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05. Lagrangian Mechanics I

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Abstract
Part five of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

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5. Lagrangian Mechanics I

- Pendulum with sliding pivot: reduction to quadrature [mex233]
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Challenges for Newtonian Mechanics [mln75]

Newton's second law, $\mathbf{F} = m\mathbf{a}$, relates cause and effect.

Methodological challenge: Not all causes are explicitly known prior to the solution of the problem.

Prominent among unknown causes are *forces of constraint*.

The problem is often obscured by ad-hoc ways of circumnavigation.

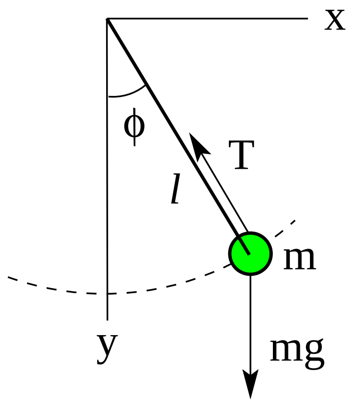
Example: Plane pendulum.

Position vector: $\mathbf{r} = (x, y)$.

Equation of motion: $m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{T}$.

Known force: $m\mathbf{g}$ (weight).

Unknown force: $\mathbf{T}$ (tension).



Different approaches to solving the plane-pendulum problem:

1. Stick to Newtonian mechanics. This is awkward. You have to deal with four equations for four unknowns. After eliminating three of the unknowns you end up with one second-order differential equation for the remaining variable. [mex132]
2. Invoke D'Alembert's principle. This is advantageous. You arrive at the same second-order differential equation more directly. [mex134]
3. Start from Lagrangian. This is even better and more elegant. You arrive at the same second-order differential equation (the Lagrange equation) yet more directly, but you still have to solve it, which is more easily said than done.
4. Handle the constraint as learned from the previous two methods and use energy conservation. This is smart. You end up with a first-order differential equation, which is almost always preferable to one of second order. [mex146] [mex147]
5. Infer the Hamiltonian from the Lagrangian and find the canonical transformation to action-angle coordinates. This is super-elegant. It gives you deep insight into mechanics, but more work is needed to get the same solution as in the previous method. [mex200]

Holonomic Constraints [mln36]

Consider a system of N particles moving in 3D space.
$3N$ Cartesian coordinates: $\mathbf{r}_i = (x_i, y_i, z_i),\ i = 1, \ldots, N.$

Equations of motion: $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(ext)} + \sum_{j \neq i} \mathbf{F}_{ji},\ i = 1, \ldots, N.$

Some of the forces may be due to constraints and are not given.

Consider k holonomic constraints: $f_j(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0,\ j = 1, \ldots, k.$
The system is then said to have $3N - k$ degrees of freedom. Its configuration space is a $(3N - k)$-dimensional manifold.

Holonomic constraints that do not depend on time are named *scleronomic* (rigid), those that do depend on time are named *rheonomic* (flowing).

Transformation to *generalized coordinates*: $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_n, t),\ n = 3N - k.$

- Plane pendulum: $N = 1,\ k = 2\ \Rightarrow\ n = 1.$
  Constraints: $z = 0,\ x^2 + y^2 = L^2.$
  Generalized coordinate ϕ: $\mathbf{r} = (L \sin\phi, L\cos\phi, 0).$
- Rigid body (made of N atoms):
  Constraints: $(\mathbf{r}_i - \mathbf{r}_j)^2 = c_{ij} = \text{const},\ i, j = 1, \ldots, N.$
  Degrees of freedom: $n = 6$ (3 translations plus 3 rotations).

Q: What is the general structure of the equations of motion for the generalized coordinates?
A: A set of n 2^{nd} order ODEs, named *Lagrange equations*, for the n generalized coordinates q_i.

Q: Do the Lagrange equations depend on the forces of constraint?
A: Not explicitly. They can be solved without knowledge of the forces of constraint.

Q: What if I wish to know the forces of constraint?
A: They can be determined either from the solution of the Lagrange equations or from a modified set of $n + k$ equations of motion that yield the time evolution of the generalized coordinates and the forces of constraint simultaneously.

Q: How are the Lagrange equations derived?
A: Lagrange's equations can be derived from Newton's equations by invoking *D'Alembert's principle*, which exploits the fact that the forces of constraint do not perform any work. Lagrange's equations can also be inferred from *Hamilton's principle*, an extremum principle whose scope is wider than that of Newtonian mechanics.

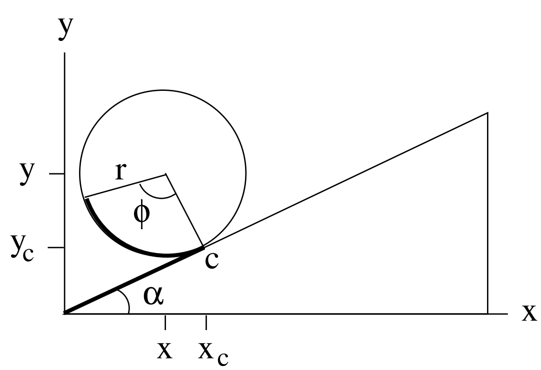
Disk Rolling Along Incline [mln76]

Rigid body in (x, y, z)-space has six degrees of freedom.

Reduction to three degrees of freedom by implicit constraints:

- Translational motion constrained to (x, y)-plane (down one).
- Rotation constrained to plane of disk (down another two).

Coordinates of free disk in plane:
center-of-mass position (x, y) and orientation (ϕ).



The requirement that the disk roll along the incline amounts to two additional constraints:

$$x = x_c - r\sin\alpha, \quad y = y_c + r\cos\alpha,$$

where $x_c = r\phi\cos\alpha$, $y_c = r\phi\sin\alpha$.

$$\Rightarrow \; x = r\phi\cos\alpha - r\sin\alpha, \quad y = r\phi\sin\alpha + r\cos\alpha.$$

The position and orientation of the rolling disk can be described by one independent variable (ϕ). The rolling disk has one degree of freedom left.

Differential form of constraint (in the context of [mln37]):

$$dx = r\cos\alpha\, d\phi, \quad dy = r\sin\alpha\, d\phi$$

.

Differential Constraints [mln37]

A general class of constraints can be expressed in differential form:

$$\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, \quad j = 1, \ldots, m.$$

n: number of generalized coordinates (Cartesian, polar, or other).
m: number of independent differential constraints.
$n - m$: number of degrees of freedom.

Integrability condition of differential constraints:

$$\frac{\partial a_{ji}}{\partial q_k} = \frac{\partial a_{jk}}{\partial q_i}, \quad \frac{\partial a_{ji}}{\partial q_t} = \frac{\partial a_{jt}}{\partial q_i}, \quad j = 1, \ldots, m, \quad i, k = 1, \ldots, n.$$

Holonomic constraints: Integrability condition is satisfied.

$$a_{ji} = \frac{\partial f_j}{\partial q_i}, \quad a_{jt} = \frac{\partial f_j}{\partial t} \;\Rightarrow\; f_j(q_1, \ldots, q_n, t) = 0, \; j = 1, \ldots, m.$$

A set of m generalized coordinates q_i can be eliminated.

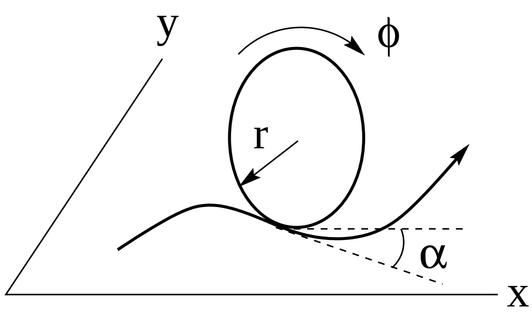
Nonholonomic constraints: Integrability condition is violated.

All n generalized coordinates are required. The kinematic effect of a nonholonomic constraint is to restrict the direction of the allowable motions at any given point in n-dimensional configuration space. This restriction does not reduce the dimensionality of the configuration space.

Example: Disk rolling upright without slipping on horizontal plane.

Generalized coordinates:
x, y (position),
ϕ, α (orientation).

Constraints:
$dx - r\cos\alpha\, d\phi = 0$,
$dy + r\sin\alpha\, d\phi = 0$.



None of these coordinates can be eliminated. It is possible to arrive at any configuration (x, y, α, ϕ) from any other configuration via a path that satisfies the two constraints.

[mex235] Heading toward moving target

A particle moves in the xy-plane under the constraint that its velocity vector is always directed toward the point $(x = f(t), y = 0)$, where $f(t)$ is differentiable. Make a sketch of the situation described here and show that this constraint is nonholonomic.

Solution:

Newtonian mechanics in the presence of holonomic constraints [mln5]

Equations of motion: $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{Z}_i, \quad i = 1, \ldots, N.$

$\mathbf{F}_i$: applied forces (known),

$\mathbf{Z}_i$: forces of constraint (unknown).

Equations of constraint: $f_j(\mathbf{r}_1, \ldots, \mathbf{r}_N) = 0,\ j = 1, \ldots, k$ (scleronomic).

The solution of this dynamical problem within the framework of Newtonian mechanics proceeds as follows:

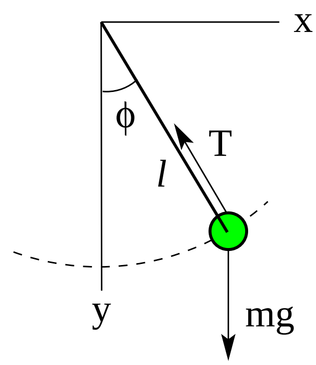
- The number of degrees of freedom is reduced to $3N - k$.
- The number of equations of motion is $3N$ with $6N$ unknowns $(x_i, y_i, z_i, Z_{ix}, Z_{iy}, Z_{iz}),\ i = 1, \ldots, N$.
- The geometrical restrictions imposed by the constraints on the orbit yield $3N$ additional relations between the unknowns. Among them are the k equations of constraint.
- A unique solution depends on $6N$ initial conditions. Because of the constraints, the number of independent initial conditions is smaller than $6N$.
- The reduction of the number of degrees of freedom from $3N$ to $3N - k$ can be taken into account by introducing $n = 3N - k$ generalized coordinates $q_1, \ldots, q_n$ such that the functions $\mathbf{r}_i(q_1, \ldots, q_n),\ i = 1, \ldots, N$ satisfy the equations of constraint.
- The number of unknowns is thus reduced to $6N - k$. The number of equations of motion stays at $3N$. The number of additional relations due to the constraints is reduced to $3N - k$.

Examples:

- Plane pendulum I [mex132]
- Heavy particle sliding inside cone I [mex133]

[mex132] Plane pendulum I

Derive the equation of motion $\ddot{\phi} + (g/\ell)\sin\phi = 0$ for the (generalized) angular coordinate ϕ within the framework of Newtonian mechanics by proceeding as follows:
(a) State the two equations of motion for the Cartesian coordinates x, y in terms of the known applied force $m\mathbf{g}$ and the unknown force of constraint (tension $\mathbf{T}$).
(b) Derive two additional relations between the four unknowns x, y, T_x, T_y geometrically from the constraint.
(c) Introduce the angular coordinate ϕ and derive from the four equations established previously one equation for ϕ.

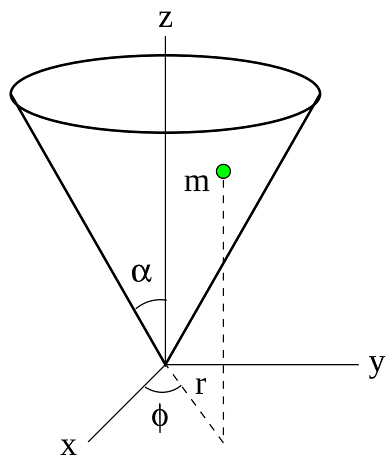


Solution:

[mex133] Heavy particle sliding inside cone I

Consider a conical surface with vertical axis (z) and apex with angle 2α at the bottom in a uniform gravitational field g. A particle of mass m is free to slide on the inside of the cone.
(a) State the three equations of motion for the Cartesian coordinates x, y, z in terms of the known applied force $m\mathbf{g}$ and the unknown force of constraint $\mathbf{Z}$ (normal force).
(b) Derive three additional relations between the six unknowns x, y, z, Z_x, Z_y, Z_z geometrically from the constraint.
(c) Introduce cylindrical coordinates r, ϕ, z and derive from the six equations established previously the following two equations of motion for the two independent generalized coordinates:

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0, \quad (\tan\alpha + \cot\alpha)\ddot{r} - r\dot{\phi}^2\tan\alpha + g = 0.$$



Solution:

D'Alembert's Principle [mln7]

Consider virtual displacements $\delta\mathbf{r}_i$:

- they are infinitesimal;
- they satisfy the equations of constraint;
- they are instantaneous ($\delta t = 0$).

For scleronomic constraints, the paths of real and virtual displacements are the same; for rheonomic constraints, they differ.

Newton's equations of motion: $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{Z}_i, \quad i = 1, \ldots, N.$

D'Alembert's principle: $\sum_{i=1}^{N} \mathbf{Z}_i \cdot \delta\mathbf{r}_i = 0.$
The forces of constraint perform zero net work.

A consequence of D'Alembert's principle is
D'Alembert's equation: $\sum_{i=1}^{N} (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta\mathbf{r}_i = 0.$
It does no longer contain the forces of constraint.

Transformation to independent (generalized) coordinates,

$$\sum_{j=1}^{3N-k} \left[\sum_{i=1}^{N} (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j = 0,$$

results in $3N - k$ equations of motion, one for each remaining degree of freedom:

$$\sum_{i=1}^{N} (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = 0, \quad j = 1, \ldots, 3N - k.$$

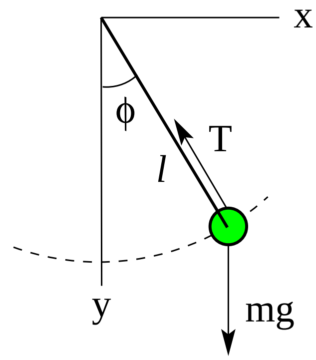
Applications:

- Plane pendulum II [mex134]
- Heavy particle sliding inside cone II [mex135]

[mex134] Plane pendulum II

Derive the equation of motion $\ddot{\phi} + (g/\ell)\sin\phi = 0$ for the (generalized) angular coordinate ϕ from D'Alembert's equation,

$$(m\ddot{\mathbf{r}} - m\mathbf{g}) \cdot \delta\mathbf{r} = 0, \quad \mathbf{r} = (x, y) = (\ell \sin\phi, \ell \cos\phi).$$

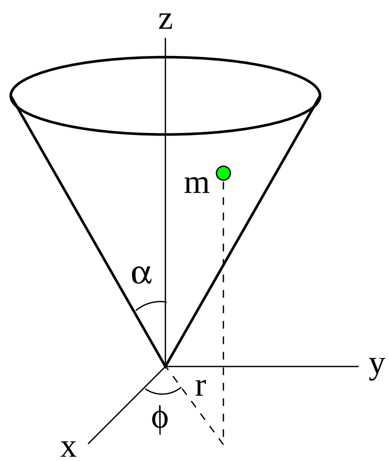


Solution:

[mex135] Heavy particle sliding inside cone II

Consider a conical surface with vertical axis (z) and apex with angle 2α at the bottom in a uniform gravitational field g. A particle of mass m is free to slide on the inside of the cone. Derive the equations of motion $2\dot{r}\dot{\phi} + r\ddot{\phi} = 0$, $(\tan\alpha + \cot\alpha)\ddot{r} - r\dot{\phi}^2\tan\alpha + g = 0$ for the two independent generalized coordinates ϕ, r from D'Alembert's equation,

$$(m\ddot{\mathbf{r}} - m\mathbf{g})\cdot\delta\mathbf{r} = 0, \quad \mathbf{r} = (x, y, z) = (r\cos\phi, r\sin\phi, r\cot\alpha).$$



Solution:

[mex146] Plane pendulum III: librations

The plane pendulum consists of a point mass m constrained by a massless rod to move in a vertical circle of radius ℓ in a uniform gravitational field g.
(a) By reduction to quadrature find the angle $\theta(t)$ of the librational motion (at energy $E < 2mg\ell$). Establish the familiar result of harmonic oscillation for very low energies ($E \ll 2mg\ell$).
(b) Find the period of oscillation T as a function of energy. Expand the exact expression $T(E)$ at low energies and derive an expression $T(\theta_0)$, where θ_0 is the amplitude, which includes the leading anharmonic correction.

Solution:

[mex147] Plane pendulum IV: separatrix motion and rotations

The plane pendulum consists of a point mass m constrained by a massless rod to move in a vertical circle of radius ℓ in a uniform gravitational field g.
(a) By reduction to quadrature find the angle $\theta(t)$ of the separatrix motion (at energy $E = 2mg\ell$).
(b) By the same method find the angle $\theta(t)$ of the rotational motion (at energy $E > 2mg\ell$). Establish the familiar result of uniform rotation at very high energies ($E \gg 2mg\ell$)
(c) Find the period of rotation T as a function of energy. Expand the exact expression $T(E)$ at high energies to include the leading correction due to gravity.

Solution:

Lagrange equations derived from D'Alembert's principle [mln8]

D'Alembert's equation: $\displaystyle\sum_{j=1}^{3N-k}\left[\sum_{i=1}^{N}(m_i\ddot{\mathbf{r}}_i-\mathbf{F}_i)\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right]\delta q_j=0$

- $\displaystyle\mathbf{F}_i(\mathbf{r}_j,\dot{\mathbf{r}}_j,t)\quad\Rightarrow\quad\sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=Q_j,\quad j=1,\ldots,3N-k.$

  $\displaystyle Q_j(q_1,\ldots,q_{3N-k},\dot{q}_1,\ldots,\dot{q}_{3N-k},t)\doteq\sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}.$

- $\displaystyle\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}\right)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial\dot{\mathbf{r}}_i}{\partial q_j}\right].$

  $\displaystyle\frac{d\mathbf{r}_i}{dt}=\sum_{l=1}^{3N-k}\frac{\partial\mathbf{r}_i}{\partial q_l}\dot{q}_l+\frac{\partial\mathbf{r}_i}{\partial t}=\dot{\mathbf{r}}_i(q_j,\dot{q}_j,t)\ \Rightarrow\ \frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j}.$

  $\displaystyle\sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_j}=\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)\right]-\frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right).$

Kinetic energy: $\displaystyle T(q_j,\dot{q}_j,t)=\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2.$

$\displaystyle\Rightarrow\ \sum_{j=1}^{3N-k}\left(\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j\right)\delta q_j=0$ with independent δq_j.

$\displaystyle\Rightarrow\ \frac{d}{dt}\frac{\partial T}{\partial\dot{q}_j}-\frac{\partial T}{\partial q_j}-Q_j=0,\quad j=1,\ldots,3N-k.$

Assumption: $\mathbf{F}_i=-\nabla_i\tilde{V},\quad\tilde{V}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=V(q_1,\ldots,q_{3N-k},t).$

$\displaystyle\Rightarrow\ Q_j=Q_j(q_1,\ldots,q_{3N-k},t)=-\frac{\partial V}{\partial q_j},\quad\frac{\partial V}{\partial\dot{q}_j}=0.$

Lagrangian: $L(q_j,\dot{q}_j,t)\doteq T(q_j,\dot{q}_j,t)-V(q_j,t).$

Lagrange equations: $\displaystyle\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_j}-\frac{\partial L}{\partial q_j}=0,\quad j=1,\ldots,3N-k.$

Generalized momenta: $\displaystyle p_j\doteq\frac{\partial L}{\partial\dot{q}_j},\quad j=1,\ldots,3N-k.$

Simple Applications of Lagrangian Mechanics [mln77]

☐ **Plane pendulum:** one degree of freedom.

Lagrangian: $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy.$

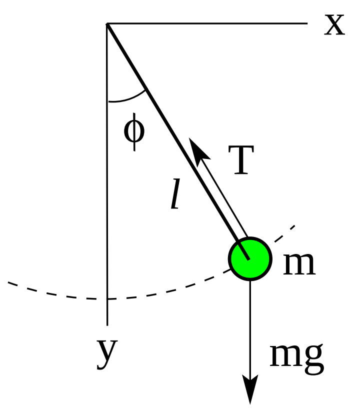
Generalized coordinate: $x = \ell \sin\phi,\ y = \ell\cos\phi.$

$\Rightarrow\ L = \frac{1}{2}m\ell^2\dot{\phi}^2 + mg\ell\cos\phi.$

Lagrange equation: $\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0.$

$\frac{\partial L}{\partial\phi} = -mgl\sin\phi, \quad \frac{\partial L}{\partial\dot{\phi}} = m\ell^2\dot{\phi}, \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = m\ell^2\ddot{\phi}.$

$\Rightarrow\ \ddot{\phi} + \frac{g}{\ell}\sin\phi = 0.$



☐ **Particle sliding inside cone:** two degrees of freedom.

Lagrangian: $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz.$

Generalized coordinates: $x = r\cos\phi,\ y = r\sin\phi,\ z = r\cot\alpha.$

$\Rightarrow\ L = \frac{1}{2}m\left[\dot{r}^2\left(1 + \cot^2\alpha\right) + r^2\dot{\phi}^2\right] - mgr\cot\alpha.$

Lagrange equations:

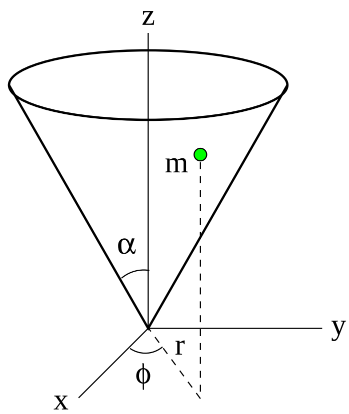
$\frac{\partial L}{\partial\phi} = 0, \quad \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi}, \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi}.$

$\Rightarrow\ 2\dot{r}\dot{\phi} + r\ddot{\phi} = 0.$

$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - mg\cot\alpha, \quad \frac{\partial L}{\partial\dot{r}} = m\dot{r}\left(1 + \cot^2\alpha\right),$

$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}} = m\ddot{r}\left(1 + \cot^2\alpha\right).$

$\Rightarrow\ \ddot{r}\left(\tan\alpha + \cot\alpha\right) - r\dot{\phi}^2\tan\alpha + g = 0.$



[mex79] Invariance under point transformations of Lagrange equations

Consider the point transformation $q_i = q_i(Q_1, \ldots, Q_n, t), i = 1, \ldots, n$ between two sets of generalized coordinates and the two functions $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \tilde{L}(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t)$. Show by substitution of coordinates that if the q_i satisfy the Lagrange equations (1) then the Q_i satisfy the Lagrange equations (2):

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, \ldots, n, \quad (1) \qquad \frac{\partial \tilde{L}}{\partial Q_i} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{Q}_i} = 0, \quad i = 1, \ldots, n. \quad (2)$$

Solution:

[mex21] Gauge invariance of Lagrange equations

A dynamical system with n degrees of freedom is specified by some Lagrangian $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. Show by direct substitution that the Lagrangian

$$L' = L + \frac{d}{dt}F(q_1, \ldots, q_n, t)$$

yields the same Lagrange equations if F is an arbitrary differentiable function of its arguments.

Solution:

[mex22] Find a simpler Lagrangian

Consider a dynamical system with one degree of freedom specified by the Lagrangian

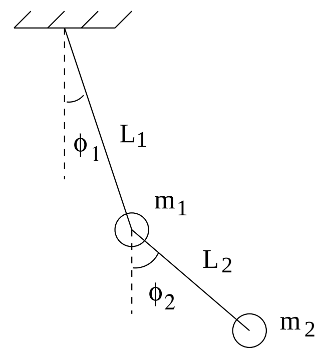
$$L = \frac{1}{12}m^2\dot{x}^4 + m\dot{x}^2W(x) - W^2(x),$$

where m is the mass of a particle that can move along the x-axis and $W(x)$ is a differentiable function. Determine the equation of motion for the dynamical variable x and explain the role of the function $W(x)$. Find a simpler Lagrangian that leads to the same equation of motion.

Solution:

[mex20] Lagrangian of plane double pendulum

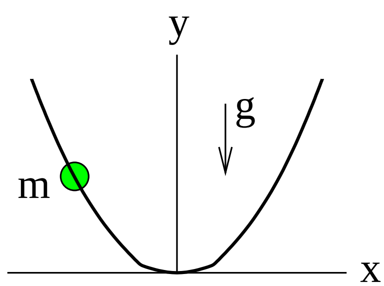
Consider a plane pendulum consisting of two point masses m_1 and m_2 and two rods of negligible mass and lengths L_1 and L_2, respectively. Determine the Lagrangian of this dynamical system as a function of the generalized coordinates ϕ_1, ϕ_2 and the associated generalized velocities $\dot{\phi}_1, \dot{\phi}_2$.



Solution:

[mex131] Parabolic slide

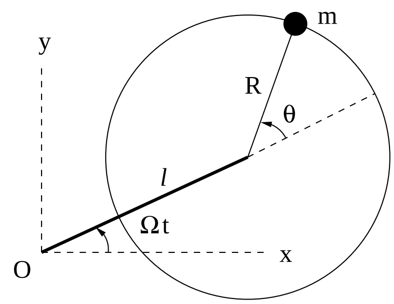
A bead of mass m slides without friction along a wire of parabolic shape, $y = Ax^2$, in a uniform gravitational field g pointing in the negative y-direction.
(a) Construct the Lagrangian $L(x, \dot{x})$.
(b) Derive the Lagrange equation.
(c) Linearize the Lagrange equation and determine a harmonic small-amplitude solution.



Solution:

[mex25] Pendulum without gravity

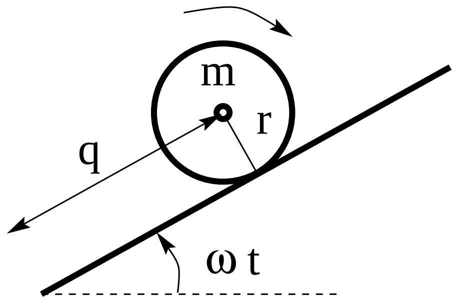
A bead of mass m is free to slide along a circular wire of radius R in a horizontal plane. The wire is forced to rotate with constant angular velocity Ω about a vertical axis at O, separated a distance ℓ from the axis of the circle. Determine the Lagrangian $L(\theta, \dot{\theta})$ of this dynamical system. Show that the Lagrange equation for θ describes the motion of a plane pendulum with angular frequency that depends on ℓ, R, Ω.



Solution:

[mex116] Disk rolling on rotating track

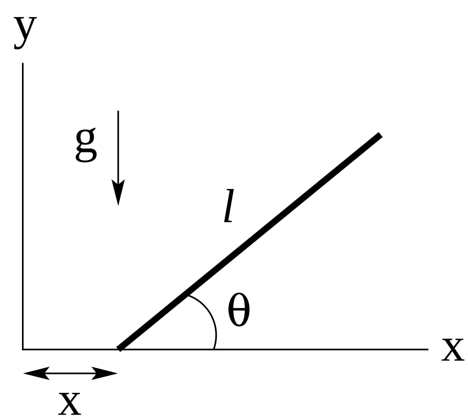
A disk of radius r and mass m rolls without slipping on a track of negligible mass which rotates with constant angular velocity ω.
(a) Find the kinetic energy $T(q,\dot{q})$.
(b) Find the solution $q(t)$ for initial conditions $q(0) = q_0,\ \dot{q}(0) = 0$.



Solution:

[mex115] Rotating and sliding

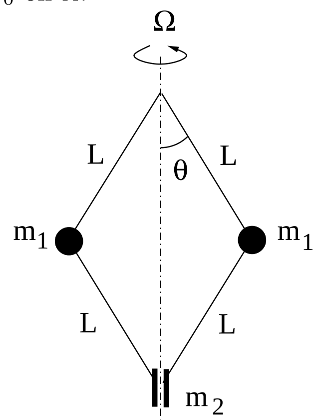
A rod of mass m and length ℓ moves in a vertical plane with one end constrained to slide along the x-axis.
(a) Find the Lagrangian $L(x, \theta, \dot{x}, \dot{\theta})$.
(b) Find the conserved generalized momentum β_x associated with the cyclic coordinate x.
(c) Find the Routhian function $R(\theta, \dot{\theta}, \beta_x)$.



Solution:

[mex23] Pendulum under forced rotation

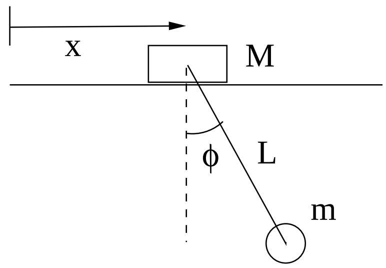
Consider a pendulum consisting of two masses m_1 and one mass m_2 connected by four rods of negligible mass and length L. Mass m_2 is constrained to move along the vertical axis. The masses m_1 are forced to rotate with constant angular velocity Ω about the vertical axis.
(a) Determine the Lagrangian $L(\theta, \dot{\theta})$ and derive the Lagrange equation for the variable θ.
(b) Determine the condition for the existence of a stable rotating mode with nonzero $\theta = \theta_0 = const$ and determine the dependence of θ_0 on Ω.



Solution:

[mex24] Pendulum with sliding pivot: Lagrange equations

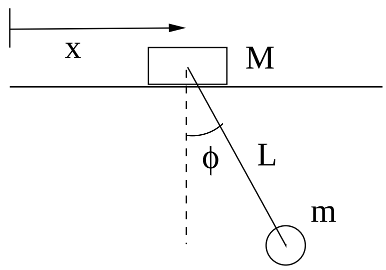
A block of mass M is free to slide horizontally on an airtrack with negligible friction. Suspended from the block by a rod of negligible mass and length L is a mass m swinging in a vertical plane.
(a) Determine the Lagrangian $L(x, \phi, \dot{x}, \dot{\phi})$ and derive the Lagrange equations for x and ϕ.
(b) Determine the angular frequency ω_0 of small-amplitude oscillations of this system.



Solution:

[mex233] Pendulum with sliding pivot: reduction to quadrature

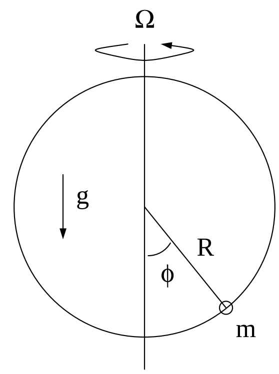
A block of mass M is free to slide horizontally on an airtrack with negligible friction. Suspended from the block by a rod of negligible mass and length L is a mass m swinging in a vertical plane. Reduce the solution $\phi(t), x(t)$ to quadrature for the case $\dot{x}(0) = \dot{\phi}(0) = 0$, $\phi(0) > 0$.



Solution:

[mex39] Pendulum oscillations in rotating plane

A particle of mass m is constrained (without friction) to move on a circular path of radius R which rotates about its vertical diameter with constant angular velocity Ω of the particle. (a) Determine the stable equilibrium position $\phi_0(\Omega, g, R)$. (b) Determine the angular frequency $\omega(\Omega, g, R)$ of small oscillations of the particle about the stable equilibrium position.



Solution:

[mex148] Chain sliding off the edge of table without friction

A uniform chain of total length A has a portion B ($0 < B < A$) hanging over the edge of a table with a smooth (frictionless) surface. Show that the time it takes the chain to slide off the table if it starts from rest is

$$T = \sqrt{\frac{A}{g}} \ln\left(\frac{A}{B} + \sqrt{\frac{A^2}{B^2} - 1}\right).$$

Solution:

[mex149] Chain sliding off the edge of table with friction

A uniform chain of total length A has a portion B $(0 < B < A)$ hanging over the edge of a table with a rough surface. The coefficient of kinetic friction is μ. Show that the time it takes the chain to slide off the table if it starts from rest is

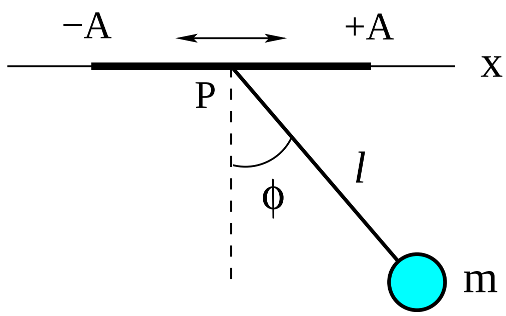
$$T = \sqrt{\frac{A}{g(1+\mu)}} \ln\left(\frac{A + \sqrt{A^2 - [B(1+\mu) - A\mu]^2}}{B(1+\mu) - A\mu}\right).$$

Solution:

[mex248] Plane pendulum with periodically driven pivot I

Consider a mathematical pendulum (mass m, length ℓ) with the pivot P oscillating horizontally, $x_P = A\cos\omega t$. Show that the Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + mA\omega^2\ell\cos\omega t\sin\phi + mg\ell\cos\phi.$$

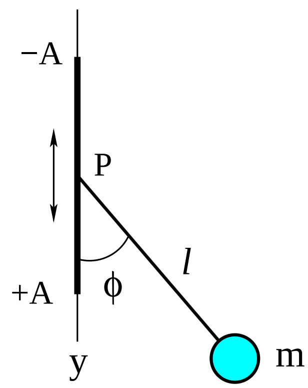


Solution:

[mex249] Plane pendulum with periodically driven pivot II

Consider a mathematical pendulum (mass m, length ℓ) with the pivot P oscillating vertically, $y_P = A\cos\omega t$. Show that the Lagrangian is

$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + mA\omega^2\ell\cos\omega t\cos\phi + mg\ell\cos\phi.$$

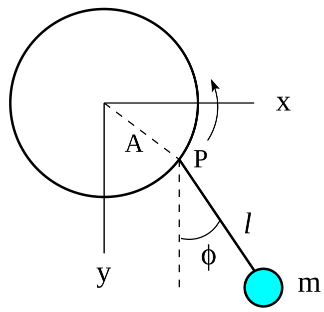


Solution:

[mex250] Plane pendulum with periodically driven pivot III

Consider a mathematical pendulum (mass m, length ℓ) with the pivot P rotating counterclockwise along a circle in a vertical plane, $x_P = A\cos\omega t$, $y_P = -A\sin\omega t$. Show that the Lagrangian is

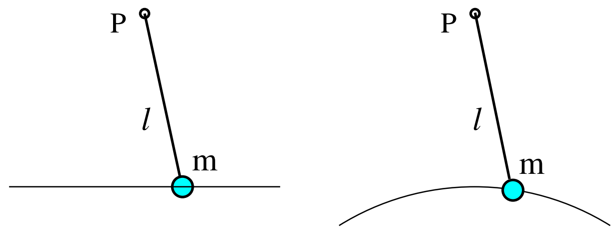
$$L = \frac{1}{2}m\ell^2\dot{\phi}^2 + mA\omega^2\ell\sin\big(\phi - \omega t\big) + mg\ell\cos\phi.$$



Solution:

[mex251] Restoring force of elastic string

An elastic string of stiffness k and negligible mass has length ℓ_0 when relaxed. One end of the string is fixed to the fixed pivot P and the other end to a block of mass m that can slide without friction
(a) along a straight line as shown on the left,
(b) along a circular line of radius r as shown on the right.
In the rest position of the block the string is stretched to length $\ell = 3\ell_0/2$. Find the angular frequency ω of small-amplitude oscillations of the block about its rest position.



Solution: