0\}.$$

These sets are used extensively in Ref. [9] when referring to the k^{th} order rational difference equation. Similarly, we shall make extensive use of this notation.

In the following proof, we use finite subsets of the set $\{1, \ldots, k\}$ as indexing sets in sums. For example, we may write $\sum_{i \in I_B \cap I_\beta} \beta_i$. In the case where $I_B \cap I_\beta = \{1, 2, 3\}$ then $\sum_{i \in I_B \cap I_\beta} \beta_i = \beta_1 + \beta_2 + \beta_3$. Let us point out the notational convention that if $I_B \cap I_\beta = \emptyset$, then $\sum_{i \in I_B \cap I_\beta} \beta_i = 0$. The notation is similar for all such sums indexed in this way.

Condition 1. *We say that Condition 1 is satisfied if, for some* $p \in \mathbb{N}$, $p | \gcd(I_\beta \setminus I_B)$. *We also must have disjoint sets* $B, L \subset \{0, \ldots, p-1\}$ *with* $B \neq \emptyset$ *and with the following properties.*

1. *For all* $b \in B$, $\{(b - j) \mod p : j \in I_B\} \subset L$.

2. *For all* $\ell \in L$, *there exists* $j \in I_B$ *so that* $(\ell - j) \mod p \in B$.

We now present Theorem 4, which makes use of Condition 1. In the remainder of this section, we will verify Condition 1 for a number of special cases of the fourth order rational difference equation, thereby confirming several conjectures in Refs. [2] and [1].

Theorem 4. *Consider the* k^{th} *order rational difference equation,*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}. \tag{2.2.1}$$

Assume nonnegative parameters and nonnegative initial conditions. Further assume that

$$\frac{4(\sum_{i\in I_B\cap I_\beta}\beta_i)\sum_{j=1}^k B_j}{\min_{j\in I_B}(B_j)} < \left(\sum_{i\in I_\beta\setminus I_B}\beta_i\right) - A,$$

and that Condition 1 is satisfied for Equation (2.2.1)*. Then unbounded solutions of Equation* (2.2.1) *exist for some initial conditions.*

Proof. By assumption, we may choose $p \in \mathbb{N}$ and $B, L \subset \{0, \ldots, p-1\}$ so that Condition 1 is satisfied. Choose initial conditions x_{-m}, where $m \in \{0, \ldots, k-1\}$, so that the following holds. If $(-m \mod p) \in B$, then

$$x_{-m} > \frac{4\alpha\sum_{j=1}^k B_j}{(\min_{j\in I_B}(B_j))((\sum_{i\in I_\beta\setminus I_B}\beta_i) - A)} + \frac{2\sum_{i\in I_\beta\setminus I_B}\beta_i}{\min_{j\in I_B}(B_j)}.$$

If $(-m \mod p) \in L$, then

$$x_{-m} < \frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i) - A}{2\sum_{j=1}^k B_j}.$$

Also, assume $x_{-m} > 0$ for all $m \in \{0, \ldots, k-1\}$. Under this choice of initial conditions, our solution $\{x_n\}$ has the following properties.

(a)

$$x_n > \frac{4\alpha\sum_{j=1}^k B_j}{(\min_{j\in I_B}(B_j))((\sum_{i\in I_\beta\setminus I_B}\beta_i) - A)} + \frac{2\sum_{i\in I_\beta\setminus I_B}\beta_i}{\min_{j\in I_B}(B_j)}$$

whenever $(n \mod p) \in B$.

(b)

$$x_n < \frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i) - A}{2\sum_{j=1}^k B_j}$$

whenever $(n \mod p) \in L$.

(c) $x_n > 0$ for all $n \in \mathbb{N}$.

We prove this using induction on n, our initial conditions provide the base case. Assume that the statement is true for all $n \leq N-1$. We show the statement for $n = N$. This induction proof has three cases. Let us begin by assuming ($N \mod p) \in B$.

Condition 1.1 tells us that in this case, $\{(N-j) \mod p : j \in I_B\} \subset L$. Hence,

$$x_{N-j} < \frac{(\sum_{i \in I_\beta \setminus I_B} \beta_i) - A}{2\sum_{j=1}^{k} B_j}$$

for all $j \in I_B$. Hence, we have the following:

$$x_N = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{N-i}}{A + \sum_{j=1}^{k} B_j x_{N-j}} \geq \frac{\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + (\sum_{j=1}^{k} B_j)\frac{(\sum_{i \in I_\beta \setminus I_B} \beta_i) - A}{2\sum_{j=1}^{k} B_j}} \left(\min_{i \in I_\beta \setminus I_B} (x_{N-i}) \right)$$

$$= \frac{\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \frac{(\sum_{i \in I_\beta \setminus I_B} \beta_i) - A}{2}} \left(\min_{i \in I_\beta \setminus I_B} (x_{N-i}) \right) = \frac{2\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \sum_{i \in I_\beta \setminus I_B} \beta_i} \left(\min_{i \in I_\beta \setminus I_B} (x_{N-i}) \right).$$

To complete this case, notice that since

$$0 \leq \frac{4(\sum_{i \in I_B \cap I_\beta} \beta_i) \sum_{j=1}^{k} B_j}{\min_{j \in I_B}(B_j)} < \left(\sum_{i \in I_\beta \setminus I_B} \beta_i \right) - A,$$

we have that $A < \sum_{i \in I_\beta \setminus I_B} \beta_i$. Thus,

$$\frac{2\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \sum_{i \in I_\beta \setminus I_B} \beta_i} > 1.$$

So, this gives us

$$x_N > \min_{i \in I_\beta \setminus I_B} (x_{N-i}).$$

Since $p | gcd(I_\beta \setminus I_B)$, $N \mod p = (N-i) \mod p$ for all $i \in I_\beta \setminus I_B$. Thus, for all $i \in I_\beta \setminus I_B$,

$$x_{N-i} > \frac{4\alpha \sum_{j=1}^{k} B_j}{(\min_{j \in I_B}(B_j))((\sum_{i \in I_\beta \setminus I_B} \beta_i) - A)} + \frac{2\sum_{i \in I_\beta \setminus I_B} \beta_i}{\min_{j \in I_B}(B_j)}.$$

Thus,

$$x_N > \frac{4\alpha \sum_{j=1}^{k} B_j}{(\min_{j \in I_B}(B_j))((\sum_{i \in I_\beta \setminus I_B} \beta_i) - A)} + \frac{2\sum_{i \in I_\beta \setminus I_B} \beta_i}{\min_{j \in I_B}(B_j)}.$$

This finishes case (a).

We now assume $(N \mod p) \in L$. Since $p|gcd(I_\beta \setminus I_B)$, $N \mod p = (N-i) \mod p$ for all $i \in I_\beta \setminus I_B$. Hence

$$x_{N-i} < \frac{(\sum_{i\in I_\beta\setminus I_B} \beta_i) - A}{2\sum_{j=1}^k B_j}$$

for all $i \in I_\beta \setminus I_B$. Condition 1.2 guarantees that there exists $j \in I_B$ so that

$$x_{N-j} > \frac{4\alpha \sum_{j=1}^k B_j}{(\min_{j\in I_B}(B_j))((\sum_{i\in I_\beta\setminus I_B}\beta_i) - A)} + \frac{2\sum_{i\in I_\beta\setminus I_B}\beta_i}{\min_{j\in I_B}(B_j)}$$

. Hence, we have the following:

$$x_N = \frac{\alpha + \sum_{i=1}^k \beta_i x_{N-i}}{A + \sum_{j=1}^k B_j x_{N-j}}$$

$$< \frac{\alpha + (\sum_{i\in I_\beta\setminus I_B}\beta_i)\left(\frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i)-A}{2\sum_{j=1}^k B_j}\right)}{(\min_{j\in I_B}(B_j))\left(\frac{4\alpha\sum_{j=1}^k B_j}{(\min_{j\in I_B}(B_j))((\sum_{i\in I_\beta\setminus I_B}\beta_i)-A)} + \frac{2\sum_{i\in I_\beta\setminus I_B}\beta_i}{\min_{j\in I_B}(B_j)}\right)} + \frac{\sum_{i\in I_B\cap I_\beta}\beta_i}{\min_{j\in I_B}(B_j)}$$

$$= \frac{\alpha + (\sum_{i\in I_\beta\setminus I_B}\beta_i)\left(\frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i)-A}{2\sum_{j=1}^k B_j}\right)}{2\left(\frac{2\alpha\sum_{j=1}^k B_j}{(\sum_{i\in I_\beta\setminus I_B}\beta_i)-A} + \sum_{i\in I_\beta\setminus I_B}\beta_i\right)} + \frac{\sum_{i\in I_B\cap I_\beta}\beta_i}{\min_{j\in I_B}(B_j)}$$

$$= \frac{\frac{2\alpha\sum_{j=1}^k B_j}{(\sum_{i\in I_\beta\setminus I_B}\beta_i)-A} + \sum_{i\in I_\beta\setminus I_B}\beta_i}{2\left(\frac{2\alpha\sum_{j=1}^k B_j}{(\sum_{i\in I_\beta\setminus I_B}\beta_i)-A} + \sum_{i\in I_\beta\setminus I_B}\beta_i\right)}\left(\frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i) - A}{2\sum_{j=1}^k B_j}\right) + \frac{\sum_{i\in I_B\cap I_\beta}\beta_i}{\min_{j\in I_B}(B_j)}$$

$$= \frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i) - A}{4\sum_{j=1}^k B_j} + \frac{\sum_{i\in I_B\cap I_\beta}\beta_i}{\min_{j\in I_B}(B_j)}.$$

Since we have assumed that

$$\frac{4(\sum_{i\in I_B\cap I_\beta}\beta_i)\sum_{j=1}^k B_j}{\min_{j\in I_B}(B_j)} < \left(\sum_{i\in I_\beta\setminus I_B}\beta_i\right) - A,$$

we get that

$$x_N \leq \frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i) - A}{4\sum_{j=1}^k B_j} + \frac{\sum_{i\in I_B\cap I_\beta}\beta_i}{\min_{j\in I_B}(B_j)} < \frac{(\sum_{i\in I_\beta\setminus I_B}\beta_i) - A}{2\sum_{j=1}^k B_j}.$$

This completes case (b). It is clear that if $x_n > 0$ for $n < N$, then $x_N > 0$ so case (c) is trivial.

We now use the facts we obtained from our induction to prove that a particular subsequence is unbounded. Take $b \in B$. We now show that $\{x_{mp+b}\}_{m=1}^{\infty}$ diverges to ∞. We explained earlier that

$$x_{mp+b} \geq \frac{2\sum_{i\in I_\beta\setminus I_B}\beta_i}{A+\sum_{i\in I_\beta\setminus I_B}\beta_i}\left(\min_{i\in I_\beta\setminus I_B}(x_{mp+b-i})\right).$$

This implies that the following inequality holds for $C = \frac{2\sum_{i\in I_\beta\setminus I_B}\beta_i}{A+\sum_{i\in I_\beta\setminus I_B}\beta_i} > 1$.

$$x_{mp+b} \geq C \min_{i\in\left\{1,\ldots,\left\lfloor\frac{k}{p}\right\rfloor\right\}}(x_{mp+b-ip}).$$

This is a difference inequality which holds for the subsequence $\{x_{mp+b}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in [7]. We set $z_m = x_{mp+b}$ for $m \in \mathbb{N}$. As we have just shown, $\{z_m\}$ satisfies the following difference inequality,

$$z_m \geq C \min_{i\in\left\{1,\ldots,\left\lfloor\frac{k}{p}\right\rfloor\right\}}(z_{m-i}), \quad m \geq k.$$

Using the results of Ref. [7], particularly Theorem 6, we have that for $m \geq k$,

$$\min\left(z_{m-1},\ldots,z_{m-\left\lfloor\frac{k}{p}\right\rfloor}\right) \geq \min\left(y_{\left\lfloor\frac{m-k}{\left\lfloor\frac{k}{p}\right\rfloor}\right\rfloor},\ldots,y_{m-k}\right).$$

Where $\{y_m\}_{m=0}^{\infty}$ is a solution of the difference equation,

$$y_m = Cy_{m-1}, \quad m \in \mathbb{N}. \tag{2.2.2}$$

With $y_0 = \min\left(z_{k-1},\ldots,z_{k-\left\lfloor\frac{k}{p}\right\rfloor}\right)$. Clearly every positive solution diverges to ∞ for the simple difference equation (2.2.2), since $C > 1$. Hence, using the inequality we have obtained, $\{z_m\}_{m=1}^{\infty}$ diverges to ∞. Hence, with given initial conditions, there is a subsequence of our solution $\{x_n\}_{n=1}^{\infty}$, namely $\{x_{mp+b}\}_{m=1}^{\infty}$, which diverges to ∞. Hence, our solution $\{x_n\}_{n=1}^{\infty}$ is unbounded. So, we have exhibited an unbounded solution under our current assumptions. ■

Now, let us apply Theorem 4 to resolve some conjectures in Ref. [2].

Corollary 1. *Consider the 4^{th} order rational difference equation,*

$$x_n = \frac{\alpha + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3} + \epsilon x_{n-4}}{A + Bx_{n-1} + Cx_{n-2} + Ex_{n-4}}, \quad n \in \mathbb{N}. \tag{2.2.3}$$

Assume nonnegative initial conditions and assume that $B, C, E, \delta > 0$ and that all other parameters are allowed to take on arbitrary nonnegative values. Further choose δ large enough so that

$$A + \frac{4(\beta + \gamma + \epsilon)(B + C + E)}{\min(B, C, E)} < \delta.$$

Then unbounded solutions of Equation (2.2.3) *exist for some initial conditions.*

Proof. We check the hypotheses of Theorem 4 and then apply Theorem 4. Since we have chosen δ large enough so that

$$A + \frac{4(\beta + \gamma + \epsilon)(B + C + E)}{\min(B, C, E)} < \delta,$$

we need only check that Condition 1 is satisfied. Notice that for all choices of parameters we have allowed here $3 \in I_\beta$ and $I_\beta \setminus I_B = \{3\}$. This is not accidental. In this case, we let $p = 3$, $B = \{1\}$, and $L = \{0, 2\}$. Notice that $B \cap L = \emptyset$ and notice the following :

$$\begin{aligned} (1-1) &\mod 3 = 0 \in L, \\ (1-2) &\mod 3 = 2 \in L, \\ (1-4) &\mod 3 = 0 \in L, \\ (0-2) &\mod 3 = 1 \in B, \\ (2-1) &\mod 3 = 1 \in B. \end{aligned}$$

Thus Condition 1 is satisfied and so Theorem 4 applies and unbounded solutions of Equation (2.2.3) exist for some initial conditions.

■

Corollary 1 establishes the conjectures 620, 621, 622, 623, 632, 633, 634, 635, 636, 637, 638, 639, 872, 873, 874, 875, 876, 877, 878, 879, 888, 889, 890, 891, 892, 893, 894, and 895.

Corollary 2. *Consider the* 4^{th} *order rational difference equation,*

$$x_n = \frac{\alpha + \gamma x_{n-2} + \delta x_{n-3} + \epsilon x_{n-4}}{A + C x_{n-2} + E x_{n-4}}, \quad n \in \mathbb{N}. \tag{2.2.4}$$

Assume nonnegative initial conditions and assume that $C, E, \delta > 0$ *and that all other parameters are allowed to take on arbitrary nonnegative values. Further choose* δ *large enough so that*

$$A + \frac{4(\gamma + \epsilon)(C + E)}{\min(C, E)} < \delta.$$

Then unbounded solutions of Equation (2.2.4) *exist for some initial conditions.*

Proof. We check the hypotheses of Theorem 4 and then apply Theorem 4. Since we have chosen δ large enough so that

$$A + \frac{4(\gamma + \epsilon)(C + E)}{\min(C, E)} < \delta,$$

we need only check that Condition 1 is satisfied. Notice that for all choices of parameters we have allowed here $3 \in I_\beta$ and $I_\beta \setminus I_B = \{3\}$. This is not accidental. In this case, we let $p = 3$, $B = \{1\}$, and $L = \{0, 2\}$. Notice that $B \cap L = \emptyset$ and notice the following :

$$\begin{aligned} (1-2) &\mod 3 = 2 \in L, \\ (1-4) &\mod 3 = 0 \in L, \\ (0-2) &\mod 3 = 1 \in B, \\ (2-4) &\mod 3 = 1 \in B. \end{aligned}$$

Thus, Condition 1 is satisfied and so Theorem 4 applies and unbounded solutions of Equation (2.2.4) exist for some initial conditions.

■

Corollary 2 establishes the conjectures 624,625,864,865,880, and 881. Theorem 4 resolves Open Problem 6.1 in Ref. [2]. Theorem 4 generalizes Theorem 6.1 in Ref. [2] through the use of modulo class techniques, such as those used in Ref. [8].

2.3 Unboundedness by Iteration

The technique of iteration has been a very useful tool for proving that every solution is bounded in many special cases of the k^{th} order rational difference equation. See Refs. [3] and [7] for a discussion on the technique of boundedness by iteration. In the following theorem, we use the technique of iteration in order to obtain bounds for certain subsequences of the solutions which we study. Obtaining bounds for these subsequences is critical in order to create an unbounded solution. For this reason, we label this technique unboundedness by iteration.

Theorem 5. *Consider the following fourth order rational difference equation*

$$x_n = \frac{\alpha + \beta x_{n-1} + \gamma x_{n-2} + \delta x_{n-3} + \epsilon x_{n-4}}{A + Cx_{n-2} + Ex_{n-4}}, \quad n \in \mathbb{N},$$

Suppose that all of the following conditions hold

(i) $A, C, E > 0$,

(ii) $\delta > 2(A + C + E)$,

(iii) $\frac{1}{\min(C,E)} + \frac{\gamma}{C} + \frac{\epsilon}{E} + \frac{\alpha}{A} + \frac{\alpha\beta}{A^2} + \frac{\beta^2}{AC} + \frac{\beta\gamma}{AC} + \frac{\beta\epsilon}{AE} + \frac{\beta\delta}{AE} \leq 1.$

Then, under a proper choice of initial conditions, $lim_{n\to\infty} x_{3n+1} = \infty$*. So, the above difference equation has unbounded solutions.*

Proof. We choose initial conditions in such a way that for $n < 1$, $x_n \leq 1$ if $n \not\equiv 1 \mod 3$, and $x_n \geq \delta$ if $n \equiv 1 \mod 3$. This provides the base case in the following induction proof. Suppose that for all $n < N$, $x_n \leq 1$ if $n \not\equiv 1 \mod 3$, and $x_n \geq \delta$ if $n \equiv 1 \mod 3$. We shall then prove that $x_N \leq 1$ if $N \not\equiv 1 \mod 3$, and $x_N \geq \delta$ if

$N \equiv 1 \mod 3$. The proof is divided into two cases. The case $N \equiv 1 \mod 3$ and the case $N \not\equiv 1 \mod 3$.

Consider the case where $N \equiv 1 \mod 3$, then

$$x_N = \frac{\alpha + \beta x_{N-1} + \gamma x_{N-2} + \delta x_{N-3} + \epsilon x_{N-4}}{A + Cx_{N-2} + Ex_{N-4}} \geq \frac{\delta x_{N-3}}{A + Cx_{N-2} + Ex_{N-4}}.$$

Now since $N \equiv 1 \mod 3$, $N-2 \not\equiv 1 \mod 3$ and $N-4 \not\equiv 1 \mod 3$. Thus, by our induction hypothesis, $x_{N-2}, x_{N-4} \leq 1$. Thus, we have

$$x_N \geq \frac{\delta x_{N-3}}{A + C + E} \geq 2x_{N-3}.$$

Since $N-3 \equiv 1 \mod 3$ by our induction hypothesis we get,

$$x_N \geq 2x_{N-3} \geq \delta.$$

Now consider the case where $N \not\equiv 1 \mod 3$, then

$$x_N = \frac{\alpha + \beta x_{N-1} + \gamma x_{N-2} + \delta x_{N-3} + \epsilon x_{N-4}}{A + Cx_{N-2} + Ex_{N-4}}$$

$$\leq \frac{\delta x_{N-3}}{A + Cx_{N-2} + Ex_{N-4}} + \frac{\beta x_{N-1}}{A + Cx_{N-2} + Ex_{N-4}} + \frac{\gamma}{C} + \frac{\epsilon}{E} + \frac{\alpha}{A}.$$

Since $N \not\equiv 1 \mod 3$, either $N-2 \equiv 1 \mod 3$ or $N-4 \equiv 1 \mod 3$. Thus, by our induction hypothesis, either $x_{N-2} \geq \delta$ or $x_{N-4} \geq \delta$. Further, since $N \not\equiv 1 \mod 3$, $N-3 \not\equiv 1 \mod 3$ so $x_{n-3} \leq 1$. These facts combine to give us

$$x_N \leq \frac{\delta}{\min(C,E)\delta} + \frac{\beta x_{N-1}}{A + Cx_{N-2} + Ex_{N-4}} + \frac{\gamma}{C} + \frac{\epsilon}{E} + \frac{\alpha}{A}.$$

To complete the proof, we iterate the x_{N-1} term. In other words, we substitute in a copy of our difference equation one step back for this term. Notice that since $N \not\equiv 1 \mod 3$ in this case and $N \in \mathbb{N}$, we have that $N \geq 2$ in this case and so this is permissable. Once we iterate, we get

$$x_N \leq \frac{1}{\min(C,E)} + \frac{\gamma}{C} + \frac{\epsilon}{E} + \frac{\alpha}{A} + \frac{\beta(\alpha + \beta x_{N-2} + \gamma x_{N-3} + \delta x_{N-4} + \epsilon x_{N-5})}{(A + Cx_{N-2} + Ex_{N-4})(A + Cx_{N-3} + Ex_{N-5})}$$

$$\leq \frac{1}{\min(C,E)} + \frac{\gamma}{C} + \frac{\epsilon}{E} + \frac{\alpha}{A} + \frac{\alpha\beta}{A^2} + \frac{\beta^2}{AC} + \frac{\beta\gamma}{AC} + \frac{\beta\epsilon}{AE} + \frac{\beta\delta}{AE} \leq 1.$$

This completes the induction proof. Notice that along the way we proved that, for all $n \equiv 1 \mod 3$,

$$x_n \geq 2x_{n-3}.$$

Thus $lim_{n\to\infty} x_{3n+1} = \infty$. This concludes our proof of unboundedness. ■

Notice that the only parameters which are required to be positive for the given equation in Theorem 5 are $A, C, E, \delta > 0$. B and D must be zero as shown above. The rest are allowed to take on arbitrary nonnegative values. Thus, Theorem 5 establishes that there exist unbounded solutions for some choice of parameters and some choice of initial conditions for the special cases: 614, 615, 626, 627, 630, 631, 866, 867, 870, 871, 882, 883, 886, and 887.

2.4 The equation $x_n = \frac{\alpha + x_{n-3}}{Bx_{n-1} + x_{n-4}}$

For the following difference equation

$$x_n = \frac{\alpha + x_{n-3}}{Bx_{n-1} + x_{n-4}}, \quad n \in \mathbb{N},$$

we show that whenever $B > 2^8$ and $\alpha < \frac{1}{B^3}$ unbounded solutions exist for some choice of nonnegative initial conditions. Our proof will establish the Conjecture 585 in Ref. [2]. We make use of the argument structure presented in Lemma 1 of Ref. [5]. The following lemma is an adaptation to be used for our case here.

Lemma 2. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $[0, \infty)$. Suppose that there exists $D > 1$ and hypotheses $H_1, \ldots, H_k$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \mathbb{N}$ so that the following holds. Whenever x_{n-i} satisfies H_i for all $i \in \{1, \ldots, k\}$, then x_{n+p_n-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{n+p_n-3} \geq Dx_{n-3}$. Further assume that for some $N \in \mathbb{N}$, x_{N-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{N-3} > 0$. Then $\{x_n\}_{n=1}^{\infty}$*

is unbounded. Particularly $\{x_{z_m-3}\}_{m=1}^{\infty}$ *is a subsequence of* $\{x_n\}_{n=1}^{\infty}$ *which diverges to* ∞*, where* $z_m = z_{m-1} + p_{z_{m-1}}$ *and* $z_0 = N$*.*

Proof. Recall from the assumptions that for all $n \in \mathbb{N}$ there exists $p_n \in \mathbb{N}$ so that the following holds. Whenever x_{n-i} satisfies H_i for all $i \in \{1, \ldots, k\}$, then x_{n+p_n-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{n+p_n-3} \geq Dx_{n-3}$. This p_n may not necessarily be unique for any given $n \in \mathbb{N}$. If there is more than one value that can act as p_n for a given n, then take the smallest such value and call that value p_n.

Let $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$, this is well defined from the prior explanation. Using induction, we prove that given $m \in \mathbb{N}$ the following holds. $x_{z_m-3} \geq D^m x_{N-3}$ and x_{z_m-i} satisfies H_i for all $i \in \{1, \ldots, k\}$. By assumption, x_{N-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{N-3} \geq D^0 x_{N-3}$. This provides the base case. Assume $x_{z_{m-1}-i}$ satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{z_{m-1}-3} \geq D^{m-1} x_{N-3}$. Using our earlier assumption, this implies that there exists $p_{z_{m-1}}$ so that $x_{z_{m-1}+p_{z_{m-1}}-i}$ satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{z_{m-1}+p_{z_{m-1}}-3} \geq Dx_{z_{m-1}-3} \geq (D)D^{m-1}x_{N-3} = D^m x_{N-3}$.

So we have shown that $x_{z_m-3} \geq D^m x_{N-3}$ for all $m \in \mathbb{N}$. Hence, the subsequence $\{x_{z_m-3}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ clearly diverges to ∞, since $D > 1$. ■

Theorem 6. *Consider the fourth order rational difference equation,*

$$x_n = \frac{\alpha + x_{n-3}}{Bx_{n-1} + x_{n-4}}, \quad n \in \mathbb{N}. \tag{2.4.1}$$

Suppose $B > 2^8$ *and* $\alpha < \frac{1}{B^3}$*, then Equation* (2.4.1) *has unbounded solutions for some initial conditions.*

Proof. We choose initial conditions so that

$$x_{-2} > B, \qquad x_{-3} < \frac{1}{4},$$

and one of the following holds

1. $x_0 < \frac{1}{4B}$ and $x_{-1} < \frac{1}{B}$,

2. $\frac{1}{4B} \leq x_0 \leq 2x_{-2}$ and $x_{-1} < \frac{2}{B^2 x_{-2}}$,

3. $x_0 > 2x_{-2}$ and $x_{-1} < \frac{2}{B^2 x_{-2}}$.

We show that there exists $D = 2$ so that for all $n \in \mathbb{N}$, there exists $p_n \in \{2, 3, 5\}$ so that the following holds.

Whenever

$$x_{n-3} > B, \qquad x_{n-4} < \frac{1}{4},$$

and one of the following holds

1. $x_{n-1} < \frac{1}{4B}$ and $x_{n-2} < \frac{1}{B}$,

2. $\frac{1}{4B} \leq x_{n-1} \leq 2x_{n-3}$ and $x_{n-2} < \frac{2}{B^2 x_{n-3}}$,

3. $x_{n-1} > 2x_{n-3}$ and $x_{n-2} < \frac{2}{B^2 x_{n-3}}$.

Then, we have

$$x_{n+p_n-3} > Dx_{n-3} > B, \qquad x_{n+p_n-4} < \frac{1}{4},$$

and one of the following holds

1. $x_{n+p_n-1} < \frac{1}{4B}$ and $x_{n+p_n-2} < \frac{1}{B}$,

2. $\frac{1}{4B} \leq x_{n+p_n-1} \leq 2x_{n+p_n-3}$ and $x_{n+p_n-2} < \frac{2}{B^2 x_{n+p_n-3}}$,

3. $x_{n+p_n-1} > 2x_{n+p_n-3}$ and $x_{n+p_n-2} < \frac{2}{B^2 x_{n+p_n-3}}$.

First assume

$$x_{n-1} < \frac{1}{4B}, \qquad x_{n-2} < \frac{1}{B}, \qquad x_{n-3} > B, \qquad x_{n-4} < \frac{1}{4}.$$

In this case, $p_n = 3$. Since $B > 2^8$, we have

$$x_{n+p_n-4} = x_{n-1} < \frac{1}{4B} < \frac{1}{4}.$$

Since $x_{n-4} < \frac{1}{4}$ and $x_{n-1} < \frac{1}{4B}$, we have

$$x_{n+p_n-3} = x_n = \frac{\alpha + x_{n-3}}{Bx_{n-1} + x_{n-4}} \geq \frac{x_{n-3}}{2\max\left(Bx_{n-1}, x_{n-4}\right)} > 2x_{n-3} > B.$$

Since $x_{n-2} < \frac{1}{B}$ and $\alpha < \frac{1}{B^3}$,

$$x_{n+p_n-2} = x_{n+1} = \frac{\alpha + x_{n-2}}{Bx_n + x_{n-3}} \leq \frac{\frac{1}{B^3} + \frac{1}{B}}{Bx_n} < \frac{2}{B^2 x_n} < \frac{2}{B^3} < \frac{1}{B}.$$

Hence, regardless of the value of x_{n+p_n-1}, one of our requirements is satisfied. If $x_{n+p_n-1} < \frac{1}{4B}$, then requirement (1) is satisfied. If $\frac{1}{4B} \leq x_{n+p_n-1} \leq 2x_{n+p_n-3}$, then requirement (2) is satisfied. If $x_{n+p_n-1} > 2x_{n+p_n-3}$, then requirement (3) is satisfied.

Next assume

$$\frac{1}{4B} \leq x_{n-1} \leq 2x_{n-3}, \qquad x_{n-2} < \frac{2}{B^2 x_{n-3}}, \qquad x_{n-3} > B, \qquad x_{n-4} < \frac{1}{4}.$$

In this case, $p_n = 5$. Since $B > 2^8$ and $\alpha < \frac{1}{B^3}$, we have

$$x_{n+p_n-4} = x_{n+1} = \frac{\alpha + x_{n-2}}{Bx_n + x_{n-3}} < \frac{1}{B^3 x_{n-3}} + \frac{x_{n-2}}{x_{n-3}}$$

$$< \frac{1}{B^3 x_{n-3}} + \frac{2}{B^2 x_{n-3}^2} < \frac{4}{B^3 x_{n-3}} < \frac{1}{4}.$$

Since $x_{n-2} < \frac{2}{B^2 x_{n-3}}$ and $B > 2^8$, we have

$$x_{n+p_n-3} = x_{n+2} = \frac{\alpha + x_{n-1}}{Bx_{n+1} + x_{n-2}} \geq \frac{x_{n-1}}{2\max\left(Bx_{n+1}, x_{n-2}\right)}$$

$$> \frac{x_{n-1}}{2\max\left(\frac{4}{B^2 x_{n-3}}, \frac{2}{B^2 x_{n-3}}\right)} \geq \frac{B^2 x_{n-3}}{2^5 B} > 2x_{n-3} > B.$$

Also notice that,

$$x_{n+p_n-2} = x_{n+3} = \frac{\alpha + x_n}{Bx_{n+2} + x_{n-1}}$$

$$\leq \frac{\alpha}{Bx_{n+2}} + \frac{\alpha}{B^2 x_{n+2} x_{n-1}} + \frac{x_{n-3}}{(Bx_{n+2} + x_{n-1})(Bx_{n-1} + x_{n-4})}$$

$$< \frac{1}{Bx_{n+2}} + \frac{x_{n-3}}{(Bx_{n+2} + x_{n-1})(Bx_{n-1} + x_{n-4})} < \frac{1}{Bx_{n+2}} + \frac{x_{n-3}}{Bx_{n+2}(Bx_{n-1} + x_{n-4})}$$

$$< \frac{1}{Bx_{n+2}} + \frac{2^5 x_{n-3}}{B^2 x_{n-3}(Bx_{n-1} + x_{n-4})} < \frac{1}{Bx_{n+2}} + \frac{2^5}{B^3 x_{n-1}} < \frac{1}{B^2} + \frac{2^7}{B^2} < \frac{1}{B}.$$

Notice that since $B > 2^8$, $\alpha < \frac{1}{B^3}$, $x_{n-3} > B$, $x_{n-1} \leq 2x_{n-3}$, and $x_{n+1} < \frac{4}{B^3 x_{n-3}}$,

$$x_{n+p_n-1} = x_{n+4} = \frac{\alpha + x_{n+1}}{Bx_{n+3} + x_n} < \frac{\alpha}{x_n} + \frac{4}{(B^3 x_{n-3})(Bx_{n+3} + x_n)} < \frac{\alpha}{x_n} + \frac{4}{B^3 x_{n-3} x_n}$$

$$\leq \frac{\alpha(Bx_{n-1} + x_{n-4})}{x_{n-3}} + \frac{4Bx_{n-1} + 4x_{n-4}}{B^3 x_{n-3}^2} < \frac{2Bx_{n-3} + .25}{B^3 x_{n-3}} + \frac{8Bx_{n-3} + 1}{B^3 x_{n-3}^2}$$

$$< \frac{16Bx_{n-3} + 2}{B^3 x_{n-3}} < \frac{16}{B^2} + \frac{2}{B^3 x_{n-3}} < \frac{1}{4B}.$$

Hence, requirement (1) is satisfied in this case. Finally, assume

$$x_{n-1} > 2x_{n-3}, \qquad x_{n-2} < \frac{2}{B^2 x_{n-3}}, \qquad x_{n-3} > B, \qquad x_{n-4} < \frac{1}{4}.$$

In this case, $p_n = 2$. Immediately, we have

$$x_{n+p_n-4} = x_{n-2} < \frac{2}{B^2 x_{n-3}} < \frac{1}{4}.$$

Also by assumption,

$$x_{n+p_n-3} = x_{n-1} > 2x_{n-3} > B.$$

Furthermore, since $x_{n-1} > 2x_{n-3}$ and $\alpha < \frac{1}{B^3}$,

$$x_{n+p_n-2} = x_n = \frac{\alpha + x_{n-3}}{Bx_{n-1} + x_{n-4}} < \frac{\alpha}{Bx_{n-1}} + \frac{x_{n-3}}{Bx_{n-1}} < \frac{1}{2B^5} + \frac{1}{2B} < \frac{1}{B}.$$

Furthermore,

$$x_{n+p_n-1} = x_{n+1} = \frac{\alpha + x_{n-2}}{Bx_n + x_{n-3}} < \frac{\alpha}{x_{n-3}} + \frac{x_{n-2}}{x_{n-3}} < \frac{1}{B^3 x_{n-3}} + \frac{2}{B^2 x_{n-3}^2} < \frac{1}{4B}.$$

Hence, requirement (1) is satisfied in this case, so after application of Lemma 2 the proof is complete. ■

2.5 Conclusion

Our work here resolves more than one third of the outstanding conjectures regarding the existence of unbounded solutions for 4^{th} order rational difference equations. However, there remain 84 special cases of third and fourth order, for which E. Camouzis and G. Ladas have conjectured that there exist unbounded solutions and the conjecture has not been established yet. We include a list of the remaining cases in the attached Appendix A. One special case of particular interest is special case #70, since it is the only remaining third order rational difference equation whose boundedness character is yet to be determined. Special case #70 is the following equation,

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

The technique used here in Theorem 6 and originally developed in Ref. [5] is a new approach that we have found useful for tackling some of the particularly thorny cases. Perhaps this type of approach will conquer the special case #70. For the most recent ideas regarding boundedness character, the reader should look to Refs. [1,5,8]. Much of our work here is built off of the ideas in these papers. For readers wishing to expand into higher order rational difference equations, for example fifth order and beyond, Conjecture 1 of Ref. [8] may be of interest. We feel that further work in this direction would be best focused on resolving any of the conjectures we have just mentioned.

2.6 Appendix A

In this appendix, we present the remaining 84 special cases of order less than or equal to four, for each of which E. Camouzis and G. Ladas have conjectured that the equation has unbounded solutions in some range of the parameters and for some initial conditions.

#70:	$x_{n+1} = (\alpha + x_n)/(Cx_{n-1} + x_{n-2})$
#292:	$x_{n+1} = (\beta x_n + \epsilon x_{n-3})/(Cx_{n-1})$
#293:	$x_{n+1} = (\alpha + \beta x_n + \epsilon x_{n-3})/(Cx_{n-1})$
#294:	$x_{n+1} = (\beta x_n + \epsilon x_{n-3})/(A + Cx_{n-1})$
#295:	$x_{n+1} = (\alpha + \beta x_n + \epsilon x_{n-3})/(A + Cx_{n-1})$
#297:	$x_{n+1} = (\alpha + \epsilon x_{n-3})/(Bx_n + Cx_{n-1})$
#308:	$x_{n+1} = (\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(Cx_{n-1})$
#309:	$x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(Cx_{n-1})$
#310:	$x_{n+1} = (\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(A + Cx_{n-1})$
#311:	$x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(A + Cx_{n-1})$
#328:	$x_{n+1} = (\delta x_{n-2} + \epsilon x_{n-3})/(Bx_n)$
#329:	$x_{n+1} = (\alpha + \delta x_{n-2} + \epsilon x_{n-3})/(Bx_n)$
#330:	$x_{n+1} = (\delta x_{n-2} + \epsilon x_{n-3})/(A + Bx_n)$
#331:	$x_{n+1} = (\alpha + \delta x_{n-2} + \epsilon x_{n-3})/(A + Bx_n)$
#332:	$x_{n+1} = (\beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(Bx_n)$
#333:	$x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(Bx_n)$
#334:	$x_{n+1} = (\beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(A + Bx_n)$
#335:	$x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(A + Bx_n)$
#352:	$x_{n+1} = (\delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$
#353:	$x_{n+1} = (\alpha + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$
#354:	$x_{n+1} = (\delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$
#355:	$x_{n+1} = (\alpha + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$
#356:	$x_{n+1} = (\beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$
#357:	$x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$
#358:	$x_{n+1} = (\beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$

#359: $x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$

#368: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$

#369: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$

#370: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$

#371: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$

#372: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$

#373: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1})$

#374: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$

#375: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + Cx_{n-1})$

#404: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(Dx_{n-2})$

#405: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(Dx_{n-2})$

#406: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(A + Dx_{n-2})$

#407: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \epsilon x_{n-3})/(A + Dx_{n-2})$

#417: $x_{n+1} = (\alpha + \epsilon x_{n-3})/(Cx_{n-1} + Dx_{n-2})$

#420: $x_{n+1} = (\beta x_n + \epsilon x_{n-3})/(Cx_{n-1} + Dx_{n-2})$

#421: $x_{n+1} = (\alpha + \beta x_n + \epsilon x_{n-3})/(Cx_{n-1} + Dx_{n-2})$

#517: $x_{n+1} = (\alpha + \beta x_n)/(Ex_{n-3})$

#532: $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(Ex_{n-3})$

#533: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(Ex_{n-3})$

#537: $x_{n+1} = (\alpha + \gamma x_{n-1})/(Bx_n + Ex_{n-3})$

#548: $x_{n+1} = (\beta x_n)/(Cx_{n-1} + Ex_{n-3})$

#549: $x_{n+1} = (\alpha + \beta x_n)/(Cx_{n-1} + Ex_{n-3})$

#564: $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(Cx_{n-1} + Ex_{n-3})$

#577: $x_{n+1} = (\alpha + \delta x_{n-2})/(Ex_{n-3})$

#580: $x_{n+1} = (\beta x_n + \delta x_{n-2})/(Ex_{n-3})$

#581: $x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2})/(Ex_{n-3})$

#592: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2})/(E x_{n-3})$

#593: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2})/(E x_{n-3})$

#594: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2})/(A + E x_{n-3})$

#595: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2})/(A + E x_{n-3})$

#596: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2})/(E x_{n-3})$

#597: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2})/(E x_{n-3})$

#598: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2})/(A + E x_{n-3})$

#599: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2})/(A + E x_{n-3})$

#600: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2})/(B x_n + E x_{n-3})$

#601: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2})/(B x_n + E x_{n-3})$

#612: $x_{n+1} = (\beta x_n + \delta x_{n-2})/(C x_{n-1} + E x_{n-3})$

#613: $x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2})/(C x_{n-1} + E x_{n-3})$

#628: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2})/(C x_{n-1} + E x_{n-3})$

#629: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2})/(C x_{n-1} + E x_{n-3})$

#644: $x_{n+1} = (\beta x_n)/(D x_{n-2} + E x_{n-3})$

#645: $x_{n+1} = (\alpha + \beta x_n)/(D x_{n-2} + E x_{n-3})$

#660: $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(D x_{n-2} + E x_{n-3})$

#661: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(D x_{n-2} + E x_{n-3})$

#676: $x_{n+1} = (\beta x_n)/(C x_{n-1} + D x_{n-2} + E x_{n-3})$

#677: $x_{n+1} = (\alpha + \beta x_n)/(C x_{n-1} + D x_{n-2} + E x_{n-3})$

#692: $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(C x_{n-1} + D x_{n-2} + E x_{n-3})$

#848: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(E x_{n-3})$

#849: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(E x_{n-3})$

#850: $x_{n+1} = (\gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + E x_{n-3})$

#851: $x_{n+1} = (\alpha + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + E x_{n-3})$

#852: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(E x_{n-3})$

#853: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Ex_{n-3})$

#854: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + Ex_{n-3})$

#855: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(A + Ex_{n-3})$

#868: $x_{n+1} = (\beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1} + Ex_{n-3})$

#869: $x_{n+1} = (\alpha + \beta x_n + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1} + Ex_{n-3})$

#884: $x_{n+1} = (\beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1} + Ex_{n-3})$

#885: $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2} + \epsilon x_{n-3})/(Cx_{n-1} + Ex_{n-3})$

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MANUSCRIPT 3

Unboundedness results for systems

by

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Abstract.

We study k^{th} order systems of two rational difference equations

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N}.$$

In particular, we assume non-negative parameters and non-negative initial conditions. We develop several approaches, which allow us to prove that unbounded solutions exist for certain initial conditions in a range of the parameters.

3.1 Introduction

There has been a recent interest in the study of systems of rational difference equations. Our goal is to provide several general theorems, which prove the existence of unbounded solutions for systems of rational difference equations. It is important to realize that these theorems only apply in a range of the parameters and that certain assumptions are placed on the initial conditions in order to achieve unbounded solutions.

We will proceed in the following manner. First, we will introduce the reader to the source of the idea for the theorem. For example, if the idea arose from the study of cerain special cases, we will present these cases and describe how they motivate the subsequent theorem. If the idea was adapted from prior results, which do not originally apply to systems, we will of course cite the result, and then describe in detail the adaptations necessary.

Before beginning, let us look closely at our notation. We find that often times for rational difference equations the behavior can change in dramatic ways depending on whether a particular parameter is zero or positive. It is for this reason that we adopt a notation similar to that presented in Theorem 6 of Ref.

[3]. So we let $I_\beta = \{i \in \{1,\ldots,k\} | \beta_i > 0\}$, $I_\gamma = \{i \in \{1,\ldots,k\} | \gamma_i > 0\}$, $I_\delta = \{i \in \{1,\ldots,k\} | \delta_i > 0\}$, $I_\epsilon = \{i \in \{1,\ldots,k\} | \epsilon_i > 0\}$, $I_B = \{j \in \{1,\ldots,k\} | B_j > 0\}$, $I_C = \{j \in \{1,\ldots,k\} | C_j > 0\}$, $I_D = \{j \in \{1,\ldots,k\} | D_j > 0\}$, and $I_E = \{j \in \{1,\ldots,k\} | E_j > 0\}$. This also proves beneficial later when we adapt an unboundedness result from Ref. [5] as the author of Ref. [5] uses a similar notation.

3.2 Unboundedness Results Involving Modulo Classes

Here we will present several general theorems, which prove unboundedness for systems of two rational difference equations. We feel that it will be helpful for the reader to see some of the special cases which led to the forthcoming Theorem 7 even though these cases are eventually subsumed by Theorem 7. Here is the first example.

Example 1. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_2 x_{n-2} + \gamma_2 y_{n-2}}{A + B_2 x_{n-2}}, \quad y_n = \frac{p + \delta_2 x_{n-2} + \epsilon_2 y_{n-2}}{q + E_2 y_{n-2}}, \quad n = 0, 1, 2, \ldots.$$

We assume non-negative parameters and non-negative initial conditions. We further assume the following

1. $\beta_2, \gamma_2, B_2, \delta_2, \epsilon_2, E_2 > 0$,
2. $\frac{\gamma_2}{A+B_2} > 2$ *and* $\frac{\delta_2}{q+E_2} > 1$,
3. $\frac{\alpha+1+\beta_2}{B_2} < 1$ *and* $\frac{p+1+\epsilon_2}{E_2} < 1$,

then the solutions x_n and y_n are unbounded for some non-negative initial conditions.

Proof. We first prove by induction that under certain non-negative initial conditions $y_{4n} > \max(1, \gamma_2, \delta_2)$ and $x_{4n} < 1$. We choose the initial conditions to provide the base case. Let $y_0 > \max(1, \gamma_2, \delta_2)$ and $x_0 < 1$. Now, let us prove the inductive step. Assume $y_{4n-4} > \max(1, \gamma_2, \delta_2)$ and $x_{4n-4} < 1$, then we have

$$x_{4n-2} = \frac{\alpha + \beta_2 x_{4n-4} + \gamma_2 y_{4n-4}}{A + B_2 x_{4n-4}} \geq \frac{\gamma_2 y_{4n-4}}{A + B_2 x_{4n-4}} > \frac{\gamma_2 y_{4n-4}}{A + B_2}.$$

We have assumed that $\frac{\gamma_2}{A+B_2} > 2$, so we have that $x_{4n-2} > 2y_{4n-4}$. Furthermore we have the following

$$y_{4n-2} = \frac{p + \delta_2 x_{4n-4} + \epsilon_2 y_{4n-4}}{q + E_2 y_{4n-4}} \leq \frac{p + \delta_2 x_{4n-4} + \epsilon_2 y_{4n-4}}{E_2 y_{4n-4}}$$
$$< \frac{p y_{4n-4} + y_{4n-4} + \epsilon_2 y_{4n-4}}{E_2 y_{4n-4}} = \frac{p + 1 + \epsilon_2}{E_2} < 1.$$

Thus, $y_{4n-2} < 1$. Now, we use these facts to get the following

$$y_{4n} = \frac{p + \delta_2 x_{4n-2} + \epsilon_2 y_{4n-2}}{q + E_2 y_{4n-2}} \geq \frac{\delta_2 x_{4n-2}}{q + E_2 y_{4n-2}} > \frac{\delta_2 x_{4n-2}}{q + E_2}.$$

We have assumed that $\frac{\delta_2}{q+E_2} > 1$, so we have that $y_{4n} > x_{4n-2} > 2y_{4n-4} > \max(1, \gamma_2, \delta_2)$. Also, since $x_{4n-2} > 2y_{4n-4} > \max(1, \gamma_2, \delta_2)$, we have the following

$$x_{4n} = \frac{\alpha + \beta_2 x_{4n-2} + \gamma_2 y_{4n-2}}{A + B_2 x_{4n-2}} \leq \frac{\alpha x_{4n-2} + \beta_2 x_{4n-2} + x_{4n-2}}{B_2 x_{4n-2}} = \frac{\alpha + \beta_2 + 1}{B_2}.$$

We have assumed that $\frac{\alpha+1+\beta_2}{B_2} < 1$, so we have that $x_{4n} < 1$. Thus, we have shown that $y_{4n} > \max(1, \gamma_2, \delta_2)$ and $x_{4n} < 1$ for all $n \in \mathbb{N}$. Notice, we have already shown that this implies $y_{4n} > x_{4n-2} > 2y_{4n-4}$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} y_{4n} = \infty$ and $\lim_{n\to\infty} x_{4n+2} = \infty$. ■

Replacing second order with k^{th} order, the second example proceeds similarly.

Example 2. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_k x_{n-k} + \gamma_k y_{n-k}}{A + B_k x_{n-k}}, \quad y_n = \frac{p + \delta_k x_{n-k} + \epsilon_k y_{n-k}}{q + E_k y_{n-k}}, \quad n = 0, 1, 2, \ldots.$$

We assume non-negative parameters and non-negative initial conditions. We further assume the following

1. $\beta_k, \gamma_k, B_k, \delta_k, \epsilon_k, E_k > 0$,

2. $\frac{\gamma_k}{A+B_k} > 2$ *and* $\frac{\delta_k}{q+E_k} > 1$,

3. $\frac{\alpha+1+\beta_k}{B_k} < 1$ *and* $\frac{p+1+\epsilon_k}{E_k} < 1$,

then the solutions x_n and y_n are unbounded for some non-negative initial conditions.

Proof. We first prove by induction that under certain non-negative initial conditions $y_{2kn} > \max(1, \gamma_k, \delta_k)$ and $x_{2kn} < 1$. We choose the initial conditions to provide the base case. Let $y_0 > \max(1, \gamma_k, \delta_k)$ and $x_0 < 1$. Now, let us prove the inductive step. Assume $y_{2kn-2k} > \max(1, \gamma_k, \delta_k)$ and $x_{2kn-2k} < 1$, then we have

$$x_{2kn-k} = \frac{\alpha + \beta_k x_{2kn-2k} + \gamma_k y_{2kn-2k}}{A + B_k x_{2kn-2k}} \geq \frac{\gamma_k y_{2kn-2k}}{A + B_k x_{2kn-2k}} > \frac{\gamma_k y_{2kn-2k}}{A + B_k}.$$

We have assumed that $\frac{\gamma_k}{A+B_k} > 2$, so we have that $x_{2kn-k} > 2y_{2kn-2k}$. Furthermore, we have the following

$$y_{2kn-k} = \frac{p + \delta_k x_{2kn-2k} + \epsilon_k y_{2kn-2k}}{q + E_k y_{2kn-2k}} \leq \frac{p + \delta_k x_{2kn-2k} + \epsilon_k y_{2kn-2k}}{E_k y_{2kn-2k}}$$
$$< \frac{p y_{2kn-2k} + y_{2kn-2k} + \epsilon_k y_{2kn-2k}}{E_k y_{2kn-2k}} = \frac{p + 1 + \epsilon_k}{E_k} < 1.$$

Thus, $y_{2kn-k} < 1$. Now, we use these facts to get the following

$$y_{2kn} = \frac{p + \delta_k x_{2kn-k} + \epsilon_k y_{2kn-k}}{q + E_k y_{2kn-k}} \geq \frac{\delta_k x_{2kn-k}}{q + E_k y_{2kn-k}} > \frac{\delta_k x_{2kn-k}}{q + E_k}.$$

We have assumed that $\frac{\delta_k}{q+E_k} > 1$, so we have that $y_{2kn} > x_{2kn-k} > 2y_{2kn-2k} > \max(1, \gamma_k, \delta_k)$. Also, since $x_{2kn-k} > 2y_{2kn-2k} > \max(1, \gamma_k, \delta_k)$, we have the following

$$x_{2kn} = \frac{\alpha + \beta_k x_{2kn-k} + \gamma_k y_{2kn-k}}{A + B_k x_{2kn-k}} \leq \frac{\alpha x_{2kn-k} + \beta_k x_{2kn-k} + x_{2kn-k}}{B_k x_{2kn-k}} = \frac{\alpha + \beta_k + 1}{B_k}.$$

We have assumed that $\frac{\alpha+1+\beta_k}{B_k} < 1$, so we have that $x_{2kn} < 1$. Thus, we have shown that $y_{2kn} > \max(1, \gamma_k, \delta_k)$ and $x_{2kn} < 1$ for all $n \in \mathbb{N}$. Notice, we have already shown that this implies $y_{2kn} > x_{2kn-k} > 2y_{2kn-2k}$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} y_{2kn} = \infty$ and $\lim_{n\to\infty} x_{2kn+k} = \infty$. ■

Notice that in the example above the key to the proof is that when $n \equiv 0 \mod 2k$, then x_n is small and y_n is large. On the other hand when $n \equiv k \mod 2k$, then x_n is large and y_n is small. So modulo classes play a key role in the above proof, though it was unnecessary to mention modulo classes. In the third example, the use of modulo classes becomes more explicit.

Example 3. *Consider the following system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_{k-1}x_{n-k+1} + \beta_k x_{n-k} + \gamma_{k-1}y_{n-k+1} + \gamma_k y_{n-k}}{A + B_{k-1}x_{n-k+1} + B_k x_{n-k} + C_k y_{n-k}}, \quad n = 0, 1, 2, \dots,$$

$$y_n = \frac{p + \delta_{k-1}x_{n-k+1} + \delta_k x_{n-k} + \epsilon_{k-1}y_{n-k+1} + \epsilon_k y_{n-k}}{q + D_{k-1}x_{n-k+1} + E_{k-1}y_{n-k+1} + E_k y_{n-k}}, \quad n = 0, 1, 2, \dots,$$

where $k = 3l+2$ and $l \geq 0$. We assume non-negative parameters and non-negative initial conditions. We further assume the following

1. $\gamma_{k-1}, B_k, B_{k-1}, \delta_k, E_k, E_{k-1} > 0$,
2. $\frac{\gamma_{k-1}}{A+B_{k-1}+B_k+C_k} > 2$ *and* $\frac{\delta_k}{q+D_{k-1}+E_k+E_{k-1}} > 1$,
3. $\frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_{k-1},B_k)} + \frac{\gamma_k}{C_k} < 1$ *and* $\frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1},E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$,
4. $C_k = 0$ *implies* $\gamma_k = 0$ *and* $D_{k-1} = 0$ *implies* $\delta_{k-1} = 0$,

then the solutions x_n and y_n are unbounded for some non-negative initial conditions.

Proof. We choose non-negative initial conditions x_{-m} and y_{-m} where $m \in \{1, \dots, k\}$ so that the following holds. If $-m \equiv -1 \mod 3$, then $y_{-m} > \max(1, \gamma_{k-1}, \delta_k)$ and $x_{-m} < 1$. If $-m \equiv -2 \mod 3$, then $y_{-m} < 1$ and $x_{-m} < 1$. If $-m \equiv -3 \mod 3$, then $y_{-m} < 1$ and $x_{-m} > \max(1, \gamma_{k-1}, \delta_k)$.

Under this choice of initial conditions our solutions $\{x_n\}$ and $\{y_n\}$ have the following properties:

(a) $y_n > \max(1, \gamma_{k-1}, \delta_k)$ and $x_n < 1$ whenever $n \equiv -1 \mod 3$;

(b) $y_n < 1$ and $x_n < 1$ whenever $n \equiv -2 \mod 3$;

(c) $y_n < 1$ and $x_n > \max(1, \gamma_{k-1}, \delta_k)$ whenever $n \equiv -3 \mod 3$.

We prove this using induction on n. Our initial conditions provide the base case. Assume that the statement is true for all $n \leq N-1$. We show the statement for $n = N$. This induction proof has three cases.

Case (a). Let us begin by assuming $N \equiv -1 \mod 3$. Since $N \equiv -1 \mod 3$, we have $N - k = N - 3l - 2 \equiv -3 \mod 3$. So we have that $y_{N-k} < 1$ and $x_{N-k} > \max(1, \gamma_{k-1}, \delta_k)$. Also, since $N \equiv -1 \mod 3$, we have $N-k+1 = N-3l-1 \equiv -2 \mod 3$. So, $y_{N-k+1} < 1$ and $x_{N-k+1} < 1$. From this we demonstrate the desired inequalities $y_N > \max(1, \gamma_{k-1}, \delta_k)$ and $x_N < 1$, given that $N \equiv -1 \mod 3$ holds. Using these facts, we get

$$\begin{aligned} y_N &= \frac{p + \delta_{k-1}x_{N-k+1} + \delta_k x_{N-k} + \epsilon_{k-1}y_{N-k+1} + \epsilon_k y_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}} \\ &\geq \frac{\delta_k x_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}} \\ &> \frac{\delta_k x_{N-k}}{q + D_{k-1} + E_{k-1} + E_k} > x_{N-k}, \end{aligned}$$

since we assumed that $\frac{\delta_k}{q+D_{k-1}+E_k+E_{k-1}} > 1$. Now, since $x_{N-k} > \max(1, \gamma_{k-1}, \delta_k)$, we have $y_N > \max(1, \gamma_{k-1}, \delta_k)$. Now, we will show in the forthcoming equation, that $x_N < 1$. We have

$$x_N = \frac{\alpha + \beta_{k-1}x_{N-k+1} + \beta_k x_{N-k} + \gamma_{k-1}y_{N-k+1} + \gamma_k y_{N-k}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}}$$

$$< \frac{\alpha}{B_k x_{N-k}} + \frac{\gamma_{k-1} y_{N-k+1}}{B_k x_{N-k}} + \frac{\beta_{k-1}+\beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k}$$
$$< \frac{\alpha+1}{B_k} + \frac{\beta_{k-1}+\beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} \le \frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} < 1,$$

since we assumed that $\frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_k,B_{k-1})} + \frac{\gamma_k}{C_k} < 1$. This finishes case (a).

Case (b). We now assume that $N \equiv -2 \mod 3$. Since $N \equiv -2 \mod 3$, we know $N-k = N-3l-2 \equiv -4 \equiv -1 \mod 3$. So $y_{N-k} > \max(1, \gamma_{k-1}, \delta_k)$ and $x_{N-k} < 1$. Also, since $N \equiv -2 \mod 3$, we get $N-k+1 = N-3l-1 \equiv -3 \mod 3$. So $y_{N-k+1} < 1$ and $x_{N-k+1} > \max(1, \gamma_{k-1}, \delta_k)$. Using this we demonstrate the desired inequalities $y_N < 1$ and $x_N < 1$, given that $N \equiv -2 \mod 3$ holds. We get

$$y_N = \frac{p + \delta_{k-1} x_{N-k+1} + \delta_k x_{N-k} + \epsilon_{k-1} y_{N-k+1} + \epsilon_k y_{N-k}}{q + D_{k-1} x_{N-k+1} + E_{k-1} y_{N-k+1} + E_k y_{N-k}}$$
$$< \frac{p}{E_k y_{N-k}} + \frac{\delta_k x_{N-k}}{E_k y_{N-k}} + \frac{\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}}$$
$$< \frac{p+1}{E_k} + \frac{\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} \le \frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1,$$

since we assumed that $\frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1},E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$. Now, we show that $x_N < 1$. We have

$$x_N = \frac{\alpha + \beta_{k-1} x_{N-k+1} + \beta_k x_{N-k} + \gamma_{k-1} y_{N-k+1} + \gamma_k y_{N-k}}{A + B_{k-1} x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}}$$
$$< \frac{\alpha}{B_{k-1} x_{N-k+1}} + \frac{\gamma_{k-1} y_{N-k+1}}{B_{k-1} x_{N-k+1}} + \frac{\beta_{k-1}+\beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k}$$
$$< \frac{\alpha+1}{B_{k-1}} + \frac{\beta_{k-1}+\beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} \le \frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_k, B_{k-1})} + \frac{\gamma_k}{C_k} < 1,$$

since we assumed that $\frac{\alpha+1+\beta_{k-1}+\beta_k}{\min(B_k,B_{k-1})} + \frac{\gamma_k}{C_k} < 1$. This finishes case (b).

Case (c). We now assume that $N \equiv -3 \mod 3$. Since $N \equiv -3 \mod 3$, we get $N-k = N-3l-2 \equiv -5 \equiv -2 \mod 3$. So $y_{N-k} < 1$ and $x_{N-k} < 1$. Also, since $N \equiv -3 \mod 3$, we know $N-k+1 = N-3l-1 \equiv -4 \equiv -1 \mod 3$. So

$y_{N-k+1} > \max(1, \gamma_{k-1}, \delta_k)$ and $x_{N-k+1} < 1$. From this, we demonstrate the desired inequalities $y_N < 1$ and $x_N > \max(1, \gamma_{k-1}, \delta_k)$, given that $N \equiv -3 \mod 3$ holds.

$$y_N = \frac{p + \delta_{k-1}x_{N-k+1} + \delta_k x_{N-k} + \epsilon_{k-1}y_{N-k+1} + \epsilon_k y_{N-k}}{q + D_{k-1}x_{N-k+1} + E_{k-1}y_{N-k+1} + E_k y_{N-k}}$$

$$< \frac{p}{E_{k-1}y_{N-k+1}} + \frac{\delta_k x_{N-k}}{E_{k-1}y_{N-k+1}} + \frac{\epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}}$$

$$< \frac{p+1}{E_{k-1}} + \frac{\epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} \leq \frac{p + 1 + \epsilon_{k-1} + \epsilon_k}{\min(E_{k-1}, E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1,$$

since we assumed that $\frac{p+1+\epsilon_{k-1}+\epsilon_k}{\min(E_{k-1},E_k)} + \frac{\delta_{k-1}}{D_{k-1}} < 1$. Now, we show that $x_N > max(1, \gamma_{k-1}, \delta_k)$. We have

$$x_N = \frac{\alpha + \beta_{k-1}x_{N-k+1} + \beta_k x_{N-k} + \gamma_{k-1}y_{N-k+1} + \gamma_k y_{N-k}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}}$$

$$\geq \frac{\gamma_{k-1}y_{N-k+1}}{A + B_{k-1}x_{N-k+1} + B_k x_{N-k} + C_k y_{N-k}}$$

$$> \frac{\gamma_{k-1}y_{N-k+1}}{A + B_{k-1} + B_k + C_k} > 2y_{N-k+1}$$

since we assumed that $\frac{\gamma_{k-1}}{A+B_{k-1}+B_k+C_k} > 2$. Now, since $y_{N-k+1} > \max(1, \gamma_{k-1}, \delta_k)$, we have $x_N > \max(1, \gamma_{k-1}, \delta_k)$.

We now conclude through proof by induction that

$$\lim_{n\to\infty} y_{(2k-1)n+2} = \infty \;\; and \;\; \lim_{n\to\infty} x_{(2k-1)n+k+1} = \infty.$$

We first see that

$$y_{(2k-1)n+2}$$

$$= \frac{p + \delta_{k-1}x_{(2k-1)n+3-k} + \delta_k x_{(2k-1)n+2-k} + \epsilon_{k-1}y_{(2k-1)n+3-k} + \epsilon_k y_{(2k-1)n+2-k}}{q + D_{k-1}x_{(2k-1)n+3-k} + E_{k-1}y_{(2k-1)n+3-k} + E_k y_{(2k-1)n+2-k}}$$

$$\geq \frac{\delta_k x_{(2k-1)n+2-k}}{q + D_{k-1}x_{(2k-1)n+3-k} + E_{k-1}y_{(2k-1)n+3-k} + E_k y_{(2k-1)n+2-k}}$$

$$> \frac{\delta_k x_{(2k-1)n+2-k}}{q + D_{k-1} + E_{k-1} + E_k}$$

since $(2k-1)n+2-k \equiv -3 \mod 3$ and since $(2k-1)n+3-k \equiv -2 \mod 3$. Also note that $y_{(2k-1)n+2} > x_{(2k-1)n+2-k}$, since $\frac{\delta_k}{q+D_{k-1}+E_{k-1}+E_k} > 1$ by assumption (2). Now,

$$\begin{aligned} y_{(2k-1)n+2} &> x_{(2k-1)n+2-k} \\ &\geq \frac{\gamma_{k-1} y_{(2k-1)(n-1)+2}}{A + B_{k-1} x_{(2k-1)(n-1)+2} + B_k x_{(2k-1)(n-1)+1} + C_k y_{(2k-1)(n-1)+1}} \\ &> \left(\frac{\gamma_{k-1}}{A + B_{k-1} + B_k + C_k} \right) \left(y_{(2k-1)(n-1)+2} \right) \end{aligned}$$

since $(2k-1)(n-1)+2 \equiv -1 \mod 3$ and since $(2k-1)(n-1)+1 \equiv -2 \mod 3$.

From assumption (2), we have that $\frac{\gamma_{k-1}}{A+B_{k-1}+B_k+C_k} > 2$. So that $y_{(2k-1)n+2} > x_{(2k-1)n+2-k} > 2y_{(2k-1)(n-1)+2}$ for all $n \in \mathbb{N}$, which proves that $\lim_{n\to\infty} y_{(2k-1)n+2} = \infty$. Also, since $y_{(2k-1)n+2} > x_{(2k-1)n+2-k} > 2y_{(2k-1)(n-1)+2}$, for all $n \in \mathbb{N}$, $x_{(2k-1)n+k+1} > 2y_{(2k-1)n+2} > 2x_{(2k-1)n+2-k}$, for all $n \in \mathbb{N}$, which proves that $\lim_{n\to\infty} x_{(2k-1)n+k+1} = \infty$.

■

Since we shall prove unboundedness via use of modulo classes, let us first introduce some new notation. Given a set $S \subset \mathbb{Z}$, we let S^a denote the set comprised of the residues modulo a of the elements of our set S. Written another way, $S^a = \{x \in \{0, \ldots, a-1\} | x \equiv s \mod a \text{ for some } s \in S\}$. We use this notation to keep track of how the sets of residues modulo a of our indices of our system of difference equations behave.

Theorem 7. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^k \beta_i x_{n-i} + \sum_{i=1}^k \gamma_i y_{n-i}}{A + \sum_{j=1}^k B_j x_{n-j} + \sum_{j=1}^k C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^k \delta_i x_{n-i} + \sum_{i=1}^k \epsilon_i y_{n-i}}{q + \sum_{j=1}^k D_j x_{n-j} + \sum_{j=1}^k E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Suppose that there exists a and b such that all of the following hold,

1. $0 \leq b < a$,

2. $I_B^a = \{1, \ldots, a-1\}$,

3. $I_E^a = \{1, \ldots, a-1\}$,

4. $(I_\gamma \setminus I_C)^a = \{b\}$,

5. $(I_\delta \setminus I_D)^a = \{-b \, mod \, a\}$,

6. $(I_\beta \setminus I_B)^a \subset \{0\}$,

7. $(I_\epsilon \setminus I_E)^a \subset \{0\}$,

8. $b \notin I_C^a$,

9. $-b \, mod \, a \notin I_D^a$.

Also assume the following

1. $\frac{\sum_{i \in I_\delta \setminus I_D} \delta_i}{q + \sum_{j=1}^k D_j + \sum_{j=1}^k E_j} > 1$,

2. $\frac{\sum_{i \in I_\gamma \setminus I_C} \gamma_i}{A + \sum_{j=1}^k B_j + \sum_{j=1}^k C_j} > 2$,

3. $\sum_{i \in I_C} \frac{\gamma_i}{C_i} + \frac{\alpha + 1 + \sum_{i=1}^k \beta_i}{\min_{j \in I_B}(B_j)} < 1$,

4. $\sum_{i \in I_D} \frac{\delta_i}{D_i} + \frac{p + 1 + \sum_{i=1}^k \epsilon_i}{\min_{j \in I_E}(E_j)} < 1$.

then for some choice of initial conditions $\lim_{n\to\infty} x_{an+b} = \infty$ *and* $\lim_{n\to\infty} y_{an} = \infty$.

Proof. We let our initial conditions provide the base case and use strong induction on N to prove that

$$x_{aN+b} > \max\left(1, \sum_{i=1}^{k}\delta_i, \sum_{i=1}^{k}\gamma_i\right),$$

$$y_{aN} > \max\left(1, \sum_{i=1}^{k}\delta_i, \sum_{i=1}^{k}\gamma_i\right),$$

$$x_{aN+s}, y_{aN+r} < 1, \quad for \quad s, r \in \{0, \ldots, a-1\} \quad with \quad s \neq b \quad and \quad r \neq 0.$$

So assume that the following holds for $n < N$,

$$x_{an+b} > \max\left(1, \sum_{i=1}^{k}\delta_i, \sum_{i=1}^{k}\gamma_i\right),$$

$$y_{an} > \max\left(1, \sum_{i=1}^{k}\delta_i, \sum_{i=1}^{k}\gamma_i\right),$$

$$x_{an+s}, y_{an+r} < 1, \quad for \quad s, r \in \{0, \ldots, a-1\} \quad with \quad s \neq b \quad and \quad r \neq 0.$$

Then we have

$$\begin{aligned} x_{aN+b} &= \frac{\alpha + \sum_{i=1}^{k}\beta_i x_{aN+b-i} + \sum_{i=1}^{k}\gamma_i y_{aN+b-i}}{A + \sum_{j=1}^{k} B_j x_{aN+b-j} + \sum_{j=1}^{k} C_j y_{aN+b-j}} \\ &\geq \frac{\sum_{i\in I_\gamma\setminus I_C}\gamma_i y_{aN+b-i}}{A + \sum_{j=1}^{k} B_j x_{aN+b-j} + \sum_{j=1}^{k} C_j y_{aN+b-j}} \\ &\geq \frac{(\sum_{i\in I_\gamma\setminus I_C}\gamma_i)\min_{i\in I_\gamma\setminus I_C}(y_{aN+b-i})}{A + \sum_{j=1}^{k} B_j x_{aN+b-j} + \sum_{j=1}^{k} C_j y_{aN+b-j}}. \end{aligned}$$

Now, since $b \notin I_C^a$ and $I_B^a = \{1, \ldots, a-1\}$, we have that $aN + b - j_1 \not\equiv 0 \mod a$ for all $j_1 \in I_C$ and $aN + b - j_2 \not\equiv b \mod a$ for all $j_2 \in I_B$, thus we get

$$x_{aN+b} > \frac{(\sum_{i\in I_\gamma\setminus I_C}\gamma_i)\min_{i\in I_\gamma\setminus I_C}(y_{aN+b-i})}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j}.$$

Now, since we have assumed that $\frac{\sum_{i\in I_\gamma\setminus I_C}\gamma_i}{A+\sum_{j=1}^{k} B_j+\sum_{j=1}^{k} C_j} > 2$, we get

$$x_{aN+b} > 2\min_{i\in I_\gamma\setminus I_C}(y_{aN+b-i}). \tag{3.2.1}$$

Since $(I_\gamma \setminus I_C)^a = \{b\}$, we have $aN + b - i \equiv 0 \mod a$ for all $i \in I_\gamma \setminus I_C$, thus

$$x_{aN+b} > 2 \min_{i \in I_\gamma \setminus I_C} (y_{aN+b-i}) > 2 \max\left(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i\right).$$

Also we have the following

$$\begin{aligned} y_{aN} &= \frac{p + \sum_{i=1}^{k} \delta_i x_{aN-i} + \sum_{i=1}^{k} \epsilon_i y_{aN-i}}{q + \sum_{j=1}^{k} D_j x_{aN-j} + \sum_{j=1}^{k} E_j y_{aN-j}} \\ &\geq \frac{\sum_{i \in I_\delta \setminus I_D} \delta_i x_{aN-i}}{q + \sum_{j=1}^{k} D_j x_{aN-j} + \sum_{j=1}^{k} E_j y_{aN-j}} \\ &\geq \frac{(\sum_{i \in I_\delta \setminus I_D} \delta_i) \min_{i \in I_\delta \setminus I_D} (x_{aN-i})}{q + \sum_{j=1}^{k} D_j x_{aN-j} + \sum_{j=1}^{k} E_j y_{aN-j}}. \end{aligned}$$

Now, since $-b \mod a \notin I_D^a$ and $I_E^a = \{1, \ldots, a-1\}$, we have that $aN - j_1 \not\equiv 0 \mod a$ for all $j_1 \in I_E$ and $aN - j_2 \not\equiv b \mod a$ for all $j_2 \in I_D$, thus we get

$$y_{aN} > \frac{(\sum_{i \in I_\delta \setminus I_D} \delta_i) \min_{i \in I_\delta \setminus I_D} (x_{aN-i})}{q + \sum_{j=1}^{k} D_j + \sum_{j=1}^{k} E_j}.$$

Now, since we have assumed that $\frac{\sum_{i \in I_\delta \setminus I_D} \delta_i}{q + \sum_{j=1}^{k} D_j + \sum_{j=1}^{k} E_j} > 1$, we get

$$y_{aN} > \min_{i \in I_\delta \setminus I_D} (x_{aN-i}). \tag{3.2.2}$$

Since $(I_\delta \setminus I_D)^a = \{-b \mod a\}$, we have $aN - i \equiv b \mod a$ for all $i \in I_\delta \setminus I_D$, thus

$$y_{aN} > \min_{i \in I_\delta \setminus I_D} (x_{aN-i}) > \max\left(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i\right).$$

We now prove the remaining inequalities. For $s \in \{0, \ldots, a-1\}$ with $s \neq b$,

$$\begin{aligned} x_{aN+s} &= \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+s-i} + \sum_{i=1}^{k} \gamma_i y_{aN+s-i}}{A + \sum_{j=1}^{k} B_j x_{aN+s-j} + \sum_{j=1}^{k} C_j y_{aN+s-j}} \\ &\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+s-i} + \sum_{i \in I_\gamma \setminus I_C} \gamma_i y_{aN+s-i}}{A + \sum_{j=1}^{k} B_j x_{aN+s-j} + \sum_{j=1}^{k} C_j y_{aN+s-j}} \\ &+ \sum_{i \in I_C} \frac{\gamma_i y_{aN+s-i}}{A + \sum_{j=1}^{k} B_j x_{aN+s-j} + \sum_{j=1}^{k} C_j y_{aN+s-j}} \end{aligned}$$

$$\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+s-i} + \sum_{i \in I_\gamma \setminus I_C} \gamma_i y_{aN+s-i}}{\min_{j \in I_B}(B_j) \max_{j \in I_B}(x_{aN+s-j})} + \sum_{i \in I_C} \frac{\gamma_i}{C_i}.$$

Since $(I_\beta \setminus I_B)^a \subset \{0\}$, we have that for all $i \in I_\beta$, $i \in I_B$ or $i \in \{z \in \mathbb{Z} | z \equiv 0 \mod a\}$. Thus, for all $i \in I_\beta$, either $x_{aN+s-i} \leq \max_{j \in I_B}(x_{aN+s-j})$, or $x_{aN+s-i} < 1$. Furthermore, since $I_B^a = \{1, \ldots, a-1\}$, there exists $j \in I_B$ so that $aN + s - j \equiv b \mod a$. Thus, $\max_{j \in I_B}(x_{aN+s-j}) > 1$ and $\max_{j \in I_B}(x_{aN+s-j}) > \sum_{i=1}^{k} \gamma_i$. To be clear, this means $\max_{i \in I_\beta}(x_{aN+s-i}) \leq \max_{j \in I_B}(x_{aN+s-j})$. So we get

$$x_{aN+s} \leq \frac{\sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)} + \frac{\alpha + \sum_{i \in I_\gamma \setminus I_C} \gamma_i y_{aN+s-i}}{\min_{j \in I_B}(B_j) \max_{j \in I_B}(x_{aN+s-j})} + \sum_{i \in I_C} \frac{\gamma_i}{C_i}$$

$$\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)} + \frac{\sum_{i \in I_\gamma \setminus I_C} \gamma_i y_{aN+s-i}}{\min_{j \in I_B}(B_j) \max_{j \in I_B}(x_{aN+s-j})} + \sum_{i \in I_C} \frac{\gamma_i}{C_i}.$$

Now since $(I_\gamma \setminus I_C)^a = \{b\}$, we have $aN + s - i \not\equiv 0 \mod a$ for all $i \in I_\gamma \setminus I_C$. Thus, $y_{aN+s-i} < 1$ for all $i \in I_\gamma \setminus I_C$. Hence

$$x_{aN+s} < \frac{\alpha + \sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)} + \frac{\sum_{i \in I_\gamma \setminus I_C} \gamma_i}{\min_{j \in I_B}(B_j) \max_{j \in I_B}(x_{aN+s-j})} + \sum_{i \in I_C} \frac{\gamma_i}{C_i}$$

$$< \frac{\alpha + 1 + \sum_{i=1}^{k} \beta_i}{\min_{j \in I_B}(B_j)} + \sum_{i \in I_C} \frac{\gamma_i}{C_i} < 1.$$

Now, for $r \in \{0, \ldots, a-1\}$ with $r \neq 0$,

$$y_{aN+r} = \frac{p + \sum_{i=1}^{k} \delta_i x_{aN+r-i} + \sum_{i=1}^{k} \epsilon_i y_{aN+r-i}}{q + \sum_{j=1}^{k} D_j x_{aN+r-j} + \sum_{j=1}^{k} E_j y_{aN+r-j}}$$

$$\leq \frac{p + \sum_{i \in I_\delta \setminus I_D} \delta_i x_{aN+r-i} + \sum_{i=1}^{k} \epsilon_i y_{aN+r-i}}{q + \sum_{j=1}^{k} D_j x_{aN+r-j} + \sum_{j=1}^{k} E_j y_{aN+r-j}}$$

$$+ \sum_{i \in I_D} \frac{\delta_i x_{aN+r-i}}{q + \sum_{j=1}^{k} D_j x_{aN+r-j} + \sum_{j=1}^{k} E_j y_{aN+r-j}}$$

$$\leq \frac{p + \sum_{i \in I_\delta \setminus I_D} \delta_i x_{aN+r-i} + \sum_{i=1}^{k} \epsilon_i y_{aN+r-i}}{\min_{j \in I_E}(E_j) \max_{j \in I_E}(y_{aN+r-j})} + \sum_{i \in I_D} \frac{\delta_i}{D_i}.$$

Since $(I_\epsilon \setminus I_E)^a \subset \{0\}$, we have that for all $i \in I_\epsilon$, $i \in I_E$ or $i \in \{z \in \mathbb{Z} | z \equiv 0 \mod a\}$. Thus, for all $i \in I_\epsilon$, either $y_{aN+r-i} \leq \max_{j \in I_E}(y_{aN+r-j})$, or $y_{aN+r-i} < 1$.

Furthermore, since $I_E^a = \{1, \ldots, a-1\}$, there exists $j \in I_E$ so that $aN + r - j \equiv 0 \mod a$. Thus, $\max_{j\in I_E}(y_{aN+r-j}) > 1$ and $\max_{j\in I_E}(y_{aN+r-j}) > \sum_{i=1}^{k} \delta_i$. To be clear this means $\max_{i\in I_\epsilon}(y_{aN+r-i}) \leq \max_{j\in I_E}(y_{aN+r-j})$. So we get

$$y_{aN+r} \leq \frac{\sum_{i=1}^{k} \epsilon_i}{\min_{j\in I_E}(E_j)} + \frac{p + \sum_{i\in I_\delta \setminus I_D} \delta_i x_{aN+r-i}}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})} + \sum_{i\in I_D} \frac{\delta_i}{D_i}$$

$$\leq \frac{p + \sum_{i=1}^{k} \epsilon_i}{\min_{j\in I_E}(E_j)} + \frac{\sum_{i\in I_\delta \setminus I_D} \delta_i x_{aN+r-i}}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})} + \sum_{i\in I_D} \frac{\delta_i}{D_i}.$$

Now since $(I_\delta \setminus I_D)^a = \{-b \mod a\}$, we have $aN + r - i \not\equiv b \mod a$ for all $i \in I_\delta \setminus I_D$. Thus, $x_{aN+r-i} < 1$ for all $i \in I_\delta \setminus I_D$. Hence

$$x_{aN+s} < \frac{p + \sum_{i=1}^{k} \epsilon_i}{\min_{j\in I_E}(E_j)} + \frac{\sum_{i\in I_\delta \setminus I_D} \delta_i}{\min_{j\in I_E}(E_j)\max_{j\in I_E}(y_{aN+r-j})} + \sum_{i\in I_D} \frac{\delta_i}{D_i}$$

$$< \frac{p + 1 + \sum_{i=1}^{k} \epsilon_i}{\min_{j\in I_E}(E_j)} + \sum_{i\in I_D} \frac{\delta_i}{D_i} < 1.$$

Thus, we have completed the induction proof and so

$$x_{aN+b} > \max\left(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i\right),$$

$$y_{aN} > \max\left(1, \sum_{i=1}^{k} \delta_i, \sum_{i=1}^{k} \gamma_i\right),$$

$$x_{aN+s}, y_{aN+r} < 1, \quad for \;\; s, r \in \{0, \ldots, a-1\} \;\; with \;\; s \neq b \;\; and \;\; r \neq 0.$$

for all $N \in \mathbb{N}$. Now recall from inequalities (3.2.1) and (3.2.2),

$$x_{aN+b} > 2 \min_{i\in I_\gamma \setminus I_C} (y_{aN+b-i})$$

and

$$y_{aN} > \min_{i\in I_\delta \setminus I_D} (x_{aN-i})$$

for all $N \in \mathbb{N}$. Here we use substitution and arrive at the following inequalities,

$$x_{aN+b} > 2 \min_{u\in U}(x_{aN+b-u})$$

and

$$y_{aN} > 2 \min_{u \in U}(y_{aN-u})$$

for all $N \in \mathbb{N}$, where $U = \{i_1 + i_2 | i_1 \in I_\gamma \setminus I_C \text{ and } i_2 \in I_\delta \setminus I_D\}$. Using the fact that $(I_\gamma \setminus I_C)^a = \{b\}$ and $(I_\delta \setminus I_D)^a = \{-b \mod a\}$, we find that $U \subset \left\{a\eta | \eta \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}\right\}$. Thus, the following inequalities hold for all $n \geq 2k$:

$$x_{an+b} > 2 \min_{i \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}} (x_{a(n-i)+b}),$$

$$y_{an} > 2 \min_{i \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}} (y_{a(n-i)}).$$

Now, let us make the following change of variables. $x_{an+b} = w_n$ and $y_{an} = v_n$. Thus we get the following difference inequalities for all $n \geq 2k$:

$$w_n > 2 \min_{i \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}} (w_{n-i}),$$

$$v_n > 2 \min_{i \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}} (v_{n-i}).$$

Thus, using Theorem 3 in Ref. [4] we get:

$$\min(w_{n-1}, \ldots, w_{n-2\lceil \frac{k}{a} \rceil}) \geq 2^{\left\lfloor \frac{n-2k}{2\lceil \frac{k}{a} \rceil} \right\rfloor} \min_{i \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}} (w_{2k-i}),$$

$$\min(v_{n-1}, \ldots, v_{n-2\lceil \frac{k}{a} \rceil}) \geq 2^{\left\lfloor \frac{n-2k}{2\lceil \frac{k}{a} \rceil} \right\rfloor} \min_{i \in \left\{1, \ldots, 2\left\lceil \frac{k}{a} \right\rceil\right\}} (v_{2k-i}).$$

Hence, $\lim_{n\to\infty} w_n = \infty$ and $\lim_{n\to\infty} v_n = \infty$. Thus, $\lim_{n\to\infty} x_{an+b} = \infty$ and $\lim_{n\to\infty} y_{an} = \infty$. ∎

We have just presented a general unboundedness result for systems of rational difference equations. Notice that Examples 1 and 2 are subsumed by Theorem 7 after a change of variables. We prove above that, when the hypotheses are satisfied, both $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are unbounded. However, there are known special cases where $\{x_n\}_{n=0}^{\infty}$ is unbounded and $\{y_n\}_{n=0}^{\infty}$ is bounded above by a positive constant, and vice versa. In fact, it is possible to sometimes apply similar techniques to those presented above in these cases. This is what motivates the following theorem. We prove the result for the case where $\{x_n\}_{n=0}^{\infty}$ is unbounded and $\{y_n\}_{n=0}^{\infty}$ is bounded above by a positive constant. In the other case, we advise the reader to make a change of variables.

Theorem 8. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, n \in \mathbb{N},$$

$$y_n = \frac{p + \sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Suppose that there exist initial conditions $y_0, \ldots, y_{-k+1}$ and that there exists $M > 0$ so that $y_n \leq M$ for all $n > -k$ and for all choices of initial conditions $x_0, \ldots, x_{-k+1}$. Further suppose that there exists a such that all of the following hold,

1. $I_B^a = \{1, \ldots, a-1\}$,

2. $(I_\beta \setminus I_B)^a = \{0\}$,

Also assume the following

1. $\frac{\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j M} > 2$,

2. $\frac{\alpha + 1 + \sum_{i \in I_B} \beta_i}{\min_{j \in I_B}(B_j)} < 1$.

then for some choice of initial conditions $\lim_{n\to\infty} x_{an} = \infty$.

Proof. We let our initial conditions provide the base case and use strong induction on N to prove that

$$x_{aN} > \max\left(1, \sum_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \gamma_i M\right),$$

$$x_{aN+s} < 1, \quad for \quad s \in \{1, \ldots, a-1\}.$$

So assume that the following holds for $n < N$,

$$x_{an} > \max\left(1, \sum_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \gamma_i M\right),$$

$$x_{an+s} < 1, \quad for \ \ s \in \{1, \ldots, a-1\}.$$

Then we have

$$\begin{aligned} x_{aN} &= \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN-i} + \sum_{i=1}^{k} \gamma_i y_{aN-i}}{A + \sum_{j=1}^{k} B_j x_{aN-j} + \sum_{j=1}^{k} C_j y_{aN-j}} \\ &\geq \frac{\sum_{i \in I_\beta \setminus I_B} \beta_i x_{aN-i}}{A + \sum_{j=1}^{k} B_j x_{aN-j} + \sum_{j=1}^{k} C_j y_{aN-j}}. \end{aligned}$$

Since $I_B^a = \{1, \ldots, a-1\}$ and $y_n \leq M$, we have

$$\begin{aligned} x_{aN} &\geq \frac{\sum_{i \in I_\beta \setminus I_B} \beta_i x_{aN-i}}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j M} \\ &\geq \frac{(\sum_{i \in I_\beta \setminus I_B} \beta_i)(\min_{i \in I_\beta \setminus I_B}(x_{aN-i}))}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j M}. \end{aligned}$$

Now, since we have assumed that $\frac{\sum_{i \in I_\beta \setminus I_B} \beta_i}{A + \sum_{j=1}^{k} B_j + \sum_{j=1}^{k} C_j M} > 2$, we get

$$x_{aN} > 2 \min_{i \in I_\beta \setminus I_B} (x_{aN-i}). \tag{3.2.3}$$

Since $(I_\beta \setminus I_B)^a = \{0\}$, we have $aN - i \equiv 0 \mod a$ for all $i \in I_\beta \setminus I_B$, thus

$$x_{aN} > 2 \min_{i \in I_\beta \setminus I_B} (x_{aN-i}) > 2 \max\left(1, \sum_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \gamma_i M\right).$$

We now prove the remaining inequality. For $s \in \{1, \ldots, a-1\}$,

$$\begin{aligned} x_{aN+s} &= \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+s-i} + \sum_{i=1}^{k} \gamma_i y_{aN+s-i}}{A + \sum_{j=1}^{k} B_j x_{aN+s-j} + \sum_{j=1}^{k} C_j y_{aN+s-j}} \\ &\leq \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{aN+s-i} + \sum_{i=1}^{k} \gamma_i M}{\sum_{j=1}^{k} B_j x_{aN+s-j}}. \end{aligned}$$

Since $(I_\beta \setminus I_B)^a = \{0\}$, we have that for all $i \in I_\beta \setminus I_B$, $x_{aN+s-i} < 1$. So

$$x_{aN+s} \leq \frac{\sum_{i \in I_B} \beta_i}{\min_{j \in I_B}(B_j)} + \frac{\alpha + \sum_{i \in I_\beta \setminus I_B} \beta_i + \sum_{i=1}^{k} \gamma_i M}{\min_{j \in I_B}(B_j) \max_{j \in I_B}(x_{aN+s-j})}.$$

Since $I_B^a = \{1, \ldots, a-1\}$, there exists $j \in I_B$ so that $aN + s - j \equiv 0 \mod a$. Thus, $\max_{j \in I_B}(x_{aN+s-j}) > 1$ and $\max_{j \in I_B}(x_{aN+s-j}) > \sum_{i=1}^k \beta_i + \sum_{i=1}^k \gamma_i M$. So

$$x_{aN+s} < \frac{\alpha + 1 + \sum_{i \in I_B} \beta_i}{\min_{j \in I_B}(B_j)}.$$

Thus, we have completed the induction proof and

$$x_{aN} > \max\left(1, \sum_{i=1}^k \beta_i + \sum_{i=1}^k \gamma_i M\right),$$

$$x_{aN+s} < 1, \quad for \quad s \in \{1, \ldots, a-1\}.$$

for all $N \in \mathbb{N}$. Now, recall from the inequality (3.2.3)

$$x_{aN} > 2 \min_{i \in I_\beta \setminus I_B} (x_{aN-i})$$

for all $N \in \mathbb{N}$. So we make a change of variables $x_{an} = w_n$ and we get the difference inequality

$$w_n > 2 \min_{i \in \{1, \ldots, \lfloor \frac{k}{a} \rfloor\}} (w_{n-i})$$

for all $n \geq k$. Thus, using Theorem 3 in Ref. [4], we get

$$\min\left(w_{n-1}, \ldots, w_{n-\lfloor \frac{k}{a} \rfloor}\right) \geq 2^{\left\lfloor \frac{n-k}{\lfloor \frac{k}{a} \rfloor} \right\rfloor} \min_{i \in \{1, \ldots, \lfloor \frac{k}{a} \rfloor\}} (w_{k-i}).$$

So $\lim_{n \to \infty} w_n = \infty$, thus $\lim_{n \to \infty} x_{an} = \infty$. ■

3.3 Adapting an unboundedness result to systems

Let us draw our attention to Theorem 2 case (iii) of Ref. [5]. To prove this result, the author separates the integers into two sets $A = \{n \in \mathbb{Z} : \gcd(I_\beta) | n\}$ and $B = \mathbb{Z} \setminus A$. The author then proves via induction that for proper choice of initial conditions, whenever $n \in A$ then $x_n > 0$, and whenever $n \in B$ then $x_n = 0$. The key here is that parameters are chosen in a way that makes such a proof possible. We wish to adapt such a result so that it might apply to systems. Thus, it is important to choose our parameters so that a similar idea holds. The first thing which comes to mind is to require that there does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|j$. This motivates the following theorem.

Theorem 9. *Suppose that we have a k^{th} order system of two rational difference equations*

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j} + \sum_{j=1}^{k} C_j y_{n-j}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{\sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q + \sum_{j=1}^{k} D_j x_{n-j} + \sum_{j=1}^{k} E_j y_{n-j}}, \quad n \in \mathbb{N},$$

with non-negative parameters and non-negative initial conditions. Further assume that $q, A > 0$ and that one of the following holds:

1. $A < \sum_{i=1}^{k} \beta_i$*, and* $I_\delta \neq \emptyset$*;*

2. $q < \sum_{i=1}^{k} \epsilon_i$*, and* $I_\gamma \neq \emptyset$*.*

Also suppose that there does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|j$*. Then unbounded solutions exist for both $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ for some choice of initial conditions.*

Proof. Choose initial conditions x_{-m} and y_{-m} where $m \in \{0, \ldots, k-1\}$, so that $x_{-m} = 1 = y_{-m}$ if $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|m$ and $x_{-m} = 0 = y_{-m}$ otherwise.

Under this choice of initial conditions, $\{x_n\}$ and $\{y_n\}$ have the property that $x_n > 0$ and $y_n > 0$ whenever $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n$ and $x_n = 0 = y_n$ otherwise. We prove this using induction on n, our initial conditions provide the base case. Assume that the statement is true for all $n \leq N-1$. We show the statement for $n = N$.

This argument has four cases. First assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$ then both denominators are clearly non-zero since we assumed $A, q > 0$. Since we assumed $\sum_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \gamma_i > 0$, we know that either there exists $i \in I_\beta$ so that $\beta_i > 0$, or there exists $i \in I_\gamma$ so that $\gamma_i > 0$. It is sufficient to show $x_{N-i} > 0$ and

$y_{N-i} > 0$. However, since $i \in I_\beta \cup I_\gamma$ it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Thus, $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$. Hence by our induction hypothesis $x_{N-i} > 0$ and $y_{N-i} > 0$. Thus $x_N > 0$. We have shown the first case.

Again assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$, then both denominators are clearly non-zero since we assumed $A, q > 0$. Since we assumed $\sum_{i=1}^k \delta_i + \sum_{i=1}^k \epsilon_i > 0$, we know that either there exists $i \in I_\delta$ so that $\delta_i > 0$, or there exists $i \in I_\epsilon$ so that $\epsilon_i > 0$. It is sufficient to show $x_{N-i} > 0$ and $y_{N-i} > 0$. However, since $i \in I_\delta \cup I_\epsilon$ it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Thus, $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$. Hence, by our induction hypothesis, $x_{N-i} > 0$ and $y_{N-i} > 0$. Thus, $y_N > 0$. We have shown the second case.

Now assume it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$. Again, the denominators are clearly non-zero and furthermore,

$$x_N = \frac{\sum_{i=1}^k \beta_i x_{N-i} + \sum_{i=1}^k \gamma_i y_{N-i}}{A + \sum_{j=1}^k B_j x_{N-j} + \sum_{j=1}^k C_j y_{N-j}} \leq \frac{\sum_{i=1}^k \beta_i x_{N-i} + \sum_{i=1}^k \gamma_i y_{N-i}}{A}.$$

Take $i \in I_\beta \cup I_\gamma$, it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Hence, by our assumption, we have that for all $i \in I_\beta \cup I_\gamma$ it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$. Indeed, assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$ for some $i \in I_\beta \cup I_\gamma$, then $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$ contradicting our hypothesis. Hence, by our induction hypothesis, $x_{N-i} = 0$ and $y_{N-i} = 0$ for all $i \in I_\beta \cup I_\gamma$. So, $\sum_{i=1}^k \beta_i x_{N-i} + \sum_{i=1}^k \gamma_i y_{N-i} = 0$. So, $x_N = 0$ in this case. Thus, we have shown the third case.

Again, assume it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$. The denominators are clearly non-zero and furthermore,

$$y_N = \frac{\sum_{i=1}^k \delta_i x_{N-i} + \sum_{i=1}^k \epsilon_i y_{N-i}}{q + \sum_{j=1}^k D_j x_{N-j} + \sum_{j=1}^k E_j y_{N-j}} \leq \frac{\sum_{i=1}^k \delta_i x_{N-i} + \sum_{i=1}^k \epsilon_i y_{N-i}}{q}.$$

Take $i \in I_\delta \cup I_\epsilon$, it follows that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|i$. Hence, by our assumption, we have that for all $i \in I_\delta \cup I_\epsilon$ it is not true that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$. Indeed, assume $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N - i$ for some $i \in I_\delta \cup I_\epsilon$, then $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|N$

contradicting our hypothesis. Hence, by our induction hypothesis, $x_{N-i} = 0$ and $y_{N-i} = 0$ for all $i \in I_\delta \cup I_\epsilon$. So $\sum_{i=1}^{k} \delta_i x_{N-i} + \sum_{i=1}^{k} \epsilon_i y_{N-i} = 0$. So $y_N = 0$ in this case. Thus, we have shown the fourth case.

Now, we will make use of the prior result. Choose n such that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n$. There does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|j$. Thus, there does not exist $j \in I_B \cup I_C \cup I_D \cup I_E$ so that $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n-j$. Using this and the prior result, it follows that for n where $\gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)|n$, $x_{n-j} = 0 = y_{n-j}$ for all $j \in I_B \cup I_C \cup I_D \cup I_E$. So, for this choice of n we get

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A}, \tag{3.3.1}$$

$$y_n = \frac{\sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q}. \tag{3.3.2}$$

We now have two cases to consider. In case 1, $A < \sum_{i=1}^{k} \beta_i$ and $I_\delta \neq \emptyset$, so we use Equation (3.3) and we get

$$x_n = \frac{\sum_{i=1}^{k} \beta_i x_{n-i} + \sum_{i=1}^{k} \gamma_i y_{n-i}}{A} \geq \frac{\sum_{i=1}^{k} \beta_i x_{n-i}}{A} \geq \left(\frac{\sum_{i=1}^{k} \beta_i}{A} \right) \min_{i \in I_\beta} (x_{n-i}).$$

For convienience, we now define $L = \gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)$. By our choice of n, we may write $m = \frac{n}{L} \in \mathbb{N}$. So, our inequality reduces in this case to

$$x_{mL} \geq \left(\frac{\sum_{i=1}^{k} \beta_i}{A} \right) \min_{i \in I_\beta} (x_{mL-i})$$

$$\geq \left(\frac{\sum_{i=1}^{k} \beta_i}{A} \right) \min_{i \in \{1, \ldots, \lfloor \frac{k}{L} \rfloor\}} (x_{mL-iL}).$$

This is a difference inequality which holds for the subsequence $\{x_{mL}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in Ref. [4]. We set $z_m = x_{mL}$ for $m \in \mathbb{N}$. As we have just shown, $\{z_m\}$ satisfies the following difference inequality,

$$z_m \geq \left(\frac{\sum_{i=1}^{k} \beta_i}{A} \right) \min_{i \in \{1, \ldots, \lfloor \frac{k}{L} \rfloor\}} (z_{m-i}), m \geq k.$$

Using the results of Ref. [4], particularly Theorem 3, we have that for $m \geq k$,

$$\min\left(z_{m-1}, \ldots, z_{m-\lfloor \frac{k}{L} \rfloor}\right) \geq \min\left(u_{\left\lfloor \frac{m-k}{\lfloor \frac{k}{L} \rfloor} \right\rfloor}, \ldots, u_{m-k}\right).$$

Where $\{u_m\}_{m=0}^{\infty}$ is a solution of the difference equation,

$$u_m = \left(\frac{\sum_{i=1}^{k} \beta_i}{A}\right) u_{m-1}, m \in \mathbb{N}. \tag{3.3.3}$$

With $u_0 = \min(z_{k-1}, \ldots, z_{k-\lfloor \frac{k}{L} \rfloor})$.

Since we are in case 1, we know that $0 < A < \sum_{i=1}^{k} \beta_i$ and so every positive solution diverges to ∞ for the simple difference equation (3.3.3). Hence, using the inequality we have obtained, $\{z_m\}_{m=1}^{\infty}$ diverges to ∞. Hence, with given initial conditions, there is a subsequence of our solution $\{x_n\}_{n=1}^{\infty}$, namely $\{x_{mL}\}_{m=1}^{\infty}$, which diverges to ∞. Thus, our solution $\{x_n\}_{n=1}^{\infty}$ is unbounded. Moreover, since $\{x_{mL}\}_{m=1}^{\infty}$ diverges to ∞ and $I_\delta \neq \emptyset$, by Equation (3.3.2), $\{y_{mL}\}_{m=1}^{\infty}$ diverges to ∞. So we have exhibited a solution in the case 1 where both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are unbounded. In case 2, $q < \sum_{i=1}^{k} \epsilon_i$, and $I_\gamma \neq \emptyset$ so we use Equation (3.3.2) and we get

$$y_n = \frac{\sum_{i=1}^{k} \delta_i x_{n-i} + \sum_{i=1}^{k} \epsilon_i y_{n-i}}{q} \geq \frac{\sum_{i=1}^{k} \epsilon_i y_{n-i}}{q} \geq \left(\frac{\sum_{i=1}^{k} \epsilon_i}{q}\right) \min_{i \in I_\epsilon}(y_{n-i}).$$

For convienience, we now define $L = \gcd(I_\beta \cup I_\gamma \cup I_\delta \cup I_\epsilon)$. By our choice of n, we may write $m = \frac{n}{L} \in \mathbb{N}$. So our inequality reduces in this case to

$$y_{mL} \geq \left(\frac{\sum_{i=1}^{k} \epsilon_i}{q}\right) \min_{i \in I_\epsilon}(y_{mL-i})$$

$$\geq \left(\frac{\sum_{i=1}^{k} \epsilon_i}{q}\right) \min_{i \in \{1, \ldots, \lfloor \frac{k}{L} \rfloor\}}(y_{mL-iL}).$$

This is a difference inequality which holds for the subsequence $\{y_{mL}\}$ for $m \geq k$. We now rename this subsequence and apply the methods used in Ref. [4]. We set $w_m = y_{mL}$ for $m \in \mathbb{N}$. As we have just shown, $\{w_m\}$ satisfies the following difference inequality,

$$w_m \geq \left(\frac{\sum_{i=1}^{k} \epsilon_i}{q}\right) \min_{i \in \{1,\ldots,\lfloor \frac{k}{L} \rfloor\}} (w_{m-i}), m \geq k.$$

Using the results of Ref [4], particularly Theorem 3, we have that for $m \geq k$,

$$\min\left(w_{m-1}, \ldots, w_{m-\lfloor \frac{k}{L} \rfloor}\right) \geq \min\left(v_{\left\lfloor \frac{m-k}{\lfloor \frac{k}{L} \rfloor} \right\rfloor}, \ldots, v_{m-k}\right).$$

Where $\{v_m\}_{m=0}^{\infty}$ is a solution of the difference equation,

$$v_m = \left(\frac{\sum_{i=1}^{k} \epsilon_i}{q}\right) v_{m-1}, m \in \mathbb{N}. \tag{3.3.4}$$

With $v_0 = \min(w_{k-1}, \ldots, w_{k-\lfloor \frac{k}{L} \rfloor})$.

Since we are in case 2, we know that $0 < q < \sum_{i=1}^{k} \epsilon_i$ and so every positive solution diverges to ∞ for the simple difference equation (3.3.4). Hence, using the inequality we have obtained, $\{w_m\}_{m=1}^{\infty}$ diverges to ∞. Hence, with given initial conditions, there is a subsequence of our solution $\{y_n\}_{n=1}^{\infty}$, namely $\{y_{mL}\}_{m=1}^{\infty}$, which diverges to ∞. Hence, our solution $\{y_n\}_{n=1}^{\infty}$ is unbounded. Moreover, since $\{y_{mL}\}_{m=1}^{\infty}$ diverges to ∞ by Equation and $I_\gamma \neq \emptyset$, $\{x_{mL}\}_{m=1}^{\infty}$ diverges to ∞. So we have exhibited a solution in the case 2 where both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are unbounded.

■

There is a very general idea taking place here. Look at a system of rational equations of the type presented here and look at the delays present in all of the numerators and all of the denominators. Does the greatest common divisor of all the delays in all the numerators divide some delay in one of the denominators? If the answer is no, we conjecture that a result similar to the one presented above can be shown for the system in question. The proof may be almost a duplicate of the above proof. We leave this proof to the determined reader.

3.4 Some Examples for Rational Systems in the Plane

Although these methods are intended to demonstrate unboundedness for systems of rational difference equations of order greater than one, there are several examples of rational systems in the plane where these techniques apply. Here we present all first order rational systems in the plane where Theorem 7 applies.

Example 4. *Consider the system of two rational difference equations*

$$x_n = \frac{\alpha + \beta_1 x_{n-1} + \gamma_1 y_{n-1}}{A + B_1 x_{n-1}}, \quad n \in \mathbb{N},$$

$$y_n = \frac{p + \delta_1 x_{n-1} + \epsilon_1 y_{n-1}}{q + E_1 y_{n-1}}, \quad n \in \mathbb{N},$$

with $\alpha, \beta_1, A, p, \epsilon_1, q \geq 0$*,* $\delta_1, \gamma_1, B_1, E_1 > 0$*, and non-negative initial conditions. Assume that*

1. $\frac{\delta_1}{q+E_1} > 1$,
2. $\frac{\gamma_1}{A+B_1} > 2$,
3. $\frac{\alpha+1+\beta_1}{B_1} < 1$,
4. $\frac{p+1+\epsilon_1}{E_1} < 1$.

then for some choice of initial conditions $\lim_{n\to\infty} x_{2n+1} = \infty$ *and* $\lim_{n\to\infty} y_{2n} = \infty$.

Proof. We apply Theorem 7. We let $a = 2$ and $b = 1$. We have the following

$$\begin{gathered} I_B^2 = \{1\} = \{1, \dots, 2-1\} \\ I_E^2 = \{1\} = \{1, \dots, 2-1\} \\ (I_\gamma \setminus I_C)^2 = \{1\} \\ (I_\delta \setminus I_D)^2 = \{1\} \\ (I_\beta \setminus I_B)^2 = \emptyset \\ (I_\epsilon \setminus I_E)^2 = \emptyset \\ 1 \notin \emptyset \end{gathered}$$

Thus Theorem 7 applies. ■

Notice that in the above example the parameters $\alpha, \beta_1, A, p, \epsilon_1$, and q were allowed to be either positive or zero. Thus, there are 64 rational systems in the plane for which Theorem 7 applies. Some of these cases have been covered by prior work, however the conjectures (23,23), (23,31), (23,34), (23,46), (31,31), (31,34), (31,46), (34,34), (34,46), and (46,46) in Appendix 3 of Ref. [2] are covered by Example 4.

3.5 Conclusion

We have presented here several general results which prove the existence of unbounded solutions for systems of two rational difference equations of order greater than one. We feel that a good direction for further study would be to develop similar techniques which prove the existence of unbounded solutions for systems of more than two rational difference equations. We have given some limited guidance toward this goal in Section 3.3. We would like to make reference to [1] and [2] for other work regarding systems of rational equations.

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