10-19-2015

07. Stochastic Processes: Applications

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Abstract

Part seven of course materials for Nonequilibrium Statistical Physics (Physics 626), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.
Updated with version 2 on 5/3/2016.

Contents of this Document [ntc7]

- Catalyst driven chemical reaction: total rate of reactions [nex108]
- Air in leaky tank I: generating function [nex48]
- Air in leaky tank II: probability distribution [nex109]
- Air in leaky tank III: detailed balance [nex49]
- Air in leaky tank IV: evolution of mean and variance [nex110]
- Pascal distribution and Planck radiation law [nex50]
- Effects of nonlinear death rate I: Malthus-Verhulst equation [nex111]
- Effects of nonlinear death rate II: stationarity and fluctuations [nex51]
- Modified linear birth rate I: stationarity [nex113]
- Modified linear birth rate II: evolution of mean and variance [nex114]
- Modified linear birth rate III: generating function [nex115]
- Modified linear birth rate IV: probability distribution [nex116]
- Bistable chemical system [nex52]
- Ultracold neutrons in an ideal Steyerl bottle [nex47]
- Random light switch [nex45]

[nex27] Specifications of diffusion process

Examine the diffusion process,

$$P(x|x_0;\Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}}\,\exp\left(-\frac{(x-x_0)^2}{4D\Delta t}\right),$$

as a special solution of the differential Chapman-Kolmogorov equation by determining the three specifications (two of which are zero):

$$W(x|x_0) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\,P(x|x_0;\Delta t),$$

$$A(x) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-x_0|<\epsilon} dx(x-x_0)P(x|x_0;\Delta t), \quad B(x) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-x_0|<\epsilon} dx(x-x_0)^2P(x|x_0;\Delta t).$$

Use the results for $W(x|x_0)$, $A(x)$, and $B(x)$ to simplify the differential Chapman-Kolmogorov equation from [nln56] nto the diffusion equation.

Solution:

[nex98] Specifications of Cauchy process

Examine the Cauchy process,

$$P(x|x_0;\Delta t) = \frac{1}{\pi}\frac{\Delta t}{(x-x_0)^2+(\Delta t)^2},$$

as a special solution of the differential Chapman-Kolmogorov equation by determining the three specifications (two of which are zero):

$$W(x|x_0) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\,P(x|x_0;\Delta t),$$

$$A(x) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-x_0|<\epsilon} dx(x-x_0)P(x|x_0;\Delta t), \quad B(x) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-x_0|<\epsilon} dx(x-x_0)^2P(x|x_0;\Delta t).$$

Solution:

Random Walk in One Dimension [nln60]

In the following, we analyze this very common process in different ways and also examine variations of it.

- **Walker takes units steps in unit time.** [nex34]
  Position $x_n = n\ell$ of walker at time $t = N\tau$.
  Conditional probability: $P(x_n, t_N|0, 0)$.
  Chapman-Kolmogorov equation (discretized version).
  Binomial distribution.

- **Walker takes smaller steps more frequently.** [nex100]
  Steps left or right with probabilities p and q, respectively.
  Limit: $\ell \to 0, \tau \to 0, p - q \to 0$ with $\ell^2/2\tau = D$ and $(p - q)\ell/\tau = v$.
  Fokker-Planck equation with constant drift and diffusion.

- **Walker's destination drifts and diffuses** [nex101]
  Solution of Fokker-Planck equation via characteristic equation.
  Gaussian with drifting peak and runaway broadening.

- **Walker takes discrete steps randomly in continuous time.** [nex33]
  Master equation for discrete random variable.
  Walker takes step left or right with equal probability.
  Mean time interval between steps: τ.
  Position distributed via modified Bessel function.
  Gaussian function in the limit $\ell \to 0, \tau \to 0$ with $\ell^2/2\tau = D$.

- **Walk that is random only in time.** [nex25]
  Poisson process.
  Walker takes one step in time τ on average (same direction).
  Master equation solved via characteristic equation.
  Poisson distribution rises as a power law and fades out exponentially.

- **Random in Las Vegas.** [nex40]
  Gambling is a biased random walk near a precipice.
  First bias: the odds favor the casino.
  Second bias: the casino has more resources.
  Third bias: the gambler imagines Markovian features.

[nex34] Random walk in one dimension: unit steps at unit times

Consider the conditional probability distribution $P(n, t_N|0, 0)$ describing a biased random walk in one dimension as determined by the (discrete) Chapman-Kolmogorov equation,

$$P(n, t_{N+1}|0, 0) = \sum_m P(n, t_{N+1}|m, t_N)P(m, t_N|0, 0),$$

where $t_N = N\tau$ and

$$P(n, t_{N+1}|m, t_N) = p\delta_{m,n-1} + q\delta_{m,n+1}$$

expresses the instruction that the walker takes a step of unit size forward (with probability p) or backward (with probability $q = 1-p$) after one time unit τ. Convert this equation into an equation for the characteristic function $\Phi(k, t_N) = \sum_n e^{ikn}P(n, t_N|0, 0)$, then solve solve that equation, and determine $P(n, t_N|0, 0)$ from it, all by elementary means.

Solution:

[nex33] Random walk in one dimension: unit steps at random times

Consider the conditional probability distribution $P(n,t|0,0)$ describing an unbiased random walk in one dimension as determined by the master equation,

$$\frac{d}{dt}P(n,t|0,0) = \sum_m \Big[W(n|m)P(m,t|0,0) - W(m|n)P(n,t|0,0)\Big],$$

with transition rates

$$W(n|m) = \sigma\delta_{n+1,m} + \sigma\delta_{n-1,m}.$$

Here 2σ is the time rate at which the walker takes steps of unit size. The mean time interval between steps is then $\tau = 1/2\sigma$.

(a) Convert the master equation into an ordinary differential equation for the characteristic function, $\Phi(k,t) = \sum_n e^{ikn}P(n,t|0,0)$, solve it, and determine the probability distribution $P(n,t|0,0)$ from it via inverse Fourier transform.

(b) Set $n\ell = x$ for the position of the walker, where ℓ is the step size, and consider the limit $\ell \to 0, \sigma \to \infty$ such that $\ell^2\sigma = D$. Then determine $P(x,t|0,0)$ in this limit.

(c) Plot $P(n,t|0,0)$ versus n for various fixed t in comparison with the asymptotic $P(x,t|0,0)$.

Solution:

[nex100] Random walk in one dimension: tiny steps at frequent times

Consider the conditional probability distribution $P(n, t_N|0, 0)$ describing a biased random walk in one dimension as determined by the (discrete) Chapman-Kolmogorov equation,

$$P(n, t_{N+1}|0, 0) = \sum_m P(n, t_{N+1}|m, t_N)P(m, t_N|0, 0),$$

where

$$P(n, t_{N+1}|m, t_N) = p\delta_{m,n-1} + q\delta_{m,n+1}$$

expresses the instruction that the walker takes a step of fixed size forward (with probability p) or backward (with probability $q = 1 - p$) after one time unit. Set $n\ell = x_n$ for the position of the walker and $N\tau = t_N$ for the elapsed time, where ℓ is the (constant) step size and τ is the length of the (constant) time unit between steps. Now consider the limit $\ell \to 0$, $\tau \to 0$, $p - q \to 0$ such that $\ell^2/2\tau = D$ and $(p - q)\ell/\tau = v$. Show that this limit converts the Chapman-Kolmogorov equation into a Fokker-Planck equation with constant coefficients v (drift) and D (diffusion). What is the solution $P(x, t|0, 0)$ in this limit?

Solution:

[nex40] Random walk in Las Vegas: chance and necessity

A gambler with \$1 in his pocket starts playing a game against a casino with infinite monetary resources. In each round of the game, the gambler wins \$1 (with probability p) or loses \$1 (with probability $1-p$). The game ends when the gambler is bankrupt.
(a) Express the probability P_C that the gambler goes bankrupt eventually as a function of p.
(b) Plot P_C versus p for $0 < p < 1$.
(c) For what value of p is it a fair game in the sense that the gambler has a 50% chance of staying in the game forever?

Solution:

[nex25] Poisson process.

Consider the discrete Poisson process specified by the master equation,

$$\frac{\partial}{\partial t} P(n,t) = \sum_m \Big[W(n|m)P(m,t|0,0) - W(m|n)P(n,t|0,0)\Big], \qquad W(n|m) = \lambda\delta_{n-1,m},$$

for the discrete stochastic variable $n = 0, 1, 2, \ldots$ and with the initial condition $P(n, 0|0, 0) = \delta_{n,0}$. Convert the master equation into a differential equation for the generating function $G(z,t)$, then solve that equation, and determine $P(n,t|0,0)$ via power expansion.

Applications of the Poisson process include the following: (i) *Radioactive decay.* Macroscopic sample of radioactive nuclei observed over a time interval that is short compared to the mean decay time of individual nuclei. The average decay rate is λ. $P(n,t|0,0)$ is the probability that exactly n nuclei have decayed until time t. (ii) *Shot noise.* Electrical current in a vacuum tube. Electrons arrive at the anode randomly. The average rate of arrivals is λ. $P(n,t|0,0)$ is the probability that exactly n electrons have arrived at the anode until time t.

Solution:

[nex36] Free particle with uncertain position and velocity

Consider a physical ensemble of free particles with unit mass moving along the x-axis. The initial positions and velocities, x_0, v_0, are specified by a Gaussian joint probability distribution: $P_0(x_0, v_0) = (2\pi)^{-1} \exp(-x_0^2/2 - v_0^2/2)$.
(a) Find the joint probability distribution $P(x, v; t)$ for the position and velocity at time t. Infer from this result the probability distributions $P(x; t)$, $P(v; t)$ for the position and the velocity separately. Calculate the average position $\langle x(t) \rangle$ and the variance $\langle\langle x^2(t) \rangle\rangle$ thereof.
(b) Find the conditional probability distribution $P(x|v; t)$ for the positions x at time t of particles that have velocity v. Calculate the conditional averages $\langle x^n(t)|v \rangle \equiv \int dx\ x^n P(x|v; t), n = 1, 2$ for the positions of particles that have velocity v, and infer from these results the conditional variance $\langle\langle x^2(t)|v \rangle\rangle$.

Solution:

[nex101] Fokker-Planck equation with constant coefficients

Convert the Fokker-Planck equation with constant coefficients of drift and diffusion,

$$\frac{\partial}{\partial t} P(x,t|x_0) = -A\frac{\partial}{\partial x} P(x,t|x_0) - \frac{1}{2}B\frac{\partial^2}{\partial x^2} P(x,t|x_0),$$

into an ordinary differential equation for the characteristic function,

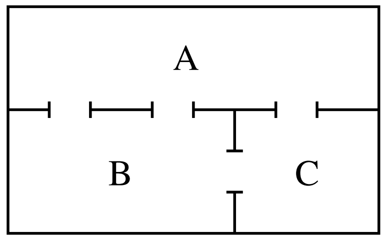
$$\Phi(k,t) \doteq \int_{-\infty}^{+\infty} dx\, e^{ikx} P(x,t|x_0).$$

(a) Solve this differential equation (by elementary means) and infer $P(x,t|x_0)$ via inverse Fourier transform. Use the initial condition $P(x,0|x_0) = \delta(x-x_0)$.
(b) Identify the mean $\langle\langle x\rangle\rangle$ and the variance $\langle\langle x^2\rangle\rangle$ in the solution $P(x,t|x_0)$.
(c) Simplify the solution $P(x,t|x_0)$ for the special case $B=0$ (no diffusion).

Solution:

[nex102] House of the mouse: two-way doors only

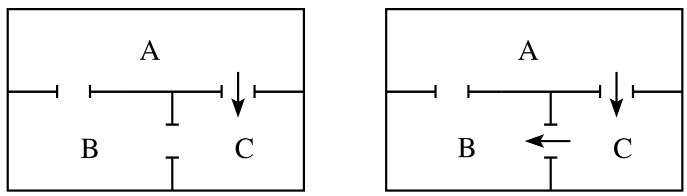
A trained mouse lives in a house with floor plan as shown. A bell rings at regular time intervals, prompting the mouse to go to an adjacent room through any door with equal probability.
(a) If the mouse starts from room A or room B or room C, with what probability is he in each room after two rings of the bell? Begin by constructing the transition matrix **W** and the initial vector $\vec{P}(0)$.
(b) Calculate the probability of the mouse being in each room after the bell has rung a great many times. Does it matter where the mouse was initially? Carry out this part in two ways: (i) Solve the left-eigenvector problem of matrix **W** to get the stationary distribution $\vec{\pi}$. (ii) Calculate the stationary distribution $\vec{\pi}$ directly from the detailed-balance condition.



Solution:

[nex103] House of the mouse: some one-way doors

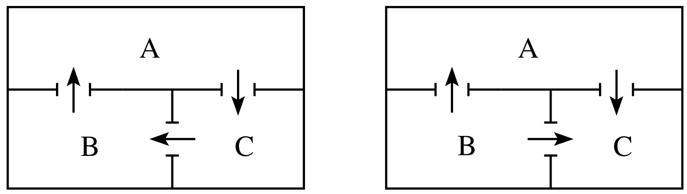
A trained mouse lives in a house with floor plan as shown in two versions. The house has three rooms and three doors. One or two doors are open one way only. A bell rings at regular time intervals, prompting the mouse to go to an adjacent room through any open door with equal probability.
(a) Construct the transition matrix $\mathbf{W}$ for both floor plans.
(b) The one-way doors are incompatible with the detailed-balance condition. Show that the transition matrix is regular, nevertheless, in both cases. For what minimum exponent s does $\mathbf{W}^s$ have no zero elements in each case?
(c) Regularity of $\mathbf{W}$ guarantees that the probability distribution for the location of the mouse is unique after the bell has rung a great many times. Calculate that stationary distribution for both cases by solving the left-eigenvector problem of matrix $\mathbf{W}$.



Solution:

[nex104] House of the mouse: one-way doors only

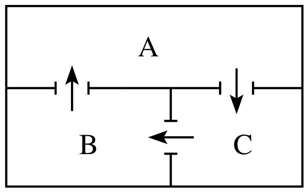
A trained mouse lives in a house with floor plan as shown in two versions. The house has three rooms and three doors. All doors are open one way only. A bell rings at regular time intervals, prompting the mouse to make an attempt to go to an adjacent room through any open door without preference.
(a) Construct the transition matrix $\mathbf{W}$ for both floor plans and calculate $\mathbf{W}^s$ for $s = 1, 2, \ldots$. Describe the resulting pattern of change in each case.
(b) Solve the left-eigenvector problem of matrix $\mathbf{W}$ in each case and interpret the results.

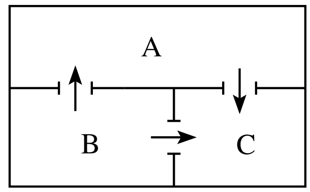


Solution:

[nex105] House of the mouse: mouse with inertia

A trained mouse lives in a house with floor plan as shown in two versions. The house has three rooms and three doors. All doors are open one way only. A bell rings at regular time intervals, prompting the mouse with equal probability to either stay put or to make an attempt to go to an adjacent room through any open door without preference.
(a) Construct the transition matrix $\mathbf{W}$ for both floor plans and calculate $\mathbf{W}^s$ for $s = 1, 2, 3$. Describe the resulting pattern of change in each case.
(b) Solve the left-eigenvector problem of matrix $\mathbf{W}$ in each case and interpret the results.

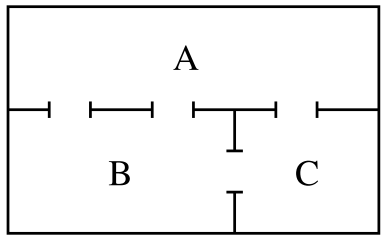




Solution:

[nex43] House of the mouse: mouse with memory

A trained mouse lives in a house with floor plan as shown. A bell rings at regular time intervals, prompting the mouse to either stay put or to go to an adjacent room through any door with no preference. When the mouse is in rooms B or C he reacts at the first ring of the bell but when he is in room A he reacts only at the second ring. What fraction of the time does the mouse spend in each room on average?



Solution:

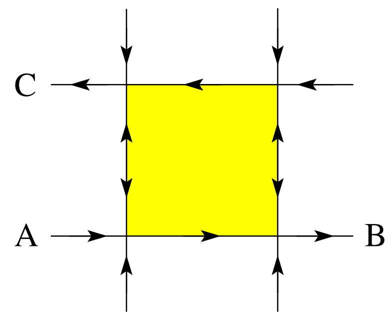
[nex42] Mixing marbles red and white

There are two white marbles in cup A and four red marbles in cup B. In every step of a Markov process a marble is selected at random from each cup and placed into the opposite cup.
(a) What are the probabilities that after three steps cup A contains (i) two white marbles, (ii) one white and one red marble, (iii) two red marbles?
(b) What is the limiting probability distribution of the three configurations after many steps?

Solution:

[nex86] Random traffic around city block

A city block is surrounded by streets with one-way and two-way traffic as shown. At intersections where drivers have a choice they choose randomly from the available options. U-turns are prohibited.
With what probability does a car that starts out at point A end up (i) at point B, (ii) at point C?



Solution:

[nex87] Modeling a Markov chain

If the following sequence of states observed in a system for the duration of 50 transitions is a Markov process, what would be a reasonable model for its transition matrix **W**? Given your model, determine the stationary distribution (π_0, π_1) of **W** and compare it with the frequency of states 0 and 1 occurring in the observed sequence.

Observed sequence (to be read in five rows from left to right):

0	1	0	1	1	1	0	1	0	0	1
	0	1	0	1	0	1	1	1	1	0
	1	0	0	1	0	1	1	0	0	1
	0	1	1	0	1	0	0	0	1	0
	1	1	0	0	1	1	0	1	1	0

Solution:

Ornstein-Uhlenbeck Process [nln62]

The Fokker-Planck equation of the Ornstein-Uhlenbeck process,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}(\kappa x P) + \frac{\gamma}{2}\frac{\partial^2 P}{\partial x^2}, \tag{1}$$

features the standard (constant) diffusion term, $B(x,t) = \gamma > 0$ and a position-dependent drift term, $A(x,t) = -\kappa x$ with $\kappa > 0$. This sort of drift is directed toward a particular position, namely $x = 0$. In a sense the drift counteracts the diffusion here. The diffusion term alone would broaden and flatten the probability distribution. A normal drift term, as in [nex101], has no effect on the broadening.

If a stationary solution $P_S(x)$ of (1) exists it must satisfy the equation

$$\frac{d}{dx}\left[\kappa x P_S(x) + \frac{\gamma}{2}\frac{d}{dx}P_S(x)\right] = 0. \tag{2}$$

Normalizability requires that $P_S(\pm\infty) = 0$ and $P'_S(\pm\infty) = 0$.
Consequence: the content of the square baracket in (2) must vanish. Separation of variables and integration then yields a Gaussian centered at $x = 0$,

$$P_S(x) = \sqrt{\frac{\kappa}{\pi\gamma}}\, e^{-\kappa x^2/\gamma}. \tag{3}$$

The ratio κ/γ in the prefactor and in the exponent represents the competition between diffusion and (restoring) drift.

For the dynamic solution of (1) we take two different approaches:

- [nex31] We derive from the second-order PDE (1) for the probability distribution $P(x,t)$ with initial condition $P(x,0) = \delta(x - x_0)$ a first-order PDE for the characteristic function and then solve that PDE. The result is a Gaussian whose mean and variance relax toward the values of the stationary solution (3). The relaxation rate is governed by the drift coefficient κ alone.

- [nex41] We search for solutions that permit a product ansatz $P(x,t) \doteq U(t)V(x)$ and aim to express the general solution as sum of such solutions. This is a reasonable goal for a linear PDE such as (1). The result expresses $U(t)$ as an exponential function and $V(x)$ as a Hermite polynomial multiplied by a Gaussian.

[nex31] Fokker-Planck equation for Ornstein-Uhlenbeck process.

Consider the Ornstein -Uhlenbeck process as specified by the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}(\kappa x P) + \frac{1}{2}\gamma \frac{\partial^2 P}{\partial x^2}, \tag{1}$$

for the conditional probability distribution $P(x,t|x_0)$, where x_0 specifies the intial value of all sample paths: $P(x,0|x_0) = \delta(x - x_0)$.
(a) Derive from the 2$^{\text{nd}}$ order PDE (1) the 1$^{\text{st}}$ order PDE for the characteristic function:

$$\frac{\partial \Phi}{\partial t} + \kappa s \frac{\partial}{\partial s}\Phi(s,t) = -\frac{1}{2}\gamma s^2 \Phi(s,t), \quad \Phi(s,t) \doteq \int_{-\infty}^{+\infty} dx\, e^{isx} P(x,t|x_0). \tag{2}$$

(b) Solve (2) by the method of characteristics,

$$\frac{1}{dt} = \frac{\kappa s}{ds} = -\frac{\frac{1}{2}\gamma s^2 \Phi}{d\Phi}. \tag{3}$$

(c) Infer from the solution $\Phi(s,t)$ an explicit expression for $P(x,t|x_0)$.

Solution:

[nex41] Ornstein−Uhlenbeck process: general solution.

(a) Show that the Fokker-Planck equation of the Ornstein-Uhlenbeck process can be solved by separation of variables and that the general solution can be expressed in terms of Hermite polynomials:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}(\kappa x P) + \frac{1}{2}\gamma\frac{\partial^2 P}{\partial x^2}; \qquad P(x,t) = \sum_{n=0}^{\infty} a_n H_n\left(\sqrt{\frac{\kappa}{\gamma}}\,x\right) e^{-n\kappa t} e^{-\kappa x^2/\gamma}.$$

(b) Show that a unique stationary solution $P_S(x)$ is approached in the limit $t \to \infty$ for arbitrary of initial conditions.
(c) Determine the expansion coefficients a_n for the particular initial distribution $P(x,0) = \delta(x-x_0)$.

Solution:

Predator-Prey System [nsl3]

A population of foxes (predator F) feeds on a population of hares (prey H). The birth rate of foxes is proportional to the fox population and to the amount of food available. Foxes die naturally, i.e. at a rate proportional to the fox population. Hares die primarily through encounters with foxes and are born at a rate proportional to the hare population.

Deterministic time evolution: Lotka-Volterra model.

$$\frac{dH}{dt} = aH - bHF, \quad \frac{dF}{dt} = bHF - dF.$$

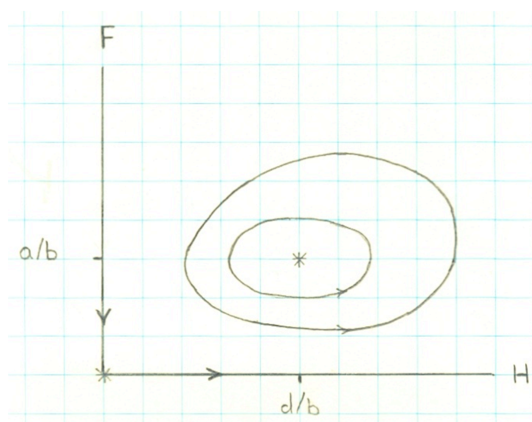
Anharmonic oscillation.

Hyperbolic fixed point at $(0,0)$.

Elliptic fixed point at $(d/b, a/b)$.

Fluctuations excluded.

Environmental effects
in initial conditions only.



Stochastic time evolution: master equation.

$$\frac{\partial}{\partial t}P(H,F,t) = \sum_{H'F'}\Big[W(H|H';F|F')P(H',F',t) - W(H'|H;F'|F)P(H,F,t)\Big],$$

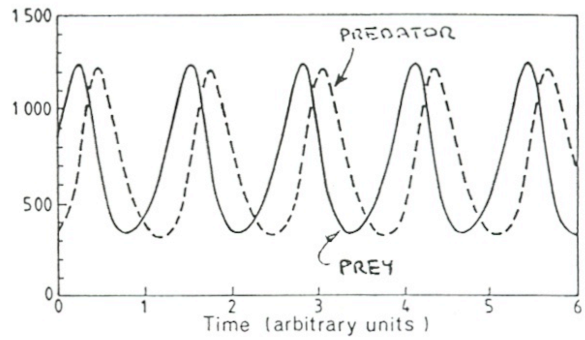
Non-vanishing transition rates:

- $W(H+1|H;F|F) = aH$ (prey is born)
- $W(H-1|H;F+1|F) = bHF$ (predator thrives on prey)
- $W(H|H;F-1|F) = dF$ (predator dies)

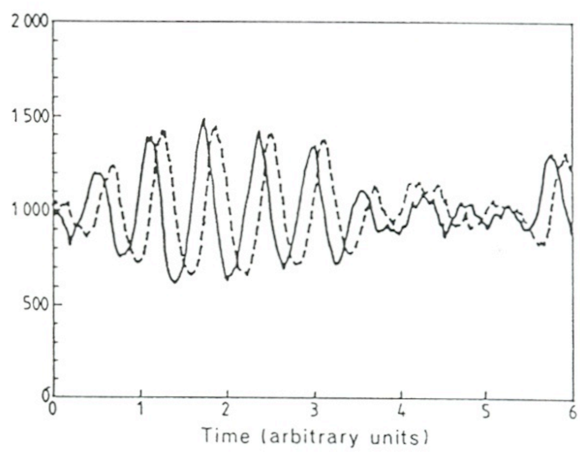
Fluctuations now included.

Environmental effects in intial conditions and in contingencies of stochastic time evolution.

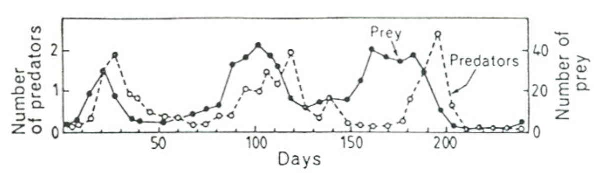
- Time evolution of deterministic system: Lotka-Volterra model



- Computer simulation of stochastic system: master equation



- Observation of real system: two species of mites



[images from Gardiner 1985]

[nex44] Populations with linear birth and death rates I

Consider the master equation

$$\frac{d}{dt}P(n,t) = \sum_m [W(n|m)P(m,t) - W(m|n)P(n,t)]$$

for the probability distribution $P(n,t)$ of the linear birth-death process. It is specified by the transition rates

$$W(m|n) = n\lambda\delta_{m,n+1} + n\mu\delta_{m,n-1},$$

where λ and μ represent the birth and death rates, respectively, of individuals in some population.
(a) Determine the jump moments $\alpha_l(m) = \sum_n (n-m)^l W(n|m)$ for $l = 1, 2$.
(b) Calculate the time evolution of the mean value $\langle n\rangle$ and the variance $\langle\langle n^2\rangle\rangle$ for the initial condition $P(n,0) = \delta_{n,n_0}$ by solving the equations of motion for the expectation values,

$$\frac{d}{dt}\langle n\rangle = \langle\alpha_1(n)\rangle, \qquad \frac{d}{dt}\langle n^2\rangle = \langle\alpha_2(n)\rangle + 2\langle n\alpha_1(n)\rangle,$$

introduced in [nln59].
(c) Plot $\langle\langle n^2\rangle\rangle$ versus t for three cases with (i) $\lambda > \mu$, (ii) $\lambda = \mu$, and (iii) $\lambda < \mu$. Interpret the shape of each curve.

Solution:

[nex112] Populations with linear birth and death rates II

Consider the master equation

$$\frac{d}{dt}P(n,t) = \sum_m [W(n|m)P(m,t) - W(m|n)P(n,t)]$$

for the probability distribution $P(n,t)$ of the linear birth-death process with initial population n_0. It is specified by the transition rates

$$W(m|n) = n\lambda\delta_{m,n+1} + n\mu\delta_{m,n-1},$$

where λ and μ represent the birth and death rates, respectively, of individuals in some population.
(a) Convert the the master equation into a linear 1^{st} order PDE for the generating function as follows:

$$\frac{\partial G}{\partial t} - (z-1)(\lambda z - \mu)\frac{\partial G}{\partial z} = 0, \quad G(z,t) \doteq \sum_{n=0}^{\infty} z^n P(n,t).$$

(b) Solve the PDE by using the method of characteristics:

$$\frac{1}{dt} = -\frac{(z-1)(\lambda z - \mu)}{dz} = \frac{0}{dG}.$$

The result reads

$$G(z,t) = \left(\frac{u(z,t) - \mu}{u(z,t) - \lambda}\right)^{n_0}, \quad u(z,t) = \frac{\lambda z - \mu}{z-1}e^{-(\lambda-\mu)t}.$$

(c) Simplify the expression of $G(z,t)$ for the special case $\lambda = \mu$.
(d) For the case $\lambda = \mu$ calculate mean $\langle\langle n(t)\rangle\rangle$ and variance $\langle\langle n^2(t)\rangle\rangle$ from derivatives of $G(z,t)$.

Solution:

[nex130] Populations with linear birth and death rates III

Consider the master equation

$$\frac{d}{dt}P(n,t) = \sum_m [W(n|m)P(m,t) - W(m|n)P(n,t)]$$

for the probability distribution $P(n,t)$ of the linear birth-death process with initial population n_0. It is specified by the transition rates

$$W(m|n) = n\lambda\delta_{m,n+1} + n\mu\delta_{m,n-1},$$

where λ and μ represent the birth and death rates, respectively, of individuals in some population. In [nex112] we have determined the following expression for the generating function pertaining to the special case $\lambda = \mu$ of equal birth and death rates:

$$G(z,t) \doteq \sum_{n=0}^{\infty} z^n P(n,t) = \left(\frac{\lambda(z-1)t - z}{\lambda(z-1)t - 1}\right)^{n_0}.$$

(a) Expand the generating function in powers of z for the special case $n_0 = 1$ to derive analytic expressions for the probabilities $P(n,t)$.
(b) Verify the normalization condition $\sum_n P(n,t) = 1$.
(c) Plot $P(n,t)$ versus λt for $n = 0, 1, \ldots, 4$ (five curves).
(d) Find the time t_n where $P(n,t)$ reaches its maximum value.
(e) Discuss the compatibility of the paradoxical results (i) $\lim_{t\to\infty} P(0,t) = 1$ (certainty of death) and from [nex44] (ii) $\langle n(t)\rangle = n_0 = 1$, (iii) $\lim_{t\to\infty}\langle\langle n^2(t)\rangle\rangle \to \infty$ (persistent signs of life and uncertainty).
(f) Explore the determination of analytic expressions of $P(n,t)$ for $n_0 = 2, 3, \ldots$ or for generic n_0.

Solution:

[nex46] Catalyst driven chemical reaction: stationary state

In the chemical reaction $A + X \leftrightarrow A + Y$, the molecule A is a catalyst at constant concentration. The total number of reacting molecules, $n_x + n_y = N$, is also constant. K_1 is the probability per unit time that a molecule X interacts with a molecule A to turn into a molecule Y, and K_2 is the probability per unit time that a Y interacts with an A to produce an X. The dynamics may be described by a master equation for $P(n,t)$, where $n \doteq n_x, n_y = N - n$. The transition rates are

$$W(m|n) = K_1 n \delta_{m,n-1} + K_2(N-n)\delta_{m,n+1}.$$

To calculate the mean value $\langle\langle n(t)\rangle\rangle$ and the variance $\langle\langle n^2(t)\rangle\rangle$ proceed as in [nex44] via the equations of motion, $d\langle n\rangle/dt = \langle\alpha_1(n)\rangle$, $d\langle n^2\rangle/dt = \langle\alpha_2(n)\rangle + 2\langle n\alpha_1(n)\rangle$, with jump moments $\alpha_l(m) = \sum_n (n-m)^l W(n|m)$.

(a) Construct the equations of motion for $\langle\langle n(t)\rangle\rangle$, $\langle\langle n^2(t)\rangle\rangle$.

(b) Infer the long-time asymptotic values $\langle\langle n(\infty)\rangle\rangle$, $\langle\langle n^2(\infty)\rangle\rangle$ directly.

(c) Plot $\langle\langle n(\infty)\rangle\rangle$, $\langle\langle n^2(\infty)\rangle\rangle$ versus γ for $K_1 = \gamma$, $K_2 = 1-\gamma$ (thus fixing the time scale) and explain the location of the maximum for each quantity.

Solution:

[nex107] Catalyst driven chemical reaction: dynamics

In the chemical reaction $A + X \leftrightarrow A + Y$, the molecule A is a catalyst at constant concentration. The total number of reacting molecules, $n_x + n_y = N$, is also constant. K_1 is the probability per unit time that a molecule X interacts with a molecule A to turn into a molecule Y, and K_2 is the probability per unit time that a Y interacts with an A to produce an X. The dynamics may be described by a master equation for $P(n,t)$, where $n \equiv n_x, n_y = N - n$. The transition rates are

$$W(m|n) = K_1 n \delta_{m,n-1} + K_2(N-n)\delta_{m,n+1}.$$

(a) Solve the equations of motion for $\langle\langle n(t)\rangle\rangle$, $\langle\langle n^2(t)\rangle\rangle$ as constructed in [nex46]. Use initial values $\langle\langle n(0)\rangle\rangle = n_0$, $\langle\langle n^2(0)\rangle\rangle = 0$.
(b) Plot $\langle\langle n(t)\rangle\rangle$, $\langle\langle n^2(t)\rangle\rangle$ in separate frames for $n_0 = 0$, $K_1 = \gamma$, $K_2 = 1-\gamma$, and various γ. This fixes the time scale. Identify any interesting features in the curves and try to explain them.

Solution:

[nex108] Catalyst driven chemical reaction: total rate of reactions

In the chemical reaction $A + X \leftrightarrow A + Y$, the molecule A is a catalyst at constant concentration. The total number of reacting molecules, $n_x + n_y = N$, is also constant. K_1 is the probability per unit time that a molecule X interacts with a molecule A to turn into a molecule Y, and K_2 is the probability per unit time that a Y interacts with an A to produce an X. The dynamics may be described by a master equation for $P(n,t)$, where $n \equiv n_x, n_y = N - n$. The transition rates are $W(m|n) = K_1 n \delta_{m,n-1} + K_2(N-n)\delta_{m,n+1}$. The total rate of chemical reactions is defined as follows:

$$R(t) \doteq \sum_{nm} W(n|m)P(m,t).$$

(a) Express $R(t)$ in terms of $\langle\langle n(t)\rangle\rangle$.
(b) Use the result of $\langle\langle n(\infty)\rangle\rangle$ from [nex46] to calculate the total rate of chemical reactions in the stationary state. Set $K_1 = \gamma$, $K_2 = 1-\gamma$ and compare the γ-dependence of $R(\infty)$ with that of $\langle\langle n^2(\infty)\rangle\rangle$ from [nex46], which is a measure of the fluctuations in the population of molecules.
(c) Use the result of $\langle\langle n(t)\rangle\rangle$ from [nex107] to calculate the time evolution of $R(t)$. Plot $R(t)$ for $n_0 = 0, K_1 = \gamma, K_2 = 1-\gamma$ and various fixed values of γ. The time scale is thus set. Compare the graph of $R(t)$ with the graph of $\langle\langle n^2(t)\rangle\rangle$ from [nex107]. Explain the similarities and differences.

Solution:

[nex48] Air in leaky tank I: generating function

At time $t = 0$ a tank of volume V contains n_0 molecules of air (disregarding chemical distinctions). The tank has a tiny leak and exchanges molecules with the environment, which has a constant density ρ of air molecules.

(a) Set up the master equation for the probability distribution $P(n,t)$ under the assumption that a molecule leaves the tank with probability $(n/V)dt$ and enters the tank with probability ρdt, implying transition rates $W(m|n) = \rho\delta_{m,n+1} + (n/V)\delta_{m,n-1}$.

(b) Derive the following linear partial differential equation (PDE) for the generating function $G(z,t)$ from that master equation:

$$\frac{\partial G}{\partial t} + \frac{z-1}{V}\frac{\partial G}{\partial z} = \rho(z-1)G, \quad G(z,t) \doteq \sum_{n=0}^{\infty} z^n P(n,t).$$

(c) Solve the PDE by the method of characteristics,

$$\frac{1}{dt} = \frac{z-1}{V dz} = \frac{\rho(z-1)G}{dG},$$

to obtain the result

$$G(z,t) = e^{V\rho(z-1)\left[1-e^{-t/V}\right]}\left[e^{-t/V}(z-1)+1\right]^{n_0}$$

for the nonequilibrium state.

(d) Find the characteristic function $G(z,\infty)$ for the equilibrium situation.

(e) If we set n_0/V (density inside) equal to ρ (density outside), the generating function still depends on time. Explain the reason.

(f) Show that for $n_0/V = \rho$ the function $G(z,t)$ at arbitrary t converges, as $\rho V \to \infty$, toward the stationary result determined in (e).

Solution:

[nex109] Air in leaky tank II: probability distribution

At time $t = 0$ a tank of volume V contains n_0 molecules of air (disregarding chemical distinctions). The tank has a tiny leak and exchanges molecules with the environment, which has a constant density ρ of air molecules.
(a) Infer from the generating function,

$$G(z,t) = e^{V\rho(z-1)\left[1-e^{-t/V}\right]}\left[e^{-t/V}(z-1)+1\right]^{n_0},$$

calculated in [nex48], via a power series expansion, the probability distribution,

$$P(n,t) = e^{-V\rho\left[1-e^{-t/V}\right]}\sum_{m=0}^{\min(n,n_0)}\frac{n_0!}{m!(n_0-m)!(n-m)!}(\rho V)^{n-m}(1-e^{-t/V})^{n+n_0-2m}e^{-mt/V},$$

for the number of molecules in the tank.
(b) Demonstrate consistency of this result with the initial condition: $\lim_{t\to 0} P(n,t) = \delta_{n,n_0}$.
(c) Show that the equilibrium state is described by a Poisson distribution by (i) expanding the equilibrium generating function $G(z,\infty)$ from [nex48] into a power series, and (ii) by taking the limit $P(n, t \to \infty)$.

Solution:

[nex49] Air in leaky tank III: detailed balance

A tank of volume V has a small leak and exchanges molecules of air with the environment. The environment has a constant density ρ of molecules. The master equation for the probability distribution $P(n,t)$ of air molecules in the container is specified by transition rates of the form

$$W(m|n) = T_+(n)\delta_{m,n+1} + T_-(n)\delta_{m,n-1}$$

with $T_+(n) = \rho$ and $T_-(n) = n/V$.
(a) Determine the stationary distribution $P_s(n) = P(n, t \to \infty)$ from the detailed balance condition, $T_-(n)P_s(n) = T_+(n-1)P_s(n-1)$, via the recurrence relation derived in [nln17].
(b) Compare the peak position n_p of the stationary distribution with the mean value $\langle n \rangle$.

Solution:

[nex110] Air in leaky tank IV: evolution of mean and variance

At time $t = 0$ a tank of volume V contains n_0 molecules of air (disregarding chemical distinctions). The tank has a tiny leak and exchanges molecules with the environment, which has a constant density ρ of air molecules. Given the transition rate $W(m|n)$ and generating function $G(z,t)$ derived in [nex48] for this process we are in a position to calculate the the time evolution of the mean $\langle\langle n(t)\rangle$ and the variance $\langle\langle n^2(t)\rangle$ in three different ways with moderate effort.
(a) Calculate $\langle n(t)\rangle_f$ and $\langle n^2(t)\rangle_f$ directly from the (known) $G(z,t)$ and infer mean and variance from these factorial moments.
(b) Infer equations of motions for the $\langle n^m(t)\rangle_f$, $m = 1, 2, \ldots$ from the (known) PDE for $G(z,t)$. Solve these equations for $m = 1, 2$. In this system a closed set of equations results for any m.
(c) Calculate the jump moments $\alpha_l(m) = \sum_n (n-m)^l W(n|m)$ from the known transition rates and show that the associated equations of motion $d\langle n\rangle/dt = \langle\alpha_1(n)\rangle$, $d\langle n^2\rangle/dt = \langle\alpha_2(n)\rangle + 2\langle n\alpha_1(n)\rangle$ for the first two moments are equivalent to those previously derived for the factorial moments.

Solution:

[nex50] Pascal distribution and Planck radiation law

Consider a quantum harmonic oscillator in thermal equilibrium at temperature T. We know from [nex22] that the population of energy levels $E_n = n\hbar\omega_0, n = 0, 1, 2, \ldots$ is described by the Pascal distribution,

$$P(n) = (1-\gamma)\gamma^n, \quad \gamma = \exp(-\hbar\omega_0/k_B T).$$

Suppose that the heat bath is provided by a blackbody radiation field of energy density $u(\omega)$ and that the oscillator interacts with that field exclusively via emission and absorption of photons with energy $\hbar\omega_0$. If we describe the aproach to thermal equilibrium of the oscillator by a master equation, the transition rates must, therefore, be of the type, $W(m|n) = T_+(n)\delta_{m,n+1} + T_-(n)\delta_{m,n-1}$, familiar from birth-death processes (see [nln17]). Here $T_+(n)$ reflects the absorption of a photon and $T_-(n)$ the emission of a photon if the oscillator is in energy level n.
Now we assume (as Einstein did) that the transition rates are of the form $T_+(n) = Bu(\omega_0)$ and $T_-(n) = A + Bu(\omega_0)$, where the term A reflects spontaneous emission and the terms $Bu(\omega_0)$ induced emission or induced absorption. Infer from the compatibility condition of these transition rates with the Pascal distribution at equilibrium the T-dependence of $u(\omega)$ at fixed frequency ω_0.

Solution:

[nex111] Effects of nonlinear death rates I: Malthus-Verhulst equation

Consider the master equation of the birth-death process with transition rates,

$$W(m|n) = n\lambda\delta_{m,n+1} + \left[n\mu + \frac{\gamma}{N}n(n-1)\right]\delta_{m,n-1}.$$

It describes a population with a linear birth rate, $n\lambda$, and a linear death rate, $n\mu$, as in [nex44]. To account for the unhealthy environment in crowded conditions ($n \simeq N$), a nonlinear death rate has been added. The nonlinearity suppresses the continued exponential growth found in [nex44] for the case $\lambda > \mu$ and stabilizes a stationary state.

(a) Construct the equations of motion for $\langle n(t)\rangle$ from the first jump moment as in [nex44]. Then neglect fluctuations by setting $\langle n^2(t)\rangle \simeq \langle n(t)\rangle^2$ and $\langle n(t)\rangle = Nx(t) + \mathrm{O}(\sqrt{N})$ to arrive (in leading order) at the Malthus-Verhulst equation,

$$\frac{dx}{dt} = (\lambda - \mu)x - \gamma x^2,$$

for the time-dependence of the (scaled) average population. Derive the analytic solution,

$$x(t) = \frac{x_0 e^{(\lambda-\mu)t}}{1 + \frac{\gamma x_0}{\lambda-\mu}\left[e^{(\lambda-\mu)t} - 1\right]},$$

pertaining to initial value $x_0 = n_0/N$ and parameters $\lambda \neq \mu$.

(b) Show how the exponential long-time asymptotics crosses over to power-law asymptotics in the limit $\lambda - \mu \to 0$. (c) Find the dependence of the stationary population $x(\infty)$ on λ, μ, γ.

(d) Plot $x(t)/x_0$ versus t for $\mu = 1$ and (i) various values of λ at fixed $\gamma = 0.2$ and (ii) various values of γ at fixed $\lambda = 2$. Interpret your results.

Solution:

[nex51] Effects of nonlinear death rate II: stationarity and fluctuations

Consider the master equation of the birth-death process with transition rates

$$W(m|n) = (n+1)\lambda\delta_{m,n+1} + \left[n\mu + \frac{\gamma}{N}n(n-1)\right]\delta_{m,n-1}.$$

It describes a population with a linear birth rate, $(n+1)\lambda$, and a linear death rate, $n\mu$. To account for the unhealthy environment under crowded circumstances ($n \simeq N$), a nonlinear death rate has been added to the process. Use the recurrence relation, $P_s(n) = [T_+(n-1)/T_-(n)]P_s(n-1)$ for the stationary distribution $P_s(n)$ derived in [nln17] from the detailed balance condition for the following tasks. Consider a system that easily accommodates $N = 20$ individuals of some population with fixed (linear) death rate $\mu = 1$, fixed birth rate $\lambda = 1.5$, and variable environmental factor γ.
(a) Compute the mean $\langle n\rangle$ and the variance $\langle\langle n^2\rangle\rangle$ for $\gamma = 0.2$.
(b) Plot the distribution $P(n)$ versus n for $\gamma = 0.2, 0.4, \ldots, 1.0$ in the same diagram.
(c) Plot the mean $\langle n\rangle$ across the range $0.2 < \gamma < 1$ and compare the result with the function $Nx(t)$ derived in [nex111] from the Malthus-Verhulst equation, which ignores fluctuations.
(d) Plot the variance $\langle\langle n^2\rangle\rangle$ across the range $0.2 < \gamma < 1$.

Solution:

[nex113] Modified linear birth rate I: stationarity

Consider a linear birth-death process with a modified birth rate,

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n\delta_{m,n-1},$$

to be used in the master equation. In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population if $\lambda < \mu$.
(a) Use the recurrence relation of [nln17] to show that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$ for $\lambda < \mu$. Use the ratio $\gamma = \lambda/\mu$ as parameter of that distribution. Show that $P_s(n)$ is the Pascal distribution (see [nex22]).
(b) Determine the stationary values of $\langle n\rangle$ and $\langle\langle n^2\rangle\rangle$. Describe the differences between the long-time asymptotics of the original linear birth-death process with $\lambda = \mu$ as analyzed in [nex130] and the results obtained here for the case of modified birth rates in the limit $\mu - \lambda \to 0$.

Solution:

[nex114] Modified linear birth rate II: evolution of mean and variance

Consider a linear birth-death process with a modified birth rate to be used in the master equation:

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n\delta_{m,n-1}, \quad \lambda < \mu.$$

In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population. In [nex113] we showed that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$: the Pascal distribution.
(a) Establish from the jump moments, $\alpha_l(n) = \sum_m (m-n)^l W(m|n)$, the equations of motion,

$$\frac{d}{dt}\langle n\rangle \doteq \langle \alpha_1(n)\rangle = (\lambda - \mu)\langle n\rangle + \lambda,$$

$$\frac{d}{dt}\langle n^2\rangle \doteq \langle \alpha_2(n)\rangle + 2\langle n\alpha_1(n)\rangle = 2(\lambda-\mu)\langle n^2\rangle + (3\lambda+\mu)\langle n(t)\rangle + \lambda,$$

for the first and second moment.
(b) Calculate the time evolution of mean $\langle\langle n(t)\rangle\rangle$ and variance $\langle\langle n^2(t)\rangle\rangle$ by solving these equations for initial conditions $\langle\langle n(0)\rangle\rangle = \langle\langle n^2(0)\rangle\rangle = 0$.

Solution:

[nex115] Modified linear birth rate III: generating function

Consider a linear birth-death process with a modified birth rate to be used in the master equation:

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n \delta_{m,n-1}, \quad \lambda < \mu.$$

In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population. In [nex113] we showed that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$: the Pascal distribution.
(a) Construct the PDE,

$$\frac{\partial G}{\partial t} - (\lambda z - \mu)(z-1)\frac{\partial G}{\partial z} = \lambda(z-1)G,$$

for the generating function $G(z,t) = \sum_n z^n P(n,t)$ and to solve that PDE for the case of zero initial population by the method of characteristics (see e.g. [nex112]). The result reads

$$G(z,t) = \frac{\lambda - \mu}{\lambda z - \mu - \lambda(z-1)e^{(\lambda-\mu)t}}.$$

(b) Calculate mean $\langle\langle n(t)\rangle\rangle$ and variance $\langle\langle n^2(t)\rangle\rangle$ from derivative of the $G(z,t)$ for comparison with the results obtained in [nex114] via a different route.
(c) Calculate the stationary distribution $P_s(n)$ from the asymptotic generating function $G(z,\infty)$.

Solution:

[nex116] Modified linear birth rate IV: probability distribution

Consider a linear birth-death process with a modified birth rate to be used in the master equation:

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n \delta_{m,n-1}, \quad \lambda < \mu.$$

In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population. In [nex113] we showed that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$: the Pascal distribution. In [nex115] we have calculated the generating function $G(z,t)$ for the case of zero initial population. Use that result here to calculate the explicit result,

$$P(n,t) = \frac{\mu - \lambda}{\mu - \lambda e^{(\lambda-\mu)t}} \left(\frac{\lambda[1 - e^{(\lambda-\mu)t}]}{\mu - \lambda e^{(\lambda-\mu)t}} \right)^n,$$

for the time-dependence of the probability distribution $P(n,t)$. Check the limit $t \to 0$ to verify the imposed initial condition and the limit $t \to \infty$ to confirm the result of $P_s(n)$ previously obtained by other means.

Solution:

[nex52] Bistable chemical system

Consider the master equation of the birth-death process specified by transition rates of the form $W(m|n) = T_+(n)\delta_{m,n+1} + T_-(n)\delta_{m,n-1}$ with

$$T_+(n) = k_1 A n(n-1) + k_3 A, \qquad T_-(n) = k_2 n(n-1)(n-2) + k_4 n.$$

This process describes two simultaneous chemical reactions $A + 2X \leftrightarrow 3X, A \leftrightarrow X$ that exhibit bistable states in a certain parameter range. The concentration of A is taken to be constant.
(a) Construct a product expression for $P_s(n)$ from the detailed-balance condition as explained in [nln17]. Use the three parameters $B = k_1 A/k_2, R = k_4/k_2, Q = k_3/k_1$.
(b) Show that that for $R/Q = 1$ we thus obtain the Poisson distribution.
(c) Plot the solution of the extremum condition $T_+(n-1) = T_-(n)$ as a graph B versus n over the range $0 \leq n \leq 100$ for the two cases (i) $Q = 100$, $R = 500$ and (ii) $Q = 100$, $R = 1200$. Identify the extremum positions n_{extr} for $B = 70$ in both cases.
(d) Plot $P_s(n)$ versus n over the range $0 \leq n \leq 100$ for $B = 70$ and cases (i), (ii) for Q, R.

Solution:

[nex47] Ultracold neutrons in an ideal Steyerl bottle

Consider a container whose walls are perfect mirrors for ultracold neutrons. At time $t = 0$ the bottle is known to contain exactly n_0 neutrons. The decay rate of a neutron is K.
(a) Set up the master equation,

$$\frac{\partial}{\partial t} P(n,t) = (n+1)KP(n+1,t) - KnP(n,t),$$

for the probability distribution $P(n,t)$, and derive the PDE,

$$\frac{\partial G}{\partial t} + K(z-1)\frac{\partial G}{\partial z} = 0,$$

for the generating function $G(z,t) \doteq \sum_n z^n P(n,t)$.
(b) Solve the PDE by the method of characteristics (see e.g. [nex112]) to obtain

$$G(z,t) = \left[(z-1)e^{-Kt} + 1\right]^{n_0}.$$

(c) Infer therefrom the probability distribution

$$P(n,t) = \frac{n_0!}{n!(n_0-n)!} \frac{\left(1-e^{-Kt}\right)^{n_0}}{\left(e^{Kt}-1\right)^n}.$$

(d) Determine (via derivatives of the generating function) the average number $\langle n(t)\rangle$ of remaining neutrons and the variance $\langle\langle n^2(t)\rangle\rangle$ thereof.
(e) Design a contour plot of $P(n,t)$ for $0 < n < n_0 = 20$ and $0 < Kt < 3$. Design a line graph of $P(n,t)$ for $0 < Kt < 5$ for fixed $n = 0, 1, 2, 5, 10, 15, 18, 19$. Interpret these graphs.
(f) Derive equations of motion for $\langle n\rangle$ and $\langle n(n-1)\rangle$ directly from the master equation and solve them to reproduce the solutions obtained in (d).

Solution:

[nex45] Random light switch.

The position of the light switch is described by the stochastic variable X, which can assume the two values $x = 0$ (lights off) and $x = 1$ (lights on). Some agent switches the lights on/off randomly at the rate γ. This means that the average interval of continuous brightness/darkness is $\tau = 1/\gamma$.
(a) Set up the master equation for $P(x, t|x_0)$ and solve it.
(b) Find the asymptotic distribution $P_s(x) = \lim_{t\to\infty} P(x, t|x_0)$.
(c) Find the conditional average $\langle X(t)|x_0\rangle \doteq \sum_x xP(x, t|x_0)$ and then $\langle X(t)\rangle_s = \lim_{t\to\infty}\langle X(t)|x_0\rangle$.
(d) Use the regression theorem $\langle X(t)X(t')\rangle_s \doteq \sum_{xx'} P(x, t|x', t')P_s(x')$ to determine the (stationary) autocorrelation function $\langle\langle X(t)X(t')\rangle\rangle_s \doteq \langle X(t)X(t')\rangle_s - \langle X(t)\rangle_s\langle X(t')\rangle_s$.

Solution: