2015

02. Newtonian Gravitation

Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Abstract
Part two of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc2]

Newton's Law of Gravitation [mln3]

Gravitational potential of a mass distribution: $\Phi(\mathbf{r}) = -G\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$;
$\rho(\mathbf{r})$: mass density;
$G \simeq 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$: universal gravitational constant.

Gravitational field (acceleration due to gravity):

$$\mathbf{g}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = -G\int d^3r' \,\frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \;\Rightarrow\; \Phi(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} d\mathbf{r}'\cdot\mathbf{g}(\mathbf{r}')$$

Gravitational force on point mass m: $\mathbf{F}(\mathbf{r}) = m\mathbf{g}(\mathbf{r})$.

Gravitational potential energy of point mass m: $V(\mathbf{r}) = m\Phi(\mathbf{r})$.

Field equation: $-\nabla\cdot\mathbf{g}(\mathbf{r}) = \nabla^2\Phi(\mathbf{r}) = G\int d^3r'\rho(\mathbf{r}')\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = 4\pi G\rho(\mathbf{r})$,
where we have used $\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = 4\pi\delta(\mathbf{r}-\mathbf{r}')$.

Gauss' theorem: $\oint d\mathbf{A}\cdot\mathbf{g}(\mathbf{r}) = \int_V d^3r\;\nabla\cdot\mathbf{g}(\mathbf{r})$.

Gauss' law: $\oint d\mathbf{A}\cdot\mathbf{g}(\mathbf{r}) = -4\pi G m_{in}$.

Gravitational self energy: $V_S = -\frac{1}{2}G\int d^3r\int d^3r'\,\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$.

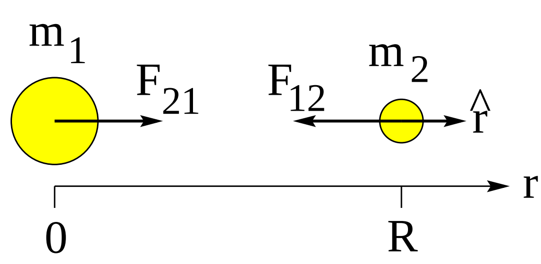
Massive spheres:

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = -G\frac{m_1m_2}{R^2}\,\hat{\mathbf{r}},$$

$$\mathbf{g}(R) = \frac{\mathbf{F}_{12}}{m_2} = -G\frac{m_1}{R^2}\,\hat{\mathbf{r}},$$

$$\Phi(R) = -\int_\infty^R dr\,g(r) = -G\frac{m_1}{R}$$

$$V_2(R) = m_2\Phi(R) = -G\frac{m_1m_2}{R}$$



[mex103] Gravitational potential of a homogeneous rod

Calculate the gravitational potential of a thin homogeneous rod of length L and mass m for the case that the field point is equidistant from the two endpoints of the rod. Show that for large distances the result approaches the potential of a point mass asymptotically. Calculate also the the first correction to the point mass potential due to the nonspherical mass distribution.

Solution:

[mex105] Gravitational field of a homogeneous massive sphere

(a) Calculate the gravitational field $\mathbf{g}(r)$ and the gravitational potential $\phi(r)$ at all $r > 0$ of a homogeneous sphere with mass M and radius R. (b) Assuming that the sphere made of a fluid in hydrostatic equilibrium, calculate the pressure $p(r)$ for $0 < r < R$.

Solution:

[mex106] Gravitational field of an inhomogeneous massive sphere

Consider a massive sphere of radius R with a spherical mass distribution. (a) Determine the mass density $\rho(r)$ which produces a gravitational field $g(r) = -\alpha = \text{const}$ for $0 < r < R$. (b) Determine the gravitational field $g(r)$ produced by that same sphere at $r > R$.

Solution:

[mex104] Gravitational self energy of a homogeneous massive sphere

(a) Determine the gravitational self energy of a homogeneous sphere of radius R and mass M. (b) Infer, by analogy, the electrostatic self energy of a homogeneous, spherical charge distribution (radius R, total charge q). (c) Under the (quite unrealistic) assumption that the total energy $E = m_e c^2$ of an electron at rest consists entirely of electrostatic self energy, determine the radius of the electron (in SI units). This value is known under the name *classical electron radius.*

Solution:

[mex3] Gravitational field and potential of interstellar dust cloud

A spherically symmetric cloud of interstellar dust is characterized by the mass density

$$\rho(r) = \frac{C}{r(r^2 + b^2)^2},$$

where C, b are constants. Find the gravitational field $g(r)$ and the gravitational potential $\Phi(r)$ of this distribution of matter.

Solution:

[mex2] Gravitational collapse of cold cloud of dust

Imagine a spherically symmetric, homogeneous cloud of dust grains. All grains are initially at rest and separated from one another by distances large compared to the grain size. Initially the cloud has radius R_0 and average mass density ρ_0. Assume that collisions between grains can be ignored, which would cause a pressure to build up.
(a) Calculate the time T it takes for the cloud to collapse to (essentially) zero radius. Note that T is independent of R_0 for given ρ_0.
(b) Show that at all times during the contraction the speed of each dust grain is proportional to its distance from the center of the cloud as in Hubble's law.

Solution:

[mex152] Gravitational potential of a homogeneous disk

Calculate the gravitational potential of a thin homogeneous disk of radius R and mass m for the case that the field point is on the axis. Show that for large distances the result approaches the potential of a point mass asymptotically. Calculate also the first correction to the point mass potential due to the non-spherical mass distribution.

Solution:

[mex153] Flat Earth versus round Earth

Consider the gravitational potential $\phi(z)$ along the axis of a flat Earth in the shape of a thin homogeneous disk of radius R as derived in [mex152]. Consider also the the gravitational potential $\phi(z) = -Gm/(R+z)$ outside a round Earth in the shape of a homogeneous sphere of mass m and radius R. If an Earthling, who does not know whether the Earth is flat or round and knows nothing about its size in either shape, is able to measure the variation of the gravitational field $g(z)$ near the Earth's surface, which power of z in an expansion of $g(z)$ will enable her to distinguish the flat shape from the round shape? Does the conclusion change if the point considered for the disk is not on the axis?

Solution: