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03. Simple Dynamical Systems

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Abstract

Part three of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc3]

One Degree of Freedom [mln71]

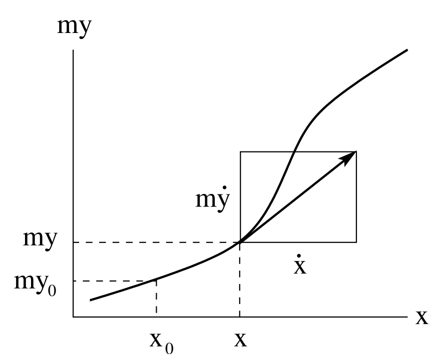
Newton's equation of motion (for autonomous system):

$$m\ddot{x} = F(x, \dot{x}). \tag{1}$$

- Conservative system: $F = F(x) \quad \Rightarrow$ solution by quadrature [mln4].
- Non-autononmous system: $F = F(x, \dot{x}, t) \Rightarrow$ transformation to autonomous system with two degrees of freedom.

Equivalent (phase-plane) representation of (1):

$$\dot{x} = y, \qquad m\dot{y} = F(x, y). \tag{2}$$



- Equations (2) determine a vector field in phase plane (x, my).
- Any solution of (2) for given initial condition (x_0, my_0) describes a trajectory in phase plane.
- Trajectories are tangential to vector field $(\dot{x}, m\dot{y})$.
- Phase plane is filled with trajectories (phase flow).
- Trajectories do not intersect (Cauchy's existence theorem).
- At fixed points $(\dot{x} = 0, m\dot{y} = 0)$ system is at rest.
- In conservative systems, all trajectories lie on lines of constant energy, $E(x, y) =$ const.
- Phase portrait describes salient features of phase flow [msl5].
- Closed trajectories describe periodic motion.
- In conservative systems the phase flow is incompressible.

Solution by Quadrature [mln 4]

Any conservative, autonomous system with one degree of freedom is solvable by quadrature.

Equation of motion: $m\ddot{x} = F(x)$ (2^{nd} order ODE).

Solution by quadrature is a three-step process:

- Identify the first integral (conserved energy):
  Define $V(x) = -\int_{x_0}^{x} dx F(x), \quad F(x) = -\dfrac{dV}{dx}$;
  write $m\dot{x}\ddot{x} - \dot{x}F(x) = \dfrac{d}{dt}\left[\dfrac{1}{2}m\dot{x}^2 + V(x)\right] = 0$;
  $\Rightarrow \; E = T + V = \dfrac{1}{2}m\dot{x}^2 + V(x) = \text{const.}$

- The first integral reduces the equation of motion to a 1^{st} order ODE:
  $\dfrac{dx}{dt} = \sqrt{2\left[E - V(x)\right]/m} \;\Rightarrow\; \int_0^t dt = \int_{x_0}^{x} \dfrac{dx}{\sqrt{2\left[E - V(x)\right]/m}}$.

- Invert the resulting function $t(x)$ to obtain the solution $x(t)$.

Application: harmonic oscillator

- $m\ddot{x} = -kx \;\Rightarrow\; m\dot{x}\ddot{x} + m\omega_0^2\dot{x}x = \dfrac{d}{dt}\left[\dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}m\omega_0^2x^2\right] = 0, \quad \omega_0^2 = \dfrac{k}{m}$.
  $\Rightarrow \; E = \dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}m\omega_0^2x^2 = \text{const.}$

- $t = \displaystyle\int_{x_0}^{x} \frac{dx}{\sqrt{2E/m - \omega_0^2x^2}} = \frac{1}{\omega_0}\arcsin\left(\frac{x\omega_0}{\sqrt{2E/m}}\right) + \text{const.}$

- $x(t) = \sqrt{\dfrac{2E}{m\omega_0^2}}\sin(\omega_0 t + \text{const}).$

[mex5] Periodic motion in quadratic and quartic potentials

Use the expression

$$T = 2\int_{x_{min}}^{x_{max}} \frac{dx}{\sqrt{2[E - V(x)]/m}}$$

to calculate the dependence on the amplitude x_{max} of the period T for the motion of a particle with mass m moving
(a) in the quadratic potential $V_2(x) = \frac{1}{2}m\omega_0^2 x^2$,
(b) in the quartic potential $V_4(x) = \frac{1}{4}\alpha x^4$.

Solution:

[mex232] Potential energy of periodic motion reconstructed

Consider a particle of mass m undergoing oscillatory motion at energy E in a symmetric potential $V(x)$ with $V(0) = 0$. Show that for given period $T(E)$ of the motion the shape of the potential energy can be reconstructed from the expression

$$x(V) = \frac{1}{2\pi\sqrt{2m}} \int_0^V dE \frac{T(E)}{\sqrt{V-E}}.$$

Note the inverse relationship of this result to that of [mex5]. Apply the above expression to the case where $T = 2\pi/\omega_0$ independent of E.

Solution:

[mex6] Periodic motion in 2D phase space

Let $S(E)$ be the area enclosed by the trajectory corresponding to a periodic motion with energy E in 2D phase space $(x, \dot{x})$.
(a) Show that the period of the motion along this trajectory is

$$T = m\frac{dS}{dE}.$$

(b) Use this relation to calculate the period T of a particle with mass m moving in the quadratic potential $V_2(x) = \frac{1}{2}m\omega_0^2 x^2$ and for a particle of mass m moving in the linear potential $V_1(x) = a|x|$.

Solution:

[mex111] Separatrix tangent lines at hyperbolic point

Consider a particle of mass m moving along the x-axis under the influence of a conservative force described by a potential energy function $V(x)$ which has a smooth maximum at $x = \bar{x}$ with curvature $|V''(\bar{x})| = k$.
(a) Find the slope of the tangent lines to the separatrix at the resulting hyperbolic fixed point $(\bar{x}, 0)$ in the $(x, \dot{x})$-plane.
(b) Calulate the time it takes the particle to move between two points x_1 and x_2 very close to the hyperbolic point $(x = \bar{x}, \dot{x} = 0)$ on the separatrix.

Solution:

Solution by Separation of Variables [mln72]

Velocity-dependent attenuation

Equation of motion: $m\ddot{x} = F$ with $F = f(\dot{x})g(t)$.

$$\Rightarrow\ m\frac{dv}{dt} = f(v)g(t) \quad \Rightarrow\ m\int_{v_0}^{v}\frac{dv}{f(v)} = \int_0^t dt\, g(t).$$

$$\text{Solve for } v(t). \quad \text{Then calculate } x(t) = \int_0^t dt\, v(t).$$

Applications: [mex15], [mex16], [mex230].

Rocket motion

Instantaneous momentum of rocket: $p_R(t) = m(t)v(t)$.

Momentum increment of exhaust gases: $\Delta p_E(t) = -[u - v(t)](-\Delta m)$.

Speed of exhaust gases relative to rocket: u.

Equation of motion: $\dot{p}_R + \dot{p}_E = F_{\text{ext}}$.

$$\Rightarrow\ m\dot{v} + \dot{m}v - (u - v)(-\dot{m}) = F_{\text{ext}}.$$

$$\Rightarrow\ m\dot{v} + \dot{m}u = F_{\text{ext}}.$$

Rocket motion in free space:

$$F_{\text{ext}} = 0 \quad \Rightarrow\ \frac{dv}{u} = -\frac{dm}{m} \quad v(t) = u\ln\frac{m_0}{m(t)}.$$

Applications: [mex17], [mex18], [mex229].

Photon rocket [mex223].

[mex18] Rocket launch in uniform gravitational field

A rocket is launched from rest against a uniform gravitational field g by burning fuel at a constant rate, $m = m_0(1 - \alpha t)$. The speed of the exhaust gases relative to the rocket is u.
(a) What is the minimum rate α_{min} at which fuel must be burned to ensure lift-off at $t = 0$.
(b) Calculate the velocity $v(t)$ of the rocket and the height $h(t)$ above ground.

Solution:

[mex101] A drop of fluid disappearing

A spherical drop of fluid with mass density ρ, initially of radius r_0, shrinks at a rate that is proportional to its size. Find the radius of the drop as a function of time.
(a) Assume that the mass decreases at a rate proportional to the surface area of the drop as a result of evaporation.
(b) Assume that the mass decreases at a rate proportional to the volume of the drop as a result of some kind of chemical instability.

Solution:

[mex15] Range and duration of attenuated motion

A particle of mass m and initial velocity v_0 moves along the x-axis under the influence of a velocity-dependent attenuation force:
(a) $F(v) = -\alpha\sqrt{v}$, (b) $F(v) = -\beta v$, (c) $F(v) = -\gamma v^2$.
In each case determine the range R of the particle (maximum displacement) and the duration T of the motion before the particle comes to a stop.

Solution:

[mex16] Projectile in resistive medium

A particle of mass m is projected vertically upward with initial velocity v_0 against a uniform gravitational field g and against a resistive force
(a) $F(v) = -\beta v$,   (b) $F(v) = -\gamma v^2$.
In each case find the maximum height h reached by the particle and the time T it takes to get there.

Solution:

[mex112] **Balancing the water level in a cone**

Water flows into a cone-shaped container at a constant rate (volume per unit time) and evaporates at a rate proportional to the free surface area.

(a) Determine the equilibrium position of the water level, expressed as the volume V_{eq} at which the two processes are in balance.

(b) Determine whether or not that stationary state is asymptotically stable.

(c) Determine the time dependence of the volume if the container is empty at first.

[mex17] Rocket motion in resistive medium

A rocket is launched from rest in a resistive medium ($F_{ext} = -\beta v$) by burning fuel at a constant rate, $m = m_0(1 - \alpha t)$. The speed of the exhaust gases relative to the rocket is u.
(a) Calculate the velocity $v(t)$ of the rocket.
(b) Take the limit $\beta \to 0$ in the result of (a) to recover the result $v(t) = u \ln[m_0/m(t)]$.

Solution:

[mex203] Position-dependent acceleration

Consider a particle of mass m moving along the x-axis. The particle experiences an acceleration that depends on its position as follows:

$$a = 6\gamma x^{1/3}, \quad \gamma = 1\mathrm{m}^{2/3}\mathrm{s}^{-2}.$$

What time does it take the particle to move from position $x = 1$m to position $x = 8$m if it has zero velocity at $x = 0$?

Solution:

[mex229] **Growth of falling raindrop**

A spherical raindrop of mass density ρ_W falling through fog of mass density ρ_F accumulates mass by absorbing all fog droplets (assumed stationary) in its way. Construct a differential equation (nonlinear second order ODE) for the radius r of the raindrop. Neglect the effects of air resistance on the raindrop.

[mex230] Modeling attenuation

An object with initial velocity v_0 is observed to grind to a halt during the time interval $0 < t < \tau$ according to the emprical law,

$$x(t) = \frac{1}{3}v_0\tau\left[1 - \left(1 - \frac{t}{\tau}\right)^3\right],$$

where τ is a constant. Construct the equation of motion in the form $m\dot{v} = f(v)$.

Solution:

[mex257] Exponential attenuation

A particle of mass m is launched at time $t = 0$ from position $x = 0$ in positive x-direction with initial velocity v_0. Acting on the particle, while it moves with $v > 0$, is the attenuating force $F = -fe^{v/c}$, where f, c are positive constants.
(a) At what time τ does the particle come to a stop?
(b) At what position R does the particle come to a stop? Hint: Use $dv/dx = (dv/dt)(dx/dt)^{-1}$.
(c) What are the maximum values of τ and R that this attenuating force permits, irrespective of how large v_0 is?
(d) For $v_0 \ll c$, the attenuating force can be interpreted as kinetic friction, $F \simeq -f = \text{const}$ with $f \doteq \mu_k mg$. What are the values of τ and R in this regime?

Solution: