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01. Newtonian Mechanics

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Abstract
Part one of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

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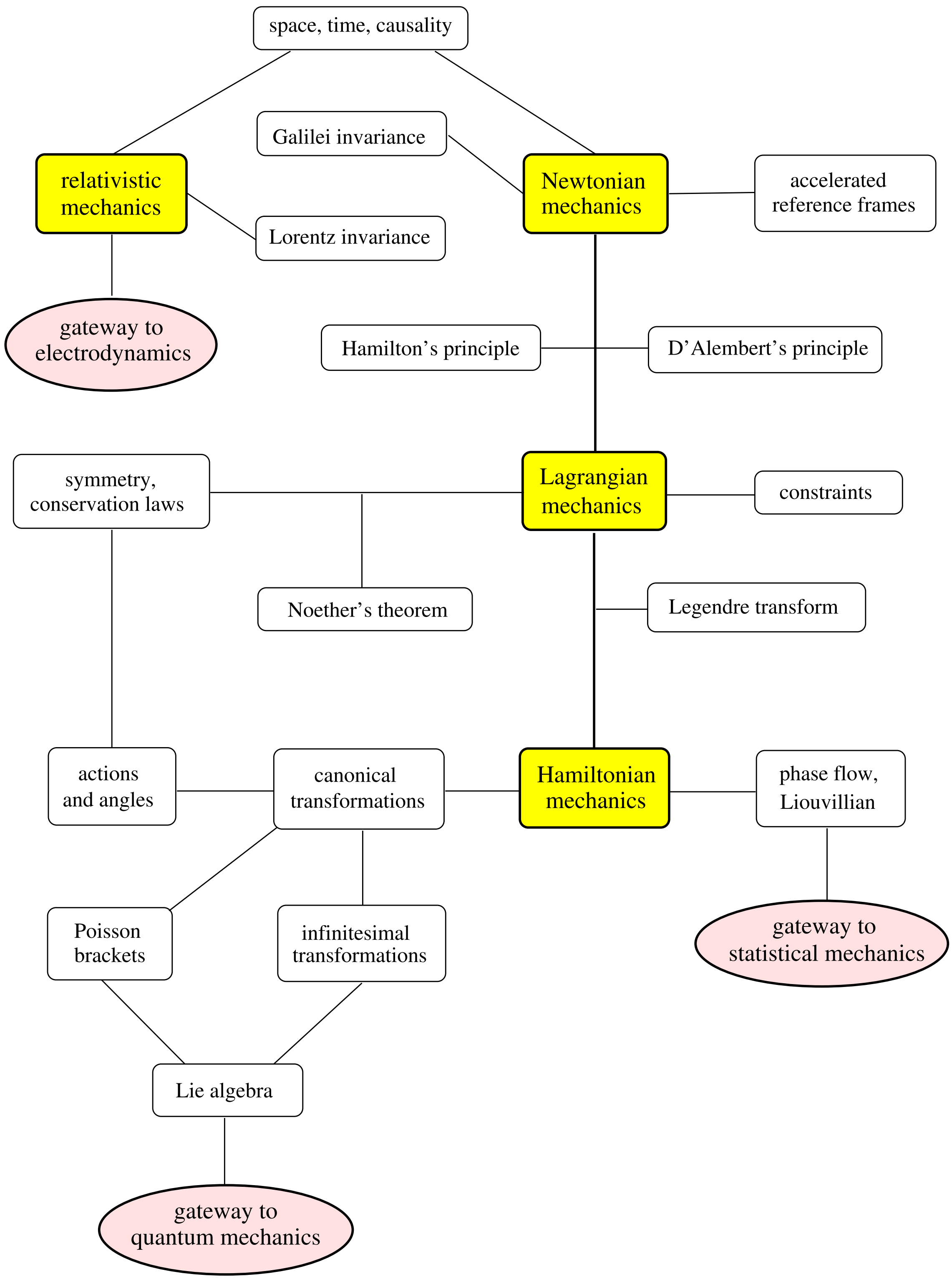


Newtonian Mechanics [mln1]

Space and Time

- Absolute space is 3-dimensional, homogeneous, and isotropic; the (Euclidean) metric is independent of the objects present in space.
- Absolute time is homogeneous.

Relativistic mechanics introduces major modifications regarding the properties of space and time.

Galilei's Principle of Relativity

There exist *inertial* coordinate systems:

- The laws of mechanics are the same in all inertial coordinate systems (invariance under Galilean transfomations).
- All coordinate systems in uniform rectilinear motion with respect to an inertial coordinate system are themselves inertial.

The laws of relativistic mechanics are invariant under Lorentz transformations. In general relativity, the restriction to inertial systems is removed.

Newton's Laws of Dynamics

1. A body remains at rest or in uniform rectilinear motion unless acted upon by a force.
2. A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force exerted on it: $d\mathbf{p}/dt = \mathbf{F}$.
3. If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction (action-reaction pair of forces).

Newton's second law states the (deterministic) relation between *cause* and *effect*. Any deterministic forecast depends on precise knowledge of initial conditions and forces.

The absolute determinism of Newtonian mechanics were subsequently undermined from two sides: by *chaos theory* and by *quantum mechanics*.

Impact of Symmetry [mln70]

Classical mechanics:

- *Continuous* symmetries have major impact on dynamics.
- *Discrete* symmetries have minor impact on dynamics in comparison.

Quantum mechanics:

- *Continuous* and *discrete* symmetries have comparable impact on spectrum and transition rates.

Inertial reference frame:

- Classical mechanics rests on the assumption that it is always possible to find a frame of reference with respect to which space is homogeneous and isotropic and time is homogeneous.

Consequences:

continuous symmetry	conservation law
space is homogeneous	linear momentum
space is isotropic	angular momentum
time is homogeneous	mechanical energy

The general relationship between continuous symmetries and conservation laws in classical mechanics is described by Noether's theorem [mln12], [mln13], [mln42].

Conservation Laws [mln2]

Single Particle

- The component of *linear momentum* $\mathbf{p} = m\mathbf{v}$ in a direction (specified by vector $\mathbf{s}$) in which the applied force $\mathbf{F}$ vanishes is a constant in time:

$$\dot{\mathbf{p}} \cdot \mathbf{s} = \mathbf{F} \cdot \mathbf{s}.$$

- The *angular momentum* $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ is a constant in time if the applied force $\mathbf{F}$ exerts zero torque $\mathbf{N}$:

$$\dot{\mathbf{L}} \equiv \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N}.$$

- If the applied force $\mathbf{F}$ is *conservative*, then the *total energy* E, which is the sum of the *kinetic energy* T and *potential energy* V, is a constant in time:

$$E = T + V; \qquad T = \frac{1}{2}mv^2, \quad V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}, \quad \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}).$$

System of Particles

External force: $\mathbf{F}^{(e)} = \sum_i \mathbf{F}_i^{(e)}$. Internal forces: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ with $\mathbf{F}_{ij} \| \mathbf{r}_{ij}$.

- The component of *total linear momentum* $\mathbf{p}$ in a direction in which the *external force* $\mathbf{F}^{(e)}$ vanishes is a constant in time:

$$\dot{\mathbf{p}} \cdot \mathbf{s} = \mathbf{F}^{(e)} \cdot \mathbf{s}.$$

- The *total angular momentum* $\mathbf{L}$ is a constant in time if the external force $\mathbf{F}^{(e)}$ exerts zero torque $\mathbf{N}^{(e)}$:

$$\dot{\mathbf{L}} \equiv \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F}^{(e)} = \mathbf{N}^{(e)}.$$

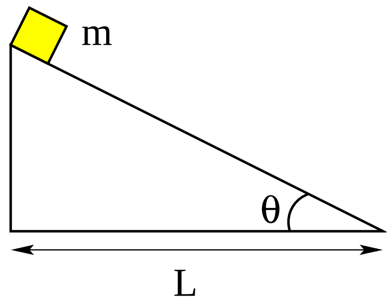
- If the forces $\mathbf{F}^{(e)}$ and $\mathbf{F}_{ij}$ are *conservative*, then the *total (mechanical) energy* E of the system is a constant in time:

$$E = T + V; \qquad T = \sum_i \frac{1}{2} m_i v_i^2, \quad V = \sum_i V_i^{(e)} + \sum_{i<j} V_{ij}.$$

Non-conservative forces (friction, attenuation) imply energy dissipation. Some mechanical energy is then converted into thermal energy or radiation.

[mex100] The shortest path is not the quickest path

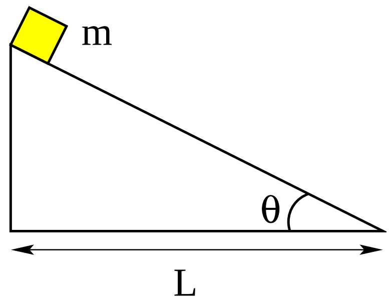
A block of mass m slides from rest down a ramp with base of fixed length L as shown. Friction is assumed to be negligibly small. Find the angle θ for which the block arrives at the bottom of the ramp in the shortest time.



Solution:

[mex154] Minimizing time of slide when friction is present

A block of mass m slides from rest down a ramp with base of fixed length L as shown. The motion is impeded by a frictional force $f = \mu N$, where N is the normal force and μ is the coefficient of kinetic friction. Find the angle θ for which the block arrives at the bottom of the ramp in the shortest time.



Solution:

[mex136] Optimized time of travel

The maximum acceleration of a train is α and its maximum deceleration is β. Show that it cannot run a distance d from rest to rest in a shorter time than

$$t = \sqrt{\frac{2d(\alpha+\beta)}{\alpha\beta}}.$$

Solution:

[mex137] Acceleration from clocking consecutive space intervals

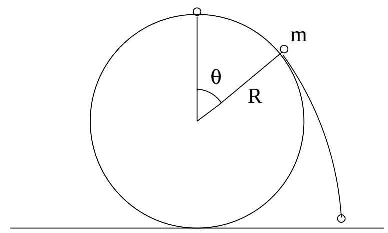
A body moving in a straight line with constant acceleration passes two consecutive equal spaces, each of length d, in times $\Delta t_1, \Delta t_2$. The body is already moving when it enters the first space. Show that the acceleration is

$$a = \frac{2d(\Delta t_1 - \Delta t_2)}{\Delta t_1 \Delta t_2 (\Delta t_1 + \Delta t_2)}.$$

Solution:

[mex1] Particle sliding down sphere

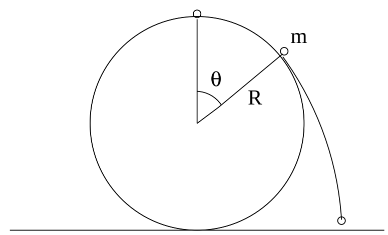
A tiny particle of mass m slides without friction down a spherical surface of radius R. The particle starts at the top with negligible speed.
(a) Identify all forces that act on the particle as it slides along the surface of the sphere.
(b) Calculate the polar angle θ at which the path of the particle leaves the sphere.



Solution:

[mex102] Time of slide and time of flight

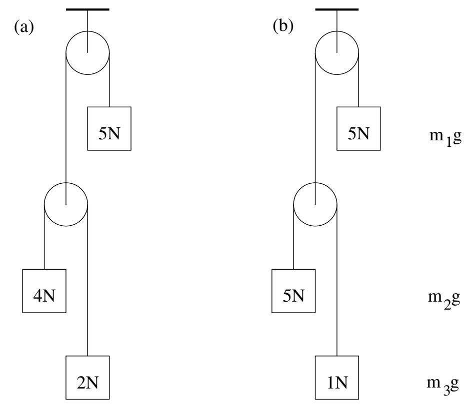
Consider the problem already encountered in [mex1]. The particle starts to slide from rest at $\theta = 0$.
(a) Find the time it takes the particle to slide from angular position $\theta_0 > 0$ to angular position θ_c, where it takes off.
(b) Find (for $R = 10\text{m}$ and $g \simeq 10\text{m/s}^2$) the time between take-off from the sphere and landing on the plane.



Solution:

[mex9] Atwood machine

Consider two Atwood machines with different weights. The pulleys and ropes have negligible mass. All weights and pulleys are initially at rest. Use $g \simeq 9.8\text{ms}^{-2}$. Determine whether the lower pulley will start to move up or down in each case. Calculate the tensions T_1 and T_2 in the upper and lower ropes, respectively. State the final results in SI units with two digits accuracy.



Solution:

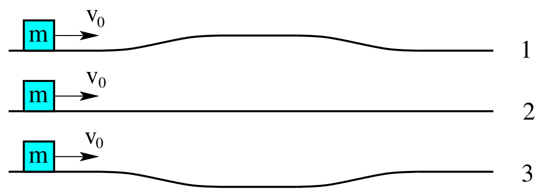
[mex204] On frozen pond

A father ($m_1 = 70$kg) and his son ($m_2 = 35$kg) are standing on the ice in the middle of a pond. The coefficient of static friction between boots and ice is the same for both persons. Who will win the race (from rest) to the edge of the pond...
(a) if both runners move as fast as possible without slipping,
(b) if each runner drags a sled ($m_s = 10$kg) by a cord, where the frictional force acting on the sled is negligibly small?
Justify your answers carefully.

Solution:

[mex205] The quick, the short, and the scenic

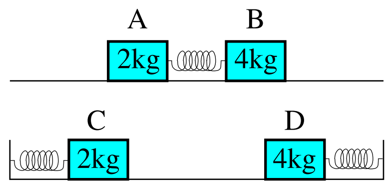
Consider three blocks of equal mass m sliding along frictionless tracks. Track 2 is strictly horizontal. Track 1 leads over a hill and track 3 through a valley. The blocks are simultaneously launched with equal initial velocity v_0 at the positions shown. The curvatures in tracks 1 and 3 are sufficiently gentle for the blocks not to lift off. In what sequence do the blocks arrive at the end of their tracks? Justify your answers carefully.



Solution:

[mex206] When push comes to shove

Consider two situations in which blocks of unequal mass initially at rest on frictionless tracks are accelerated by springs of equal stiffness and with equal initial compression as shown.
(a) Which block, A or B, C or D, has acquired more momentum when the spring in contact with it has become fully relaxed?
(b) Which block, A or B, C or D, has acquired more kinetic energy when the spring in contact with it has become fully relaxed?
Justify your answers carefully.



Solution:

[mex138] Rubber speed

A car is moving with constant velocity v along a level road. Find the instantaneous speed v_T of a point on the tread (perimeter) of the tire, which has radius R. Express v_T as a function of y, where $0 \leq y \leq 2R$ is the distance of the point on the tire from the surface of the road.

Solution:

[mex11] Water projected into air by wheel rolling on wet road

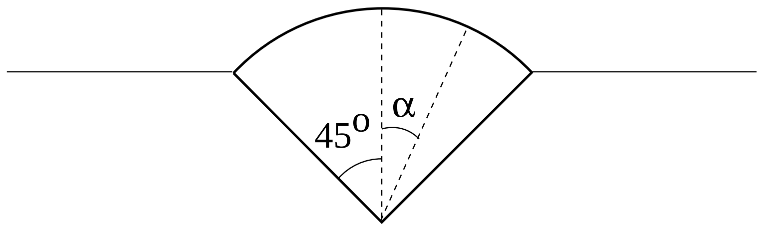
A wheel of radius R, which is part of some vehicle, is rolling along a wet road with speed v. Drops of water are being continuously thrown off the rim. Show that if $v^2 > Rg$ then the maximum height above the road attained by the drops is

$$R + \frac{v^2}{2g} + \frac{R^2 g}{2v^2}.$$

Solution:

[mex113] Design of a lawn sprinkler

The lawn sprinkler is to consist of a spherical cap with cross section as shown. The spherical surface is perforated with a large number of small holes through which water jets are ejected with initial speed v_0. How must the holes be distributed so that a circular area around the sprinkler is watered uniformly? Express this distribution a function $n(\alpha)$ describing the number of holes per unit area on a ring at angle α from the vertical. The size of the cap is small compared to the area being sprinkled and the cap is about level with the lawn. Air resistance is assumed to be negligible.



Solution:

[mex139] Longest shot from the top of a hill

A gun is mounted on a hill of height h above a level plain. Assuming that the muzzle speed is v_0 and that the path of the projectile is parabolic, show that the angle of elevation α for greatest horizontal range depends on h and v_0 as follows:

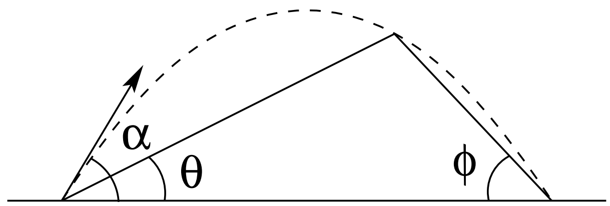
$$\frac{1}{\sin^2 \alpha} = 2\left(1 + \frac{gh}{v_0^2}\right).$$

Solution:

[mex140] Lowest shot to target across hill

Consider a hill consisting of two straight slopes with inclinations θ and ϕ. A projectile is fired along a parabolic trajectory from the foot of the hill on one side to the foot of the hill on the opposite side in such a way that it grazes the summit. Show that the angle of projection must be

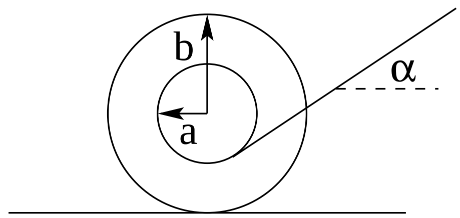
$$\alpha = \arctan(\tan\theta + \tan\phi).$$



Solution:

[mex141] Reel of thread I: statics

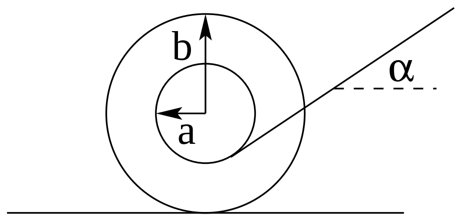
A reel of thread of weight W whose spindle and rim have radii a and b, respectively, rests on a horizontal table. The loose end of the thread passes under the spindle and leads off at an angle α above the horizontal as shown. The static frictional force between the reel and the table is $f \leq \mu_S N$, where N is the normal force and μ_S is a constant.
(a) Find the angle α_c at which a static equilibrium exists for nonzero tension T in the thread.
(b) Find the maximum value T_c of the tension for which the equilibrium holds.



Solution:

[mex142] Reel of thread II: dynamics

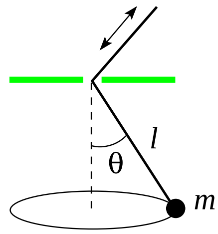
A reel of thread whose spindle and rim have radii a and b, respectively, rests on a horizontal table. The weight of the reel is mg and the moment of inertia for rotations about its axis is I. The loose end of the thread passes under the spindle and leads off at an angle α above the horizontal as shown. The static frictional force between the reel and the table during rolling motion is $f \leq \mu_S N$, where N is the normal force and μ_S is a constant. Consider the range $0 \leq \alpha \leq \pi$ of angles.
(a) For a given tension not too strong to make the reel roll without slipping, find the angular acceleration $\dot{\omega}$, the frictional force f, and the normal force N.
(b) For the three cases $\alpha = 0, \pi/2, \pi$ find the direction (clockwise or counterclockwise) of the angular acceleration $\dot{\omega}$ and the direction (left or right) of the frictional force f.
(c) For the three cases $\alpha = 0, \pi/2, \pi$ find the maximum possible value of $|\dot{\omega}|$ for rolling without slipping.



Solution:

[mex226] Spherical pendulum of slowly varying length

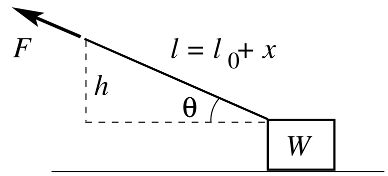
A small mass m rotates in a horizontal circle at the end of a string of length l at constant angle θ from the vertical as shown. When the length l is slowly varied the angle θ changes such that $f(l,\theta) = \text{const}$. Find the function $f(l,\theta)$.



Solution:

[mex227] Dragging block by elastic cord

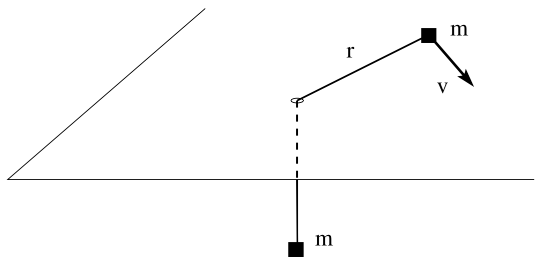
A block of weight $W = 5\text{N}$ is being dragged along a horizontal surface against kinetic friction with coefficient $\mu = 0.2$. Find the force F (in SI unit) needed to keep the block moving at constant speed if the upper end of the elastic cord is held at constant height $h = 1\text{m}$. The length of the cord is $l = l_0 + x$, where $l_0 = 1\text{m}$ and $F = kx$ with $k = 1\text{N/m}$.



Solution:

[mex228] Centripetal elevator

Two small blocks of mass m are connected by a string of length $2l$, which passes through a hole in a horizontal table. One block is free to slide on the table and the other block hangs underneath. In the initial state the block on the table is a distance $r = l$ from the hole and projected with velocity v horizontally and at right angle to the string. The hanging block is initially at rest. Ignore friction. (a) Find v such that the hanging block stays at rest. (b) Find v such that the hanging block moves up but comes to rest at the hole without going through the hole or touching the table.



Solution:

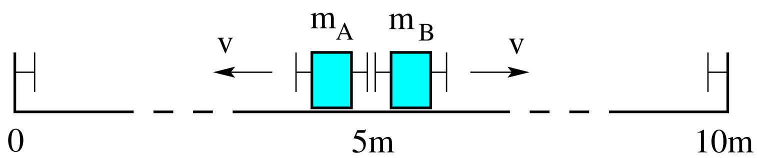
[mex231] Lateral force on hanging chain

Consider a chain of N links in the form of uniform rods of mass m and length $2l$ each, connected by frictionless hinges. At one end the chain is attached to a stationary pivot. The other end is pulled sideways by a constant force F. The links are numbered N through 1 from the pivot outwards. Find the angle θ_n between link n and the horizontal for $n = 1, 2, \ldots, N$ at equilibrium.

Solution:

[mex247] Let's meet again... and again

Two blocks of masses m_A and m_B, positioned in the middle (at $x_0 = 5m$) of a horizontal air track, are launched with equal speed v in opposite directions. They bounce off the ends of the track and off each in a succession of elastic collisions. The track does not recoil significantly and the blocks are of very small size compared to the length of the track. Find the locations x_1 and x_2 on the track where the two blocks collide the first and second times, respectively, if (i) $m_A = 2$kg, $m_B = 4$kg and (ii) $m_A = 2$kg, $m_B = 3$kg.



Solution: