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04. Fixed Points and Limit Cycles

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Abstract
Part four of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc4]

4. Fixed Points and Limit Cycles

Classification of Fixed Points in Plane

Equation of motion: $m\ddot{x} = F(x, \dot{x}) \quad \Rightarrow \quad \dot{x} = y, \ \dot{y} = F(x, y)/m$.

Velocity vector field: $\mathbf{v}(\mathbf{r}) = \mathbf{v}(x, y) = (\dot{x}, \dot{y}) = (v_x, v_y)$.

Fixed point: $\mathbf{v}(\mathbf{r}_k) = 0 \quad \Rightarrow \quad (\dot{x}, \dot{y}) = 0$ at $(x, y) = (x_k, y_k)$.

Linearized velocity field around fixed point $\mathbf{r}_k$:

$$\mathbf{v} = \mathbf{A} \cdot (\mathbf{r} - \mathbf{r}_k) + \mathrm{O}(\mathbf{r} - \mathbf{r}_k)^2$$

with Jacobian matrix

$$\mathbf{A} = \begin{pmatrix} \partial v_x/\partial x & \partial v_x/\partial y \\ \partial v_y/\partial x & \partial v_y/\partial y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Nature of fixed point depends on eigenvalues of $\mathbf{A}$:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \Rightarrow \quad \lambda^2 - \tau \lambda + \delta = 0,$$

where $\delta = ad - bc$ is the *determinant* and $\tau = a + d$ the *trace*.

Solution:

$$\lambda = \frac{\tau}{2} \pm \sqrt{\frac{\tau^2}{4} - \delta}.$$

Three types of fixed points:

☐ Type 1: $\tau^2 > 4\delta \quad \Rightarrow \quad \lambda_1 \neq \lambda_2$, real:
- $\delta > 0 \quad \Rightarrow \quad$ *node* (attractor if $\tau < 0$, repellor if $\tau > 0$),
- $\delta < 0 \quad \Rightarrow \quad$ *hyperbolic point*.

☐ Type 2: $\tau^2 < 4\delta \quad \Rightarrow \quad \lambda_1 = \lambda_2^*$, complex conjugate:
- $\Re\{\lambda\} \neq 0 \quad \Rightarrow \quad$ *spiral* (attractor if $\tau < 0$, repellor if $\tau > 0$),
- $\Re\{\lambda\} = 0 \quad \Rightarrow \quad$ *elliptic point*.

☐ Type 3: $\tau^2 = 4\delta \quad \Rightarrow \quad \lambda_1 = \lambda_2$, real:
- $b = c = 0 \quad \Rightarrow$ *star* (attractor if $\tau < 0$, repellor if $\tau > 0$),
- $b \neq 0$ or $c \neq 0 \quad \Rightarrow$ *improper node* (attr. if $\tau < 0$, rep. if $\tau > 0$).

Conservative force implies area-preserving flow.
Consequence: $\tau = 0 \quad \Rightarrow \quad$ no repellors or attractors.
Only elliptic or hyperbolic fixed points are realized.

Isoclines [mln31]

Isoclines are a simple device used to identify, in conjunction with the analysis of fixed points, all salient features in the 2D phase flow of a given pair of 1$^{\text{st}}$ order ODEs:

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2).$$

Isoclines are sets of curves on which all phase-plane trajectories have tangents with one and the same direction.

Two special directions are commonly singled out:

1. Isoclines intersected *vertically* by all trajectories are determined by the curves representing the equation $f_1(x_1, x_2) = 0$.

2. Isoclines intersected *horizontally* by all trajectories are determined by the curves representing the equation $f_2(x_1, x_2) = 0$.

All points of intersection between a curve of isocline 1 and a curve of isocline 2 are fixed points of the phase flow.

Alternative isoclines, representing locations in the phase plane where all trajectories have slope ± 1 are determined by the solutions of the equations $f_1(x_1, x_2) = \pm f_2(x_1, x_2)$.
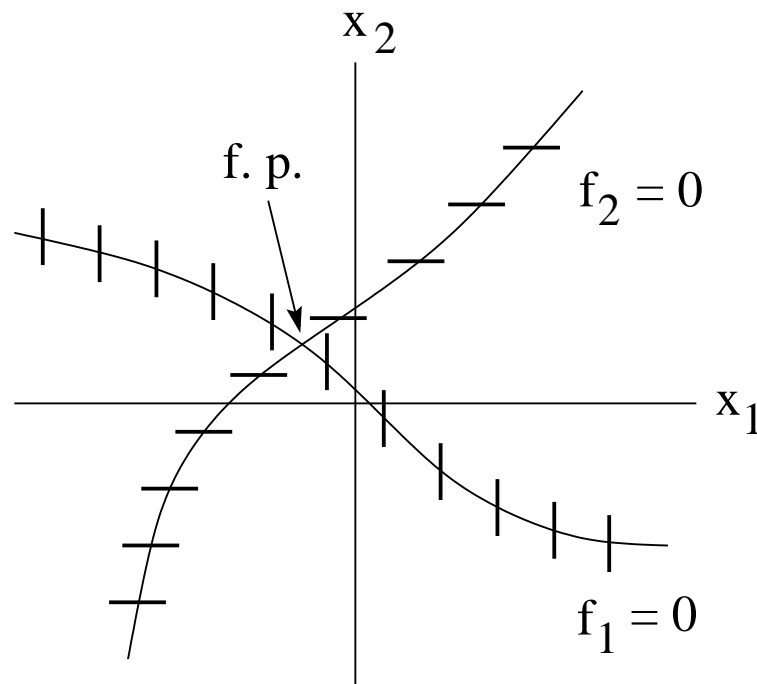
[mex12] Fixed points of the plane pendulum

Consider the equation of motion
$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2 \sin\theta = 0,$$
where $\omega_0 = \sqrt{g/L}$ is the characteristic frequency and β is the damping parameter. Determine the nature of the two fixed points (a) for zero damping ($\beta = 0$) and (b) for weak damping ($\beta < \omega_0$).

Solution:

[mex7] 2D Phase Portrait I

Consider the dynamical system characterized by the following equation of motion:

$$\ddot{x} - \dot{x}^2 + x^2 - x = 0.$$

(a) Identify all fixed points in the plane $(x, \dot{x})$ and determine the type of each fixed point.
(b) Identify the lines of vertical and horizontal isoclines.
(c) Sketch the phase portrait of this dynamical system including fixed points and isoclines.

Solution:

[mex8] 2D Phase Portrait II

Consider the dynamical system characterized by the following equation of motion:

$$\ddot{x} - \dot{x} + x^2 - 2x = 0.$$

(a) Identify all fixed points in the plane $(x, \dot{x})$ and determine the type of each fixed point.
(b) Sketch the phase portrait of this dynamical system.

Solution:

[mex13] Predator and prey

A population F of foxes feeds on a population H of hares. The birth rate of foxes is proportional to the fox population and to the amount of food available. Foxes die at a rate proportional to the fox population. Hares die primarily through encounters with foxes and are born at a rate proportional to the hare population:

$$\dot{H} = aH - bHF, \qquad \dot{F} = cHF - dF,$$

where a, b, c, d are positive constants and $H \geq 0, F \geq 0$ is assumed.

(a) Find all fixed points in the (H, F)-plane and determine their nature. Sketch the phase portrait and give an interpretation of the phase flow.

(b) If the population of hares is suddenly decimated by an epidemic disease from which the remaining hares are immune, discuss the different effects this can have on the system depending on the size of the fox population at the time the hare population is reduced by the disease.

Solution:

[mex14] Host and parasite

The populations of a host $H(t)$ and a parasite $P(t)$ are described by the following equations:

$$\dot{H} = (1 - P)H, \qquad \dot{P} = P\left(1 - \frac{2P}{1 + H}\right).$$

Find the three fixed points for $H \geq 0, P \geq 0$ and determine their nature. Sketch the phase diagram and discuss the phase flow.

Solution:

[mex108] Isoclines and fixed points

Two species of animals vie for the same food source. In isolation each species grows logistically. Through interaction they impede each other's growth. The equations of motion for the two populations N_1, N_2,

$$\dot{N}_1 = rN_1\left(1 - \frac{N_1}{K}\right) - \alpha N_1 N_2, \qquad \dot{N}_2 = sN_2\left(1 - \frac{N_2}{L}\right) - \beta N_1 N_2,$$

depend on six paramters: r, s are the per-capita growth rates, K, L the carrying capacities, for the two populations N_1, N_2, respectively, and α, β are the adverse impact parameters of the competitor population. Consider the two cases (i) $K = L = 2, r = s = 1, \alpha = \beta = 1$, and (ii) $K = L = 1, r = s = 2, \alpha = \beta = 1$.

For each case determine all curves of vertical and horizontal isoclines. Draw all these lines in a diagram N_1 versus N_2 for each case. Indicate the location of all four fixed points in each case as the intersection points between curves belonging to the vertical and horizontal isloclines. Determine the coordinates of all fixed points in the (N_1, N_2)-plane.

Solution:

[mex109] Fierce competition versus mild competition

Two species of animals vie for the same food source. In isolation each species grows logistically. Through interaction they impede each other's growth. The equations of motion for the two populations N_1, N_2,

$$\dot{N}_1 = rN_1\left(1 - \frac{N_1}{K}\right) - \alpha N_1 N_2, \qquad \dot{N}_2 = sN_2\left(1 - \frac{N_2}{L}\right) - \beta N_1 N_2,$$

depend on six paramters: r, s are the per-capita growth rates, K, L the carrying capacities for the two populations N_1, N_2, respectively, and α, β are the adverse impact parameters of the competitor population. Consider the two cases (i) $K = L = 2, r = s = 1, \alpha = \beta = 1$, and (ii) $K = L = 1, r = s = 2, \alpha = \beta = 1$.

Determine the nature of all four fixed points in each case as previously identified in [mex108]. What conclusions can be drawn from these results about the nature of the competition between the two species?

Solution:

Limit Cycles [mln74]

Not all attractors in 2D phase flow are fixed points.
There exists exactly one other type of attractor: the limit cycle.

Example: Flow in (x, y)-plane with circular limit cycle.

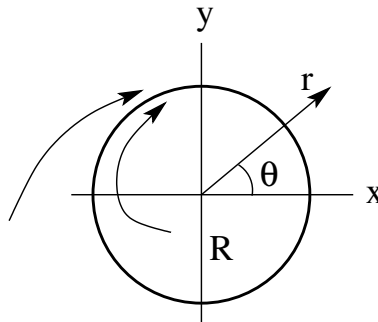Equations of motion in polar coordinates (r, θ):

$$\dot{r} = -\alpha r(r - R), \qquad \dot{\theta} = \omega = \text{const.}$$

Periodic trajectory: $r(t) = R = \text{const.}, \quad \theta(t) = \theta_0 + \omega t.$

Linearized radial equation of motion for $|r(t) - R| \doteq s \ll R$:

$$\dot{s} = -\alpha R s \quad \Rightarrow \quad s(t) = s_0 e^{-\alpha R t}.$$

$\Rightarrow$ Periodic trajectory is an attractor (limit cycle).

Note presence of fixed point at $r = 0$:

Cartesian coordinates: $x = r\cos\theta, \quad y = r\sin\theta.$

Linear anlaysis of equations of motion predict spiral repellor:

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = -\alpha\left(\sqrt{x^2 + y^2} - R\right)x - \omega y,$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -\alpha\left(\sqrt{x^2 + y^2} - R\right)y + \omega x,$$

$$\Rightarrow \mathbf{A} = \begin{pmatrix} \alpha R & -\omega \\ +\omega & \alpha R \end{pmatrix} \quad \Rightarrow \quad \lambda = \alpha R \pm i\omega.$$

[mex19] Hopf bifurcation

A simple Hopf bifurcation generates a limit cycle from a point attractor upon variation of some parameter in the equations of motion of a dynamical system. Consider the dynamical system specified (in polar coordinates) by

$$\dot{r} = -\Gamma r - r^3, \qquad \dot{\theta} = \omega,$$

where Γ and ω are constants.

(a) Find the exact solution $r(t), \theta(t)$ for initial conditions $r(0) = r_0, \theta(0) = 0$.

(b) Identify the circular periodic trajectory for $\Gamma < 0$, which plays the role of a limit cycle, and determine its radius.

(c) Determine the nature of the fixed point at $r = 0$ for $\Gamma > 0$ and $\Gamma < 0$.

(d) Produce a Mathematica Plot with three trajectories in the (x, y)-plane to illustrate the emergence of a limit cycle from a stable fixed point. The first trajectory is for $\Gamma > 0$. It will spiral into the point attractor at the origin. The second and third attractor are for $\Gamma < 1$ with iitial conditions inside and outside the limit cycle, respectively. Fine-tune the parameters and initial conditions to make the message of your graph compelling.

(e) Choose several values of t_{max} for fixed values of r_0, ω, Γ. Then plot $r(t_{max})$ versus Γ to illustrate the emergence of a bifurcation singularity in the limit $t_{max} \to \infty$. Again fine-tune your parameter values to optimize your graph for didactic effect.

Solution:

Feedback Control [mln33]

Consider the phase diagram of the plane pendulum as given in [msl8]. The upright rest position is an unstable equilibrium (hyperbolic fixed point).

Feedback control: Introduce a lateral motion of the pivot which is coupled to the instantaneous angular position and angular velocity in such a way that the upright rest position becomes a stable fixed point.

Displacement of pendulum bob along arc: $s = L\phi$.

Equation of motion: $m\ddot{s} = mg \sin \phi - m\ddot{w} \cos \phi$.

Horizontal displacement of pivot: $w(t)$.
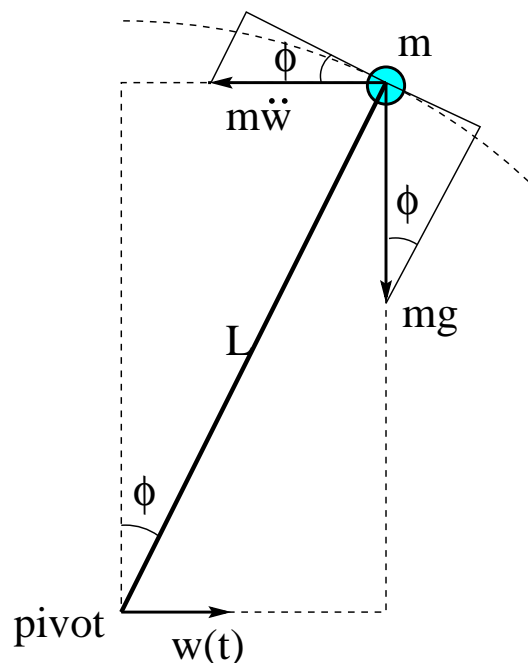
Change of variables: $x_1 = \phi, x_2 = \dot{\phi}$.

Design of feedback: $\ddot{w} = c_1 x_1 + c_2 x_2$, where c_1, c_2 are controllable parameters.

Equation of motion with feedback:

$$\dot{x}_1 = x_2, \tag{1a}$$

$$\dot{x}_2 = \frac{c_1 x_1}{L} \cos x_1 + \frac{c_2 x_2}{L} \cos x_1 + \frac{g}{L} \sin x_1. \tag{1b}$$

Goal: Find the conditions for the control parameters c_1, c_2 which make the state $\phi = \dot{\phi} = 0$, i.e. $(x_1, x_2) = (0, 0)$ a stable equilibrium.

[mex110] Balancing a heavy object on a light rod

The equations of motion

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{c_1 x_1}{L} \cos x_1 + \frac{c_2 x_2}{L} \cos x_1 + \frac{g}{L} \sin x_1$$

represent a point mass m being balanced on a rod of length L and negligible mass through lateral movement of the pivot. Here the variable x_1 represents the angle ϕ from the upright equilibrium position and the variable x_2 the associated angular velocity $\dot{\phi}$ as explained in [mln33]. (a) Analyze the nature of the fixed point at $(x_1, x_2) = (0, 0)$ for the case with zero feedback ($c_1 = c_2 = 0$). (b) Determine the conditions for the control parameters c_1, c_2 under which the the fixed point at $(x_1, x_2) = (0, 0)$ is an attractor, i.e. for which it is asymptotically stable. (c) Solve the coupled differential equations (1) by using the NDSolve option of Mathematica. Produce plots x_1 versus x_2 for trajectories that describe (i) perfect balance established and maintained, (ii) slowly lost balance, (iii) quickly lost balance, (iv) imperfect balance maintained. Discuss the relevant parameter settings for each case.

Solution:

Logistic Model (continuous version)

The (continuous) logistic model was introduced in population dynamics:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$
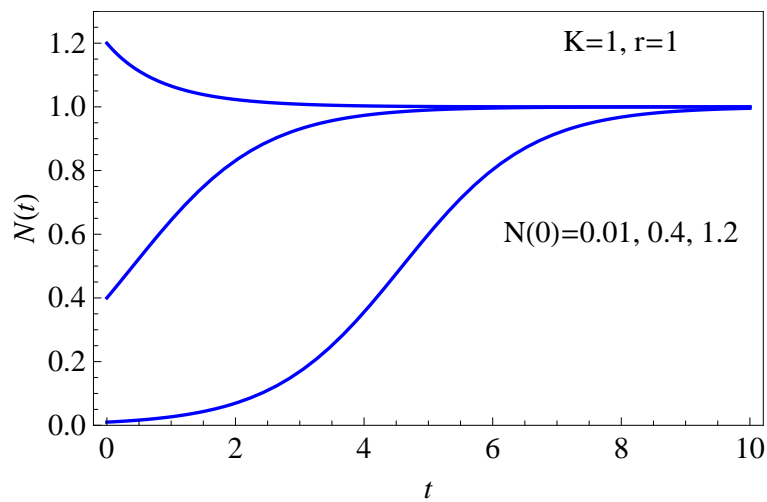
The model has one variable and two parameters:

$N(t)$: instantaneous size of population,

r: per-capita growth rate,

K: carrying capacity due to limited living space and resources.

The general solution for can be obtained by separation of variables [mex107]:

$$N(t) = \frac{N(0)e^{rt}}{1 + \frac{N(0)}{K}\left(e^{rt} - 1\right)}.$$

In the limit $K \to \infty$, the solution approaches unimpeded exponential growth: $N(t) = N(0)e^{rt}$.

A discrete version of the logistic model exhibits more complex behavior.

[mex107] Continuous logistic model

Consider the continuous logistic model in population dynamics,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where the variable $N(t)$ represents the instantaneous size of some population, the parameter r is the per-capita growth rate and the parameter K the carrying capacity due to limited living space and resources.

(a) Find the solution for initial condition $N(0)$.

(b) Find the value of N (for given $N(0), r, K$) at which the population grows most rapidly.

Solution:

Dynamical Systems with 1 Degree of Freedom

Newton's equation of motion: $m\ddot{x} = F(x, \dot{x}) \Leftrightarrow \dot{x} = y, \ \dot{y} = F(x, y)/m$.

Velocity vector field in 2D phase space: $(\dot{x}, \dot{y})$.

Solution $(x(t), y(t))$ describes *trajectory* in 2D phase space.
All trajectories are tangential to velocity vector field.
Trajectories do not intersect each other or themselves.

Orbits are projections of trajectories onto the x-axis.

Fixed points in phase space have zero phase velocity: $(\dot{x}, \dot{y}) = (0, 0)$.

Conservative system: $F = F(x) = -\dfrac{dV}{dx}, \quad V(x) = -\displaystyle\int_{x_0}^{x} dx\, F(x)$.

Integral of the motion: $E(x, \dot{x}) = \dfrac{1}{2}m\dot{x}^2 + V(x) = \text{const}$.

In conservative systems, trajectories are confined to lines of constant energy.

Separatrix: line of constant energy corresponding to local maximum of $V(x)$.

In conservative systems, there are two types of fixed points:
elliptic fixed points at energies where $V(x)$ has a local minimum.
hyperbolic fixed points at energies where $V(x)$ has a local maximum.

In dissipative systems, there are additional types of fixed points:
attractors and *repellors*.

Not all attractors are fixed points:
Spirals, stars, and *nodes* are 0D attractors in 2D phase space.
Limit cycles are 1D attractors in 2D phase space.