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07. Lagrangian Mechanics III

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Abstract
Part seven of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

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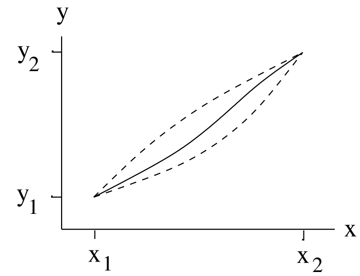
Calculus of Variation [mln78]

Given a functional $f(y, y'; x)$ with $y' = dy/dx$, determine the path $y(x)$ between fixed endpoints $y(x_1) = y_1$, $y(x_2) = y_2$ such that the following integral is an extremum:

$$J = \int_{x_1}^{x_2} dx\, f\big(y(x), y'(x); x\big).$$



Variation of path: $y(x, \alpha) = y(x, 0) + \alpha\eta(x)$ with $\eta(x_1) = \eta(x_2) = 0$.

Parametrized integral: $J(\alpha) = \displaystyle\int_{x_1}^{x_2} dx\, f\big(y(x, \alpha), y'(x, \alpha); x\big)$.

Extremum condition: $\left(\dfrac{dJ}{d\alpha}\right)_{\alpha=0} = 0$ for arbitrary $\eta(x)$.

Differentiation: $\dfrac{dJ}{d\alpha} = \displaystyle\int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)$.

Integration by parts (second term):

$$\int_{x_1}^{x_2} dx\, \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x \partial \alpha} = \left[\frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}.$$

Substitute and use $(\partial y/\partial\alpha)_{\alpha=0} = \eta(x)$:

$$\Rightarrow \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x) = 0.$$

Requirement that integral must vanish for arbitrary $\eta(x)$ implies

Euler's equation: $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\dfrac{\partial f}{\partial y'} = 0.$

Notation used in calculus of variation:

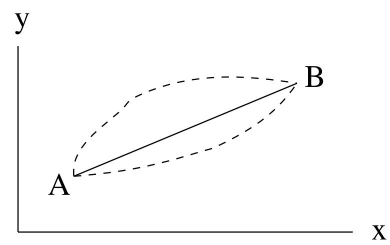
Variation of path: $\left(\dfrac{\partial y}{\partial \alpha}\right)_{\alpha=0} d\alpha \doteq \delta y.$

Variation of integral: $\left(\dfrac{dJ}{d\alpha}\right)_{\alpha=0} d\alpha \doteq \delta J.$

$$\Rightarrow \delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\delta y = 0.$$

[mex26] Shortest path between two points in a plane I

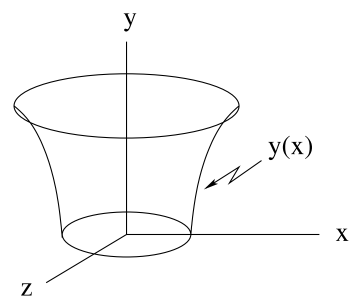
Use the calculus of variation to prove that the shortest path between two points A and B in the (x, y)-plane is a straight line $y(x) = ax + b$.



Solution:

[mex27] Economy plastic cup

A manufacturer of plastic cups receives an order for cups of given height, diameters at the top and bottom, and material thickness. Determine the profile $y(x)$ at $z = 0$ of the cup which minimizes the amount of plastic needed for each cup.



Solution:

Variational Problem with Auxiliary Condition [mln16]

Search for a function $y(x)$ that yields an extremum of the integral

$$J = \int_{x_1}^{x_2} dx\, f[y(x), y'(x); x]$$

subject to an auxiliary condition in the form of the integral constraint

$$C = \int_{x_1}^{x_2} dx\, \sigma[y(x), y'(x); x] = \text{const.}$$

Use the functional $F_\lambda[y(x), y'(x); x] = f[y(x), y'(x); x] + \lambda\sigma[y(x), y'(x); x]$, where λ is an undetermined Lagrange multiplier.

Find the extremum of $J_\lambda = \displaystyle\int_{x_1}^{x_2} dx\, F_\lambda[y(x), y'(x); x]$.

This leads to Euler's equation $\displaystyle\frac{\partial F_\lambda}{\partial y} - \frac{d}{dx}\left(\frac{\partial F_\lambda}{\partial y'}\right) = 0.$

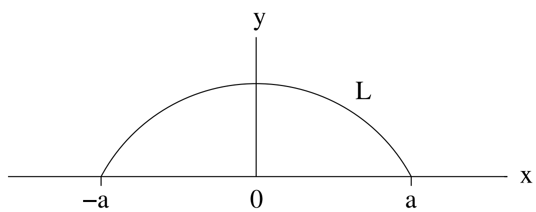
Then adjust the value of λ in the solution such that the auxiliary condition is satisfied.

Examples:

- Isoperimetric problem [mex28]
- Catenary problem [mex38]

[mex28] Isoperimetric problem

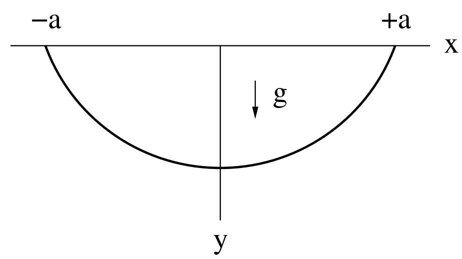
Consider a fence of length L constructed in such a manner as to connect two points of a wall that are a distance $2a$ apart. Use the calculus of variation with an auxiliary integral constraint to show that the shape of the fence must be part of a circle.



Solution:

[mex38] Catenary problem

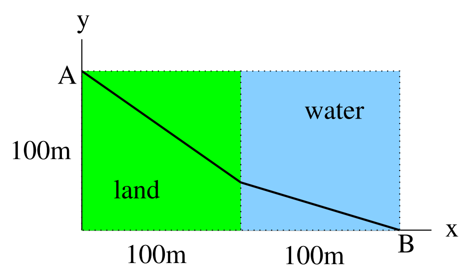
Consider a chain of length L and mass per unit length ρ_L. Its ends are suspended from two fixed points which are positioned at the same height and a distance $2a$ apart. Use the calculus of variation with an auxiliary integral constraint to show that the shape of the chain is described by the function $y(x) = A - B\cosh(x/B)$, where A, B are constants.



Solution:

[mex29] Athletic refraction

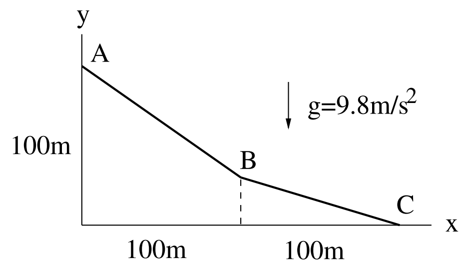
An athlete starts at point A and wants to reach point B in the shortest possible time by running over land and swimming across water. Her running speed is $v_1 = 7\text{m/s}$ and her swimming speed $v_2 = 1\text{m/s}$. (a) At which point $(x, y) = (100m, ??)$ should she dive into the water along the optimal path and in what time does she finish the race? (b) Derive Snell's law, $\sin\theta_1/\sin\theta_2 = v_1/v_2$, from this extreme-value calculation and identify the angles θ_1, θ_2 in the illustration below.



Solution:

[mex30] Brachistochrone problem I

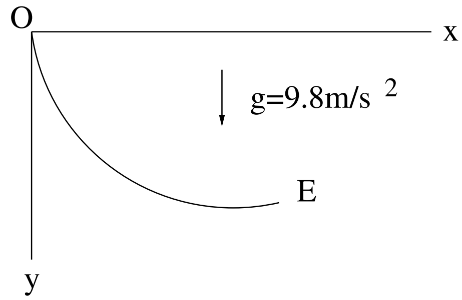
A particle of mass m slides from rest down from point A to point C along a frictionless path consisting of two straight-line segments that are joined at point B with coordinates $(x, y) = (100m, ??)$. At what height should point B be positioned to make the particle travel from A to C in the shortest time. Find the time t_O it takes the particle to travel from A to C along the optimal path and the time t_S along a straight path.



Solution:

[mex31] Brachistochrone problem II

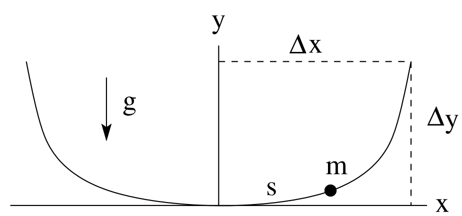
A particle of mass m slides from rest at the origin of the coordinate system down to the point E along a frictionless path. (a) Use the calculus of variation to determine the path along which the particle arrives at E in the shortest time. (b) Determine the time it takes the particle in [mex30] to travel from A to C along a path such as found in (a).



Solution:

[mex144] Isochronous potential well

A particle is constrained to move under the influence of a uniform gravitational field g on a curve $y(x)$ with a minimum at $x = 0$ in a vertical plane. Find the shape of the curve such that the oscillations of the particle about this potential minimum have a period that is independent of the amplitude. This is accomplished by requiring that the potential energy (here mgy) is proportional to the square of the arc length s from $x = 0$, just as is the case in a harmonic oscillator. Use $\frac{1}{2}ks^2 = mgy$, where k is the equivalent spring stiffness, and set $mg/k = 4a$, where a is a characteristic length of the potential well. Find the maximum half width Δx and the maximum height Δy of the potential well for which this scheme works and express these measures in units of a. Find the value of a which makes the period of oscillation one second (1s) for $g = 9.8\text{m/s}^2$.



Solution:

Geodesics [mln38]

The term *geodesic* originates from surveying the Earth's surface over distances so large that its curvature is significant.

Mathematical definition:
A *geodesic* is the shortest line between two points on any given surface.

Applications:

- Geodesics on a plane are straight lines [mex26], [mex117].
- Geodesics on a sphere lie on great circles [mex118].

Relation to dynamics:
Consider a particle of mass m that is constrained to move on a surface specified by a holonomic constraint $g(x, y, z) = 0$ and is not subject to any forces other than the forces of constraint. The path of such a particle consists of segments that are all geodesics.

Sketch of a proof: The potential energy V is identically zero and the energy E is conserved. Therefore the kinetic energy T, the speed v of the particle, and the Lagrangian $L = T - V$ are constants. Now consider Hamilton's principle for paths with constant L. The action J is then minimized if the time of travel, $t_2 - t_1$, is minimized, which, in turn, is the case on the shortest path, i.e. on a geodesic.

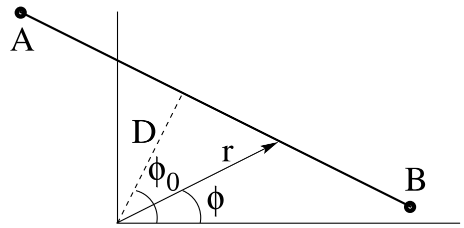
Clairaut's theorem:
Consider a surface of revolution described by cylindrical coordinates $z, \phi, r(z)$. Suppose a particle with mass m, constrained to move on that surface, is launched with a speed v_0 at $\phi = z = 0$ in a direction at an angle α_0 from the meridian. (The intersection between the surface and a plane through its axis produces two meridians.) From the conservation of kinetic energy and the conservation of angular momentum around the axis it follows that $r \sin\alpha = \text{const}$ holds along the path of the particle.

Applications:

- Dynamical trap without potential energy [mex119].
- Vertical range of particle sliding inside cone [mex120].

[mex117] Shortest path between two points in a plane II

Use the calculus of variation to prove that the shortest path between two points A and B in the (x, y)-plane is a straight line: $r = D/\cos(\phi - \phi_0)$. Perform the entire calculation using polar coordinates.



Solution:

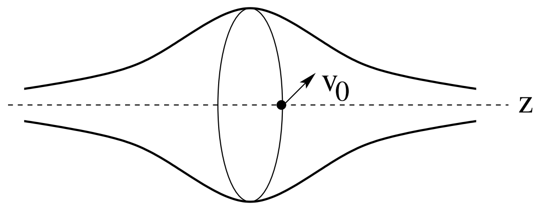
[mex118] Geodesic on a sphere

Use the calculus of variation to prove that the shortest path between two points A and B on a sphere of radius r is a great circle. A great circle is the intersection between the sphere and a plane that goes through the center of the sphere.

Solution:

[mex119] Dynamical trap without potential energy

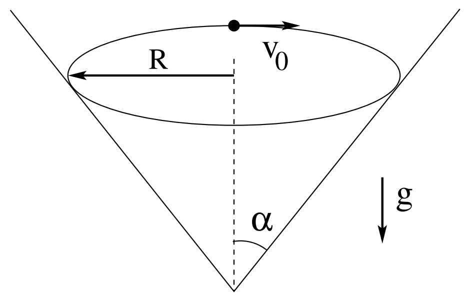
Consider a surface of revolution with cylindrical coordinates $z, \phi, r(z) = (1+z^2)^{-1}$. A particle of mass m is constrained to move on that surface without friction. It is launched at $z = 0$ with a speed v_0 in a direction at 45° relative to the meridian. Find the maximum value of $|z|$ the particle reaches along its trajectory.



Solution:

[mex120] Vertical range of particle sliding inside cone

Consider a conical surface with vertical axis (z) and apex with angle 2α at the bottom in a uniform gravitational field g. A particle of mass m is projected horizontally with velocity v_0 at a distance R from the axis on the inside of the cone. (a) How must v_0 be chosen to keep the particle on a horizontal circular path? (b) If v_0 is smaller (larger) than the value required to keep it on a horizontal circle, the resulting path will explore a band with $r_{min} \leq r \leq R$ ($R \leq r \leq r_{max}$). Find r_{min} and r_{max}.



Solution:

Extremum Principles [msl20]

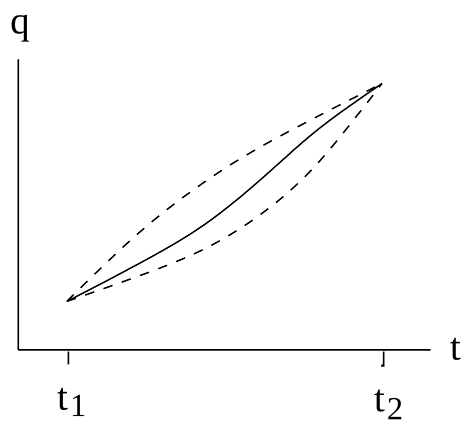
- **Hero of Alexandria** (2^{nd} century BC) : A ray of light traveling from one point to another by reflection from a plane mirror always takes the shortest possible path. $\Rightarrow$ Law of reflection.
- **Fermat** (1657): A ray of light traveling through the interface of optically different media chooses the path that requires the least time. $\Rightarrow$ Law of refraction.
- **Newton, Leibniz, Bernoulli, Euler**: Development of the calculus of variation. Solution of important extremum problems.
- **Maupertuis** (1747): The motion of a dynamical system subject to constraints proceeds in a way that minimizes the action (principle of least action). $\Rightarrow$ Equations for trajectories.
- **Hamilton** (1834): Of all possible paths along which a dynamical system may move between two points within a specified time interval and consistent with any constraints, the actual path followed is that for which the action integral is an extremum. $\Rightarrow$ Equations of motion.

Action integral: $J = \int_{t_1}^{t_2} dt\, L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t).$

Hamilton's principle: $\delta J = 0.$

Lagrange equations: $\dfrac{\partial L}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = 0, \quad j = 1, \ldots, n.$

Lagrange equations are Euler equations for Hamilton's extremum principle.



Generalized Forces of Constraint and Hamilton's Principle [mln17]

Lagrangian: $L(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$.

Holonomic constraint: $f(q_1, q_2, t) = 0$.

Action integral: $J(\alpha) = \displaystyle\int_{t_1}^{t_2} dt\, L(q_1, q_2, \dot{q}_1, \dot{q}_2, t), \quad q_i(t,\alpha) = q_i(t,0) + \alpha\eta_i(t)$.

$$\Rightarrow \frac{dJ}{d\alpha} = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\frac{\partial q_1}{\partial \alpha} + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\frac{\partial q_2}{\partial \alpha}\right]_{\alpha=0} = 0.$$

Constraint: $\dfrac{df}{d\alpha} = \dfrac{\partial f}{\partial q_1}\dfrac{\partial q_1}{\partial \alpha} + \dfrac{\partial f}{\partial q_2}\dfrac{\partial q_2}{\partial \alpha} = 0 \quad \Rightarrow \eta_2(t) = -\eta_1(t)\dfrac{\partial f/\partial q_1}{\partial f/\partial q_2}$.

$$\Rightarrow \frac{dJ}{d\alpha} = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\right]\eta_1(t) = 0.$$

$$\Rightarrow \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\left(\frac{\partial f}{\partial q_1}\right)^{-1} = \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\left(\frac{\partial f}{\partial q_2}\right)^{-1} = -\lambda(t).$$

This results in 3 equations for the unknown functions $q_1(t), q_2(t), \lambda(t)$:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \lambda(t)\frac{\partial f}{\partial q_i} = 0,\ i = 1, 2; \quad f(q_1, q_2, t) = 0.$$

Generalized forces of constraint: $Q_i(t) = \lambda(t)\dfrac{\partial f}{\partial q_i},\ i = 1, 2.$

Generalization to n coordinates and k constraints:

$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ with $f_j(q_1, \ldots, q_n, t) = 0,\ j = 1, \ldots, k.$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_j \lambda_j(t)\frac{\partial f_j}{\partial q_i} = 0,\ i = 1, \ldots, n,$$

$$\sum_i \frac{\partial f_j}{\partial q_i}dq_i + \frac{\partial f_j}{\partial t}dt = 0,\ j = 1, \ldots, k.$$

Applications:

- Static frictional force of constraint [mex32].
- Normal force of constraint [mex33]
- Particle sliding down sphere [mex34]
- Particle sliding inside cone: normal force of constraint [mex159]

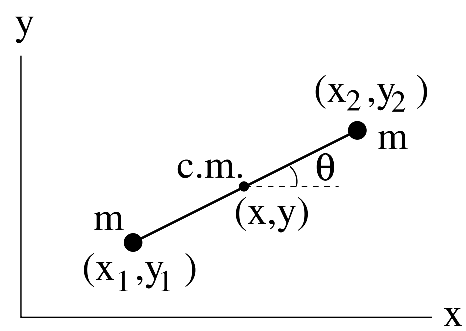
[mex160] Bead sliding down cylindrical spiral

A bead of mass m slides down (from rest and without friction) a spiral with vertical axis: $z = a\phi,\ r = R$ in cylindrical coordinates.
(a) Write the Lagrangian $L(z, r, \phi, \dot{z}, \dot{r}, \dot{\phi})$ and the two equations of holonomic constraint, $f_j(z, r, \phi) = 0, j = 1, 2$. Derive the Lagrange equations.
(b) From the three Lagrange equations and the two equations of constraint determine the three coordinates $z(t), r(t), \phi(t)$ and the two Lagrange multipliers $\lambda_j(t), j = 1, 2$.
(c) Infer the generalized force of constraint for each cylindrical coordinate.
(d) Show that the results are consistent with $\dot{J}_z = N_z$, where J_z is the angular momentum of the bead and N_z is the torque exerted by the spiral on the bead.

Solution:

[mex122] Massive dimer on skates

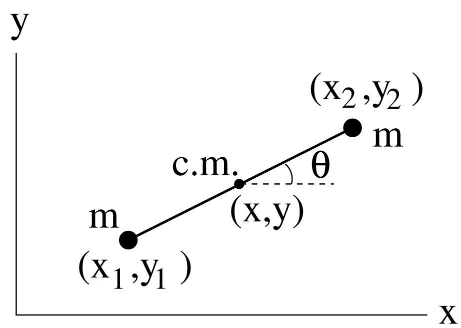
Consider two particles of mass m connected by a rigid rod of negligible mass and length ℓ which rotates freely about its center of mass. In addition, the center of mass undergoes translational motion constrained by the requirement that it must by perpendicular to the direction of the rod at all times. This system has one holonomic and one nonholonomic constraint. (a) Use the holonomic constraint to express the Lagrangian $L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$, where x, y are the center-of-mass coordinates and the θ is the angle between the rod and the x-axis. (b) Express the nonholonomic constraint as a relation between $\dot{x}, \dot{y}, \theta$. (c) Derive the equations of motion in the form of three Lagrange equations and one equation of (nonholonomic) constraint. (d) Solve these equations of motion for the initial conditions $x(0) = y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = v_0 > 0$, $\theta(0) = 0$, $\dot{\theta}(0) = \omega > 0$. (e) Determine the forces of nonholonomic constraint.



Solution:

[mex161] Massive dimer skating on incline

The massive dimer on skates described in [mex122] is now moving on an incline. The y-axis is tilted an angle α above the horizontal. (a) Write the Lagrangian $L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$, where x, y are the center-of-mass coordinates and the θ is the angle between the rod and the x-axis. (b) Express the nonholonomic constraint as a relation between $\dot{x}, \dot{y}, \theta$. (c) Derive the equations of motion in the form of three Lagrange equations and one equation of (nonholonomic) constraint. (d) Solve these equations of motion for the initial conditions $x(0) = y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = v_0 > 0$, $\theta(0) = 0$, $\dot{\theta}(0) = \omega > 0$. (e) Determine the forces of nonholonomic constraint.



Solution:

[mex162] Wave equation from Hamilton's principle

Consider a violin string of length ℓ and mass per unit length ρ under tension F. Use Hamilton's principle to derive the wave equation and determine the speed of transverse wave propagation c.

Solution: