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# 02. Coulomb force in 2D. Electric field. Superposition principle

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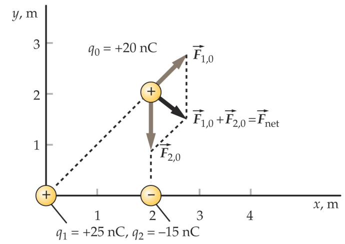
# PHY204 Lecture 2 [rln2]

## Coulomb Force in Two Dimensions (1a)




Find the magnitude and direction of the resultant force on charge $q_0$.

- Magnitude of individual forces:

$$F_{1,0} = k\frac{|q_1 q_0|}{r_{1,0}^2} = 5.62 \times 10^{-7}\text{N}, \quad F_{2,0} = k\frac{|q_2 q_0|}{r_{2,0}^2} = 6.74 \times 10^{-7}\text{N}.$$

- Components of individual forces:

$$F_{1,0}^x = F_{1,0}\cos 45^\circ, \quad F_{1,0}^y = F_{1,0}\sin 45^\circ, \quad F_{2,0}^x = 0, \quad F_{2,0}^y = -F_{2,0}.$$

tsl9

When we examine the forces between three charged particles that are not all positioned in one straight line, then we must describe these forces as vectors with more than one nonzero component. Three charged particles always fit onto a plane, here defined as the $xy$-plane as shown. None of the forces then has a $z$-component.

A good strategy for the task at hand is to first calculate the magnitude of $\vec{F}_{1,0}$ exerted by particle 1 on particle 0, which is repulsive, and the force $\vec{F}_{2,0}$ exerted by particle 2 on particle 0, which is attractive. The results are shown in the first item.

Graphically, the task is now readily completed and carried out on the slide. The directions of $\vec{F}_{1,0}$ and $\vec{F}_{2,0}$ are away from particle 1 and toward particle 2, respectively. The lengths of the two vectors (on a scale of your choice) are the magnitudes. The resultant force $\vec{F}_{net}$ is constructed as a diagonal in the completed parallelogram.

Analytically, we express both forces $\vec{F}_{1,0}$ and $\vec{F}_{2,0}$ in components, then add the components of each force to get the components of the resultant force.

The first step is carried out in the second item. We have used the fact that the angle of $\vec{F}_{1,0}$ above the horizontal is the same as the angle between the $x$-axis and the dashed line connecting particles 1 and 0.

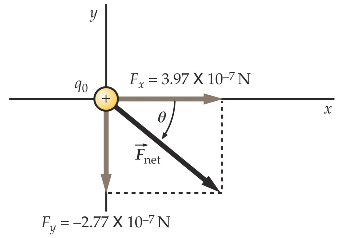
The second step continues on the next page.

## Coulomb Force in Two Dimensions (1b)

- Components of resultant force:

$$F_x = F^x_{1,0} + F^x_{2,0} = 3.97 \times 10^{-7}\text{N}, \quad F_y = F^y_{1,0} + F^y_{2,0} = -2.77 \times 10^{-7}\text{N}.$$

- Magnitude of resultant force: $F = \sqrt{F_x^2 + F_y^2} = 4.84 \times 10^{-7}\text{N}$.
- Direction of resultant force: $\theta = \arctan\left(F_y/F_x\right) = -34.9°$.




tsl325

The first item on the slide establishes the $x$- and $y$-components of the resultant force $\vec{F}_{net}$ using the results from the previous page. With this step the task is completed in one particular way. A vector is fully specified by its components.

If we wish to graphically represent this vector we complete the rectangle (right-angled parallelogram) as shown on the slide. It is the same vector as on the previous page except that the origin of the coordinate system has been shifted.

A different way of representing a vector in a plane uses polar coordinates. Instead of specifying Cartesian components $F_x$ and $F_y$, we specify the magnitude $F = F_{net}$ and the angle $\theta$ from the $x$-axis. The steps of this conversion are shown in the second and third items on the slide.

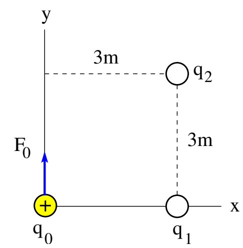
A generic arrangement of four charged particles does not fit onto a plane. The forces between them have three nonzero Cartesian components $F_x, F_y, F_z$. The strategy for calculating the resultant force on one charged particle exerted by three arbitrarily placed particles remains the same.

1. Determine the relevant distances between particles.
2. Calculate the magnitude of all relevant forces.
3. Split each force into three Cartesian components.
4. Add the $x$-components of all forces to get the $x$-component of the resultant force. Do the same with the $y$- and $z$-components.
5. The magnitude of the resultant force is $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$.
6. Its direction is encoded in two angles, the polar angle $\theta$ away from the $z$-axis and the azimuthal angle $\phi$ above the $x$-axis in the $xy$-plane.

## Coulomb Force in Two Dimensions (2)

The unknwon point charges $q_1, q_2$ exert a force $F_0 = 2\text{N}$ on the known point charge $q_0 = 1\text{nC}$. This force is directed in the positive $y$-direction as shown.

Determine first whether $q_1, q_2$ are positive or negative. Then determine the values of the two point charges.



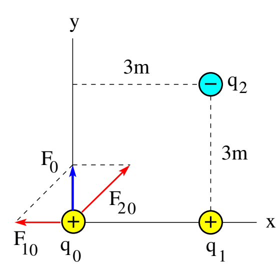

tsl10

In this problem we know that two forces are in operation, of which we only know the resultant force in both magnitude and direction:

$$\vec{F}_0 = \vec{F}_{10} + \vec{F}_{20}.$$

We know that the force $\vec{F}_{10}$ is along the line connecting $q_0$ and $q_1$ (the $x$-axis) and the force $\vec{F}_{20}$ along the line connecting $q_0$ and $q_2$.

Graphically, the directions of the two forces are found by constructing a parallelogram with sides along these lines and the diagonal equal to the given $\vec{F}_0$. This construction is shown in the diagram below.



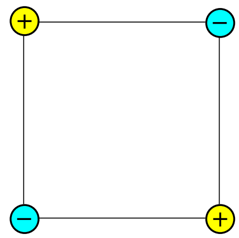

It follows immediately that $\vec{F}_{10}$ is a repulsive force and $\vec{F}_{20}$ an attractive force, implying that the charge $q_1$ is positive and the charge $q_2$ negative.

Recognizing that $F_0$ and $F_{10}$ are the sides of a square of which $F_{20}$ is the diagonal makes it easy to conclude that $F_{10} = F_0 = 2\text{N}$ and $F_{20} = 2\sqrt{2}\text{N}$.

Both forces $F_{10}$ and $F_{20}$ (now known) are Coulomb forces. For each Coulomb force we know one of two charges and the distance between them. Solving Coulomb's law for the unknown charge is straightforward.

## Coulomb Force in Two Dimensions (4)

Point charges of equal magnitude are positioned at the corners of a square.




- Copy this configuration and indicate by arrows the direction of the resultant force on each point charge.
- If the force between nearest-neighbor charges is 1N, what is the strength of the resultant force on each charge?

tsl12

Let us pick the positive charge at the top left corner and determine the net force it experiences. That force is the sum of three Coulomb forces. The three forces have different directions: vertical, horizontal, and diagonal:

$$\vec{F}_{net} = \vec{F}_{hor} + \vec{F}_{ver} + \vec{F}_{dia}.$$

The horizontal and vertical forces are attractive, thus directed right and down, respectively. The diagonal force is repulsive, directed up and left.

The problem statement says that the vertical and horizontal forces each have a magnitude of 1N. What is the magnitude of the diagonal force? The distance between the charges is larger by a factor of $\sqrt{2}$. Hence the diagonal Coulomb force is weaker by a factor of 2: it is $\frac{1}{2}$N. We thus have

$$F_{hor} = F_{ver} = 1\text{N}, \quad F_{dia} = \tfrac{1}{2}\text{N}.$$

What are direction and magnitude of $\vec{F}_{net}$? A quick way to the answers is the following. We first add $\vec{F}_{hor}$ and $\vec{F}_{ver}$, which yields a vector of magnitude,

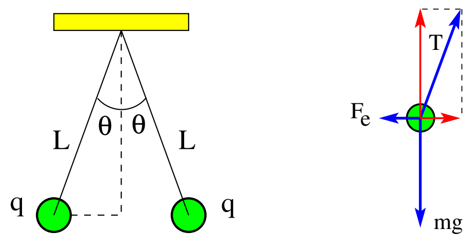
$$|\vec{F}_{hor} + \vec{F}_{ver}| = \sqrt{(1\text{N})^2 + (1\text{N})^2} = \sqrt{2}\text{N},$$

pointing along the diagonal, opposite to the direction of $\vec{F}_{dia}$. The net force $\vec{F}_{net}$, therefore, is a vector pointing along the diagonal toward the interior with magnitude,

$$F_{net} = \left(\sqrt{2} - \tfrac{1}{2}\right)\text{N}.$$

## Coulomb Force in Two Dimensions (5)

Two identical small charged spheres, each having a mass $m = 30\text{g}$, hang in equilibrium at an anlge of $\theta = 5°$ from the vertical. The length of the strings is $L = 15\text{cm}$.




- Identify all forces acting on each sphere.
- Find the magnitude of the charge $q$ on each sphere.

tsl13

The sphere on the left experiences gravity $m\vec{g}$ directed down, a tension $\vec{T}$ directed along the string, and an electrostatic repulsion $\vec{F}_e$ directed left. We know the magnitude of the gravitational force only. Since the sphere is not moving, the three forces are in balance.

Our plan is to find $\vec{F}_e$ as a first step toward finding $q$. The best strategy for this endeavor is to balance the vertical and horizontal components of all three forces separately. Balancing the vertical forces involves only one unknown, whereas balancing the horizontal forces involves two unknowns. Therefore, a smart move is to begin with the vertical forces:

$$\sum F_y = 0 \quad \Rightarrow T\cos\theta - mg = 0 \quad \Rightarrow T = \frac{mg}{\cos\theta},$$

$$\sum F_x = 0 \quad \Rightarrow T\sin\theta - F_e = 0 \quad \Rightarrow F_e = T\sin\theta = mg\tan\theta.$$

The solution for $T$ from the first balancing condition is substituted into the second balancing condition to determine $F_e$. Make sure to use SI units. Convert g into kg and cm into m. Use $g = 9.8\text{m/s}^2$. The results reads $F_e = 2.57 \times 10^{-2}\text{N}$.

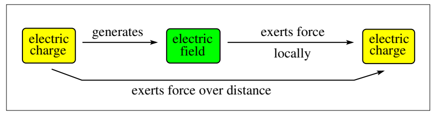
Coulomb's law,

$$F_e = \frac{kq^2}{r^2} \quad \text{with} \quad r = 2L\sin\theta,$$

can then be solved for $q = \pm\sqrt{q^2}$. The result, after a few intermediate steps, is $q = \pm 4.41 \times 10^{-8}\text{C}$.

Either both charges are positive or both are negative to ensure repulsion. To find out whether they are positive or negative requires a different experiment.

## Electric Field of a Point Charge

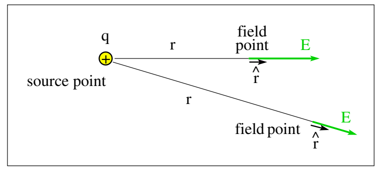



(1) Electric field $\vec{E}$ generated by point charge $q$: $\vec{E} = k\dfrac{q}{r^2}\hat{r}$

(2) Force $\vec{F}_1$ exerted by field $\vec{E}$ on point charge $q_1$: $\vec{F}_1 = q_1\vec{E}$

(1+2) Force $\vec{F}_1$ exerted by charge $q$ on charge $q_1$: $\vec{F}_1 = k\dfrac{qq_1}{r^2}\hat{r}$ (static conditions)

- $\epsilon_0 = 8.854 \times 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}$
- $k = \dfrac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \mathrm{Nm}^2\mathrm{C}^{-2}$
- SI unit of $E$: [N/C]




tsl14

How is it possible that charged particles can exert a force on each other across any distance? It is, in fact, not necessary to contemplate this possibility. The Coulomb force is just a convenient way to describe the interaction between charged particles at rest. An more general description of this interaction requires that we raise the level of abstraction one notch.

If we place an electric charge $q$ into a region of space (as shown), it generates an electric field $\vec{E}$ in the space around it. The electric field is a quantity with magnitude and direction, which both vary with position in space.

Line (1) gives the mathematical expression for $\vec{E}$ with $q$ as its source. The magnitude of $\vec{E}$ depends on $q$ and the distance $r$ of the field point from the source point. The direction of $\vec{E}$ is radial, away from the source if $q$ is positive, toward the source if $q$ is negative.

Line (2) gives the mathematical expression for the force $\vec{F}$ a point charge $q_1$ experiences when placed into the electric field $\vec{E}$. This expression holds irrespective of what the source of the field $\vec{E}$ is.

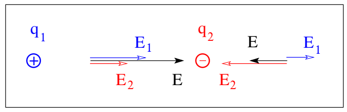
If the source of the field $\vec{E}$ is the charge $q$ as stated in line (1), then we can substitute that field expression into the force law (2) to recover Coulomb's law (1+2) for the effective force between charges at rest.

The convention is that the unit vector $\hat{r}$ is always pointing away from the source of an electric field. This ensures the correct direction of the electric field and the electric force in the expressions that contain $\hat{r}$. If the SI units of force and charge are [N] and [C], respectively, then the SI unit of the electric field is [N/C].

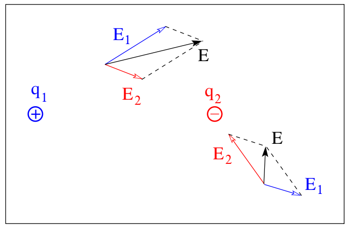
The force law (2) is generally true, whereas the source law (1) only holds if the charge $q$ is at rest. The same restriction thus applies to the Coulomb force (1+2).

## Superposition Principle for Electric Field

Electric field on line connecting two point charges:




Electric field of point charges in plane:




tsl15

A charge $q_1$ placed into a region of space generates an electric field $\vec{E}_1$ at an arbitrarily picked point in its vicinity. If we place a second charge $q_2$ at that location, it experiences a force $\vec{F}_{12} = q_2\vec{E}_1$.

Conversely, we can reason that the charge $q_2$ generates a field $\vec{E}_2$ at the position of $q_1$, which then experiences the force $\vec{F}_{21} = q_1\vec{E}_2$. The two forces $\vec{F}_{12} = -\vec{F}_{21}$ are an action-reaction pair.

There is now an electric field $\vec{E} = \vec{E}_1 + \vec{E}_2$ at any point in the vicinity of the two sources thus positioned. Electric fields from different sources add up without affecting each other. The superposition principle applies.

The vector sum of two electric fields is graphically represented on the slide for field points located on the axis defined by the two sources (top) and for field points away from that axis (bottom).

If we place a charge $q_3$ at any of these field points, it will experience an electrostatic force $\vec{F}_3 = q_3\vec{E}$.

We can now see where this leads to. Any number of charges produce, in their vicinity, an electric field that is, for a given field point, the vector sum of the fields generated by each charge at that point.

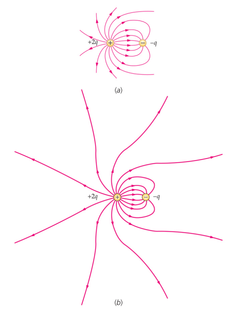
We shall recall this fact later, when we calculate the electric field generated by extended electrically charged objects. We divide the object into tiny cubes, treat them as charged particles, and use the superposition principle.

## Vector Field and Electric Field Lines

- The electric field is a vector field:

$$\vec{E}(\vec{r}) = \vec{E}(x,y,z) = E_x(x,y,z)\hat{\imath} + E_y(x,y,z)\hat{\jmath} + E_z(x,y,z)\hat{k}$$

- Electric field lines: graphical representation of
- Properties of electric field lines (electrostatics):
  - Electric field lines begin at positive charges or at infinity.
  - Electric field lines end at negative charges or at infinity.
  - The direction of $\vec{E}$ is tangential to the field line going through the field point.
  - Electric field lines bunched together indicate a strong field.
  - Electric field lines far apart indicate a weak field.
  - Field lines do not intersect.




tsl22

How can we visualize $\vec{E}(\vec{r})$, a vector that is a function of another vector, the electric field as a function of position? Both vectors have a magnitude and a direction in 3-dimensional space.

The vector $\vec{r}$ points from the origin of some chosen coordinate system to the field point. The position $\vec{r}$ of the field point has components $x, y, z$ in this coordinate system.

Now imagine a coordinate system with axes parallel to the one used for position but with its origin shifted to the field point. The electric field $\vec{E}$ at the field point has components $E_x, E_y, E_z$ in this shifted coordinate system.

At a different field point $\vec{r}$, the electric field $\vec{E}$ may have a different magnitude and direction, i.e. different components. Therefore, each field component, $E_x, E_y, E_z$, is a function of all three position components, $x, y, z$. This is what the equation on the slide expresses.

Visualizing vector functions is not an experience most of us have grown up with. Field lines are great visual aid. The itemized list on the slide is a guide for how to read an image of electric field lines.

We cannot see an electrostatic field $\vec{E}$ with our eyes. However, the electric field $\vec{E}$ exerts a force $\vec{F} = q\vec{E}$ on a charge $q$. The use of test charges is one way to recognize the presence of an electric field and to measure its magnitude and direction.

## Electric Field on Line Connecting Point Charges (1)

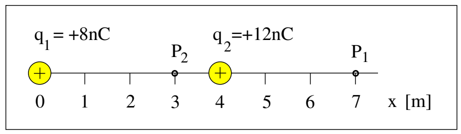
Consider the $x$-component of the electric field.

- Electric field at point $P_1$:

$$E = E_1 + E_2 = \frac{kq_1}{(7\text{m})^2} + \frac{kq_2}{(3\text{m})^2} = 1.47\text{N/C} + 12.0\text{N/C} = 13.5\text{N/C}.$$

- Electric field at point $P_2$:

$$E = E_1 + E_2 = \frac{kq_1}{(3\text{m})^2} - \frac{kq_2}{(1\text{m})^2} = 7.99\text{N/C} - 108\text{N/C} = -100\text{N/C}.$$




tsl19

When electric charges are placed along a straight line and we declare that line to be the $x$-axis, then the electric field generated by these charges has an $x$-component only. That component can be positive or negative, meaning that the field is directed right or left, respectively.

The exercise on the slide calculates the electric field of two positive charges generated at two field points.

Keep in mind that the field is directed away from positive charges. Therefore, at point $P_1$ both field are directed right (counted positively), whereas at point $P_2$ only one field is directed right. The other field is directed left, thus counted negatively.

## Electric Field of Point Charges in Plane (1)

Determine magnitude of $\vec{E}_1$ and $\vec{E}_2$
and identify directions in plane:

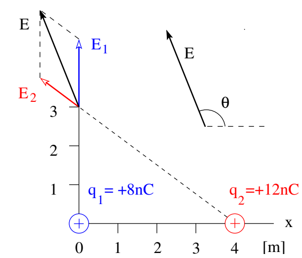
$$E_1 = \frac{k|q_1|}{(3\mathrm{m})^2} = 7.99\mathrm{N/C}, \quad E_2 = \frac{k|q_2|}{(5\mathrm{m})^2} = 4.32\mathrm{N/C}.$$

Determine $x$- and $y$-components of $\vec{E}_1$ and $\vec{E}_2$
and of the resultant field $\vec{E}$:

$$E_1^x = 0, \quad E_1^y = 7.99\mathrm{N/C};$$
$$E_2^x = -3.46\mathrm{N/C}, \quad E_2^y = 2.59\mathrm{N/C};$$
$$E_x = -3.46\mathrm{N/C}, \quad E_y = 10.6\mathrm{N/C}.$$

Determine magnitude and direction of $\vec{E}$:

$$E = \sqrt{E_x^2 + E_y^2} = 11.2\mathrm{N/C}, \quad \theta = \arctan\left(\frac{E_y}{E_x}\right) = 108^\circ.$$




tsl18

The exercise on the slide is related to the one on page 1. There we calculated the force exerted by two charged particles on a third charged particle.

Here we calculate the electric field generated by two charged particles at some field point. If we place a third particle with charge $q_3$ at the field point, it will experience a force $\vec{F} = q_3\vec{E}$, where $\vec{E} = \vec{E}_1 + \vec{E}_2$ is the combined electric field of the particles with charges $q_1$ and $q_2$.

We adopt the strategy used on page 1. There we added forces vectorially. Here we add fields vectorially. We again do it graphically and analytically.

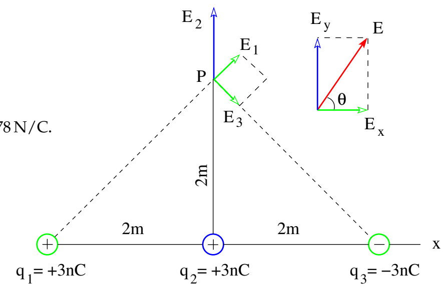
The field $\vec{E}_1$ only has a $y$-component, which is positive. The field $\vec{E}_2$ has a negative $x$-component and a positive $y$-component. How do we determine these components?

Note that the angle $\phi$ of $\vec{E}_2$ above the horizontal is the same as the angle of the dashed line with one end at $q_2$ above the horizontal. The length of that dashed line is 5m by Pythagorean theorem. Hence $\cos\phi = \frac{4}{5}$ and $\sin\phi = \frac{3}{5}$, which we use in the relations $E_2^x = -E_2\cos\phi$ and $E_2^y = E_2\sin\phi$.

The remaining steps are straightforward. We have taken equivalent steps before, on pp. 1-2, in a slightly different context.

Find magnitude and direction of the resultant electric field at point $P$.

- $E_1 = \frac{k|q_1|}{8\text{m}^2} = 3.38\,\text{N/C}.$
- $E_2 = \frac{k|q_2|}{4\text{m}^2} = 6.75\,\text{N/C}.$
- $E_3 = \frac{k|q_3|}{8\text{m}^2} = 3.38\,\text{N/C}.$
- $E_x = E_1 \cos 45^\circ + E_3 \cos 45^\circ = 4.78\,\text{N/C}.$
- $E_y = E_2 = 6.75\,\text{N/C}.$
- $E = \sqrt{E_x^2 + E_y^2} = 8.27\,\text{N/C}.$
- $\tan\theta = \frac{E_y}{E_x} = 1.41.$
- $\theta = \arctan 1.41 = 54.7^\circ.$




tsl371

The exercise on this slide is very similar to the one on the previous page. We are looking for the superposition of three electric fields at field point $P$. The three fields are generated by three point charges placed on the $x$-axis.

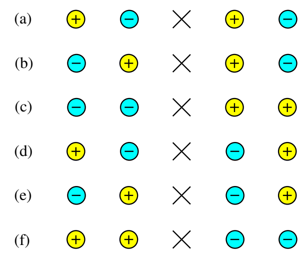
When we begin by adding field $\vec{E}_1$ and $\vec{E}_3$ we realize that the resultant field $\vec{E}_1 + \vec{E}_3$ only has an $x$-component. The $y$-components cancel.

What remains to be done is to add to it $\vec{E}_2$, which only has a $y$-component. In other words, the $x$-component of $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$ comes from $q_1$ and $q_3$, whereas the $y$-component comes from $q_2$.

The last three items calculate magnitude and direction of $\vec{E}$ from the components. The direction is indicated by the angle $\theta$ above the positive $x$-axis.

## Electric Field on Line Connecting Point Charges (2)

Four particles with charges of equal magnitude are positioned on a horizontal line in six different configurations.




Determine for each configuration the direction of the resultant electric field (left/right/zero) at the location indicated by $\times$.

tsl16

This is the quiz for lecture 2.

Here we have six situations with four point charges aligned as shown and a field point in the middle.

At the field point there is the superposition of our fields, one field from each charge in the row. There are two strong fields from the nearby charges and to weaker fields from the charges further away. Fields are directed away from positive sources and toward negative sources.

If necessary, take your cues from page 9.