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01. Newtonian Mechanics

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Abstract
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Table of Contents [mtc]

1. Newtonian Mechanics
    - Classical mechanics overview [mln69]
    - Space and time – Galilei's principle of relativity – Newton's laws of dynamics [mln1]
    - Impact of symmetry [mln70]
    - Conservation laws [mln2]
    - The shortest path is not the quickest path [mex100]
    - Minimizing time of slide when friction is present [mex154]
    - Optimized time of travel [mex136]
    - Acceleration from clocking consecutive space intervals [mex137]
    - Particle sliding down a sphere [mex1]
    - Time of slide and time of flight [mex102]
    - Atwood machine [mex9]
    - On frozen pond [mex204]
    - The quick, the short, and the scenic [mex205]
    - When push comes to shove [mex206]
    - Rubber speed [mex138]
    - Water projected into air by wheel rolling on wet road [mex11]
    - Design of a lawn sprinkler [mex113]
    - Longest shot from the top of a hill [mex139]
    - Lowest shot to target across hill [mex140]
    - Reel of thread I: statics [mex141]
    - Reel of thread II: dynamics [mex142]
    - Spherical pendulum of varying length [mex226]
    - Dragging block by elastic cord [mex227]
    - Centripetal elevator [mex228]
    - Lateral force on hanging chain [mex231]

- Let's meet again... and again [mex247]

2. Newtonian Gravitation
   - Newton's law of gravitation [mln3]
   - Gravitational potential of a homogeneous rod [mex103]
   - Gravitational field of a homogeneous massive sphere [mex105]
   - Gravitational field of an inhomogeneous massive sphere [mex106]
   - Gravitational self energy of a homogeneous massive sphere [mex104]
   - Gravitational field and and potential of interstellar dust cloud [mex3]
   - Gravitational collapse of cold cloud of dust [mex2]
   - Gravitational potential of a homogeneous disk [mex152]
   - Flat Earth versus round Earth [mex153]

3. Simple Dynamical Systems
   - One degree of freedom [mln71]
   - Solution by quadrature [mln4]
   - Phase portraits of conservative systems [msl5]
   - Periodic motion in quadratic and quartic potentials [mex5]
   - Potential energy of periodic motion reconstructed [mex232]
   - Periodic motion in 2D phase space [mex6]
   - Separatrix tangent lines at hyperbolic point [mex111]
   - Solution by separation of variables [mln72]
   - Rocket launch in uniform gravitational field [mex18]
   - A drop of fluid disappearing [mex101]
   - Range and duration of attenuated motion [mex15]
   - Projectile in resistive medium [mex16]
   - Balancing the water level in a cone [mex112]
   - Rocket motion in resistive medium [mex17]

- Position-dependent acceleration [mex203]
- Growth of falling raindrop [mex229]
- Modeling attenuation [mex230]
- Exponential attenuation [mex257]

4. Fixed Points and Limit Cycles
    - Phase portrait: particle in double-well potential [msl7]
    - Phase portrait: plane pendulum [msl8]
    - Phase portrait: magnetic pendulum [msl9]
    - Classification of fixed points in plane [mln73]
    - Table of fixed points in 2D phase space [msl10]
    - Isoclines [mln31]
    - Fixed points of the plane pendulum [mex12]
    - 2D phase portrait I [mex7]
    - 2D phase portrait II [mex8]
    - Predator and prey [mex13]
    - Host and parasite [mex14]
    - Isoclines and fixed points [mex108]
    - Fierce competition versus mild competition [mex109]
    - Limit cycles [mln74]
    - Hopf bifurcation [mex19]
    - Feedback control [mln33]
    - Balancing a heavy object on a light rod [mex110]
    - Logistic model (continuous version) [mln32]
    - Continuous logistic model [mex107]
    - Summary of properties [mln14]

5. Lagrangian Mechanics I
    - Challenges for Newtonian mechanics [mln75]

- Holonomic constraints [mln36]
- Example: disk rolling along incline [mln76]
- Differential constraints [mln37]
- Heading toward moving target [mex235]
- Newtonian mechanics in the presence of holonomic constraints [mln5]
- Plane pendulum I [mex132]
- Heavy particle sliding inside cone I [mex133]
- D'Alembert's principle [mln7]
- Plane pendulum II [mex134]
- Heavy particle sliding inside cone II [mex135]
- Plane pendulum III: librations [mex146]
- Plane pendulum IV: separatrix motion and rotations [mex147]
- Lagrange equations derived from D'Alembert's principle [mln8]
- Simple applications of Lagrangian mechanics [mln77]
- Invariance under point transformations of Lagrange equations [mex79]
- Gauge invariance of Lagrange equations [mex21]
- Find a simpler Lagrangian [mex22]
- Lagrangian of plane double pendulum [mex20]
- Parabolic slide [mex131]
- Pendulum without gravity [mex25]
- Disk rolling on rotating track [mex116]
- Rotating and sliding [mex115]
- Pendulum under forced rotation [mex23]
- Pendulum with sliding pivot: Lagrange equations [mex24]
- Pendulum with sliding pivot: reduction to quadrature [mex233]
- Pendulum oscillations in rotating plane [mex39]
- Chain sliding off the edge of table without friction [mex148]
- Chain sliding off the edge of table with friction [mex149]

- Plane pendulum with periodically driven pivot I [mex248]
- Plane pendulum with periodically driven pivot II [mex249]
- Plane pendulum with periodically driven pivot III [mex250]
- Restoring force of elastic string [mex251]

6. Lagrangian Mechanics II
    - Constants of the motion [mln10]
    - Conservation laws and symmetry [mln11]
    - Kinetic energy in Lagrangian mechanics [mex155]
    - Spherical pendulum: reduction to quadrature [mex156]
    - Routhian function [mln39]
    - Routhian function for heavy particle sliding inside cone [mex157]
    - Routhian function of 2D harmonic oscillator [mex121]
    - Noether's theorem I [mln12]
    - Noether's theorem: translation in space [mex35]
    - Noether's theorem: rotation in space [mex36]
    - Noether's theorem II [mln13]
    - Noether's theorem: pure Galilei transformation [mex37]
    - Noether's theorem III [mln42]
    - Dissipative forces in Lagrangian mechanics [mln9]
    - Motion with friction on inclined plane [mex151]
    - Linearly damped spherical pendulum [mex158]
    - Generalized forces of constraint in Lagrangian mechanics [mln15]
    - Particle sliding down sphere (revisited) [mex34]
    - Static frictional force of constraint [mex32]
    - Normal force of constraint [mex33]
    - Particle sliding inside cone: normal force of constraint [mex159]

7. Lagrangian Mechanics III

- Calculus of variation [mln78]
- Shortest path between two points in a plane I [mex26]
- Economy plastic cup [mex27]
- Variational problems with auxiliary conditions [mln16]
- Isoperimetric problem [mex28]
- Catenary problem [mex38]
- Athletic refraction [mex29]
- Brachistochrone problem I [mex30]
- Brachistochrone problem II [mex31]
- Isochronous potential well [mex144]
- Geodesics [mln38]
- Shortest path between two point in a plane II [mex117]
- Geodesics on a sphere [mex118]
- Dynamical trap without potential energy [mex119]
- Vertical range of particle sliding inside cone [mex120]
- Extremum principles [msl20]
- Generalized forces of constraint and Hamilton's principle [mln17]
- Bead sliding down cylindrical spiral [mex160]
- Massive dimer on skates [mex122]
- Massive dimer skating on incline [mex161]
- Wave equation from Hamilton's principle [mex162]

8. Central Force Motion I
    - Central force motion: two-body problem [mln66]
    - Central force motion: one-body problem [mln67]
    - Central force problem: formal solution [mln18]
    - Orbits of power-law potentials [msl21]
    - Unstable circular orbit [mex51]
    - Orbit of the inverse-square potential at large angular momentum

[mex46]

- Orbit of the inverse-square potential at small angular momentum [mex47]
- In search of some hyperbolic orbit [mex41]
- Virial theorem [mln68]
- Changing orbit by brief rocket boost [mex163]
- Discounted gravity: 50
- Bounded orbits open or closed [mln79]
- Bertrand's theorem [mln44]
- Stability of circular orbits [mex53]
- Small oscillations of radial coordinate about circular orbit [mex125]
- Angle between apsidal vectors for nearly circular orbits [mex126]
- Robustness of apsidal angles [mex127]
- Apsidal angle reinterpreted [mex128]
- Apsidal angle at very high energies [mex129]
- Apsidal angle at very low energies [mex130]

9. Central Force Motion II
    - Kepler's laws of planetary motion [msl22]
    - Orbits of Kepler problem [msl23]
    - Motion in time on elliptic orbit [mln19]
    - Cometary motion on parabolic orbit [mex44]
    - Cometary motion on hyperbolic orbit [mex234]
    - Close encounter of the first kind [mex145]
    - Kepler's second and third laws [mex43]
    - Circular and radial motion in inverse-square law potential [mex164]
    - Circular orbit of the Yukawa potential [mex54]
    - Orbital differential equation [mln46]
    - Exponential spiral orbit [mex49]

- Orbital differential equation applied to the Kepler problem [mex48]
- Linear spiral orbit [mex52]
- Crash course on circular orbit [mex50]
- Laplace-Runge-Lenz vector [mln45]
- Precession of the perihelion [mln21]
- Precession of the perihelion: orbital integral [mex165]
- Precession of the perihelion: orbital differential equation [mex166]
- The comet and the planet [mex45]
- Free fall with or without angular momentum [mex42]
- Elliptic and hyperbolic orbits [mex169]

10. Scattering from Central Force Potential
    - Scattering from stationary central force potential [msl2]
    - Determination of the scattering angle [mln20]
    - Total cross section for shower of meteorites [mex 58]
    - Rutherford scattering formula [mex56]
    - Scattering from hard spheres [mex55]
    - Elastic scattering from hard ellipsoids [mex60]
    - Scattering cross section for inverse square potential [mex59]
    - Particle experiencing soft Coulomb kick [mex10]
    - Scattering angle in the laboratory frame [msl3]
    - Loss of kinetic energy in elastic collision [mex57]
    - Elastic collision: angle between scattered particles [mex240]
    - Elastic collision: velocities of scattered particles [mex241]
    - Mechanical refraction [mex167]
    - Scattering from a spherical potential well [mex168]
    - Grazing collision between flat surfaces [mex219]
    - Absorption cross section of power-law potential [mex242]
    - Small-angle scattering [mln105]

- Small-angle scattering from-power-law potential [mex246]
- Classical inverse scattering [mln104]
- Classical inverse scattering problem I [mex243]
- Classical inverse scattering problem II [mex244]
- Classical inverse scattering problem III [mex245]
- Decay of particle I [mln102]
- Decay of particle II [mln103]
- Decay of particle: maximum kinetic energy [mex237]
- Decay of particle: directions in lab frame I [mex238]
- Decay of particle: directions in lab frame II [mex239]

11. Dynamics in Rotating Frames of Reference
    - Motion in rotating frame of reference [mln22]
    - Effect of Coriolis force on falling object [mex61]
    - Effects of Coriolis force on an object projected vertically up [mex62]
    - Foucault pendulum [mex64]
    - Effects of Coriolis force and centrifugal force on falling object [mex63]
    - Lateral deflection of projectile due to Coriolis force [mex65]
    - Effect of Coriolis force on range of projectile [mex66]
    - What is vertical? [mex170]
    - Lagrange equations in rotating frame [mex171]
    - Holonomic constraints in rotating frame [mln23]
    - Parabolic slide on rotating Earth [mex172]

12. Rigid Body Dynamics I
    - Coordinate systems used in rigid body dynamics [mln24]
    - Rotational kinetic energy [mln25]
    - Translational and rotational kinetic energies [mex67]

- Kinetic energy of rolling cylinder [mex173]
- Principal axes of inertia [mln80]
- Parallel-axis theorem [mex69]
- Perpendicular-axis theorem [mex73]
- Inertia tensor of homogeneous cube [mex68]
- Principal moment of a solid cylinder [mex252]
- Principal moments of a solid sphere [mex253]
- Principal moments of a solid ellipsoid [mex254]
- Inertia tensor of four-atomic molecule [mex255]
- Inertia tensor of a cone [mex71]
- Simulating a stick by three point masses [mex143]
- Angular momentum [mln26]
- Eulerian angles of rotation [msl25]
- Eulerian angular velocities [msl26]
- Rotating rectangular box [mex174]
- Euler's equations [mln27]
- Heavy wheels [mex175]

13. Rigid Body Dynamics II
    - Torque-free motion of symmetric top [msl27]
    - Torque-free motion of asymmetric top [msl28]
    - Stability of rigid body rotations about principal axes [mex70]
    - Steady precession of symmetric top [mex176]
    - Heavy symmetric top: general solution [mln47]
    - Heavy symmetric top: steady precession [mln81]
    - Heavy symmetric top: precession and nutation [msl49]
    - Stability of sleeping top [mex177]
    - Cube standing on edge [mex72]
    - Rolling pendulum [mex178]

- Cone on the roll [mex74]
- Make the billiard ball roll [mex4]
- From sliding to rolling motion [mex220]
- Rolling inhomogeneous disk [mex179]
- Balancing act of board on cylinder [mex75]
- Falling flat [mex256]
- Rod off balance [mex258]
- Solid sphere rolling on plane [mln106]
- Solid sphere rolling on plane [mex260]

14. Oscillations
    - Damped harmonic oscillator [mln6]
    - Harmonic oscillator with friction [mex150]
    - Harmonic oscillator with attenuation [mex261]
    - Driven harmonic oscillator I [mln28]
    - Amplitude resonance and phase angle [msl48]
    - Driven harmonic oscillator: steady state solution [mex180]
    - Driven harmonic oscillator: kinetic and potential energy [mex181]
    - Driven harmonic oscillator: power input [mex182]
    - Quality factor of damped harmonic oscillator [mex183]
    - Driven harmonic oscillator: runaway resonance [mex262]
    - Driven harmonic oscillator II [mln29]
    - Fourier coefficients of a sawtooth force [mex184]
    - Fourier coefficients of periodic seuence of rectangular pulses [mex185]
    - Driven harmonic oscillator III [mln107]
    - Driven harmonic oscillator with Coulomb damping [mex263]
    - Small oscillations [mln43]
    - Transformation to principal axes [mln30]
    - Elastic chain [mln48]

- Blocks and springs in series [mex123]
- Two coupled oscillators [mex186]
- Three coupled oscillators [mex187]
- What is the physical nature of these modes? [mex114]
- Small oscillations of the double pendulum [mex124]

15. Hamiltonian Mechanics
    - Legendre transform [tln77]
    - Hamiltonian and canonical equations [mln82]
    - Lagrangian from Hamiltonian via Legendre transform [mex188]
    - Can you find the Hamiltonian of this system? [mex189]
    - Variational principle in phase space [mln83]
    - Properties of the Hamiltonian [mln87]
    - When does the Hamiltonian represent the total energy? [mex81]
    - Hamiltonian: conserved quantity or total energy? [mex77]
    - Bead sliding on rotating rod in vertical plane [mex78]
    - Use of cyclic coordinates in Lagrangian and Hamiltonian mechanics [mln84]
    - Velocity-dependent potential energy [mln85]
    - Charged particle in electromagnetic field [mln86]
    - Velocity-dependent central force [mex76]
    - Charged particle in a uniform magnetic field [mex190]
    - Particle with position-dependent mass moving in 1D potential [mex88]
    - Pendulum with string of slowly increasing length [mex89]
    - Librations between inclines [mex259]

16. Canonical Transformations
    - Point transformations (of coordinates in configuration space) [mln88]
    - Effect of point transformation on Hamiltonian [mex80]

- Effect of point transformation on canonical equations [mex82]
- Hamiltonian of free particle in rotating frame [mex193]
- Canonical transformations (of coordinates in phase space) [mln89]
- Canonicity and volume preservation [mln90]
- Determine canonicity and generating function I [mex87]
- Determine canonicity and generating function II [mex90]
- Determine canonicity and generating function III [mex194]
- Determine canonicity and generating function IV [mex198]
- Infinitesimal canonical transformations [mln91]
- Canonicity of time evolution: Liouville theorem [tln45], [tln46]
- Canonicity of gauge transformation [mex195]
- Electromagnetic gauge transformation [mex196]
- Check the canonicity of coordinate transformations [mex84]
- Time-dependent generating functions [mex83]
- Canonical transformation from rest frame to moving frame [mex85]
- Canonical transformation applied to harmonic oscillator [mex86]

17. Action-Angle Coordinates
    - Action-angle coordinates [mln92]
    - Actions and angles for librations [mln93]
    - Actions and angles for rotations [mln94]
    - Action-angle coordinates of the harmonic oscillator [mex91]
    - Action-angle coordinates of an anharmonic oscillator [mex92]
    - Unbounded motion in piecewise constant periodic potential [mex96]
    - Unbounded motion in piecewise linear periodic potential [mex93]
    - Bounded motion in piecewise constant periodic potential [mex95]
    - Poisson brackets [msl30]
    - Specifications of Hamiltonian system [mln95]
    - Poisson's theorem [mex191]

- Poisson brackets of angular momentum variables [mex192]
- Action-angle coordinates of plane pendulum: librations [mex200]
- Hamiltonian system specified by noncanonical variables [mex94]
- Generating a pure Galilei transformation [mex197]
- Exponential potential [mex199]

18. Hamilton-Jacobi Theory
    - Hamilton's principal function [mln96]
    - Hamilton's characteristic function [mln97]
    - Hamilton-Jacobi equation for the harmonic oscillator [mex97]
    - Hamilton's principal function for central force problem [mex98]
    - Hamilton's characteristic function for central force problem [mex99]
    - Particle in time-dependent field [mex201]
    - Hamilton-Jacobi theory for projectile motion [mex202]

19. Deterministic Chaos
    - Dissipative dynamical systems [mln101]
    - Fixed points in 3D phase flow [msl16]
    - Limit cycles in 3D phase flow [msl17]
    - Toroidal attractor in 3D phase flow [msl18]
    - Strange attractor in 3D phase flow: Roessler band [msl19]
    - Integrability as a universal property [mln98]
    - Integrability as a contingent property [mln99]
    - Poincar surface of section [mln100]
    - Summary of properties [msl15]
    - Toda system (integrable) [msl12]
    - Henon-Heiles system (nonintegrable) [msl13]
    - Introduction to Hamiltonian chaos [mln108]

20. Relativistic Mechanics I
    - Relativistic versus Newtonian mechanics [mln49]
    - Relativity of space and time [mln50]
    - Relativity of simultaneity [mln51]
    - Time dilation paradox [mln52]
    - Length contraction paradox [mln53]
    - Hello Earth [mex207]
    - Who passes more quickly? [mex208]
    - Time on the fly [mex236]
    - Pion decay in accelerator [mex209]
    - Interstellar travel [mex210]
    - TGV [mex211]
    - Minkowski diagram I: relativity of simultaneity [mln54]
    - Minkowski diagram II: length contraction and time dilation [mln55]
    - Twin paradox [mln56]
    - Longitudinal Doppler effect [mln57]
    - Optical birthday cards [mex212]
    - Two views of an event [mex213]
    - Hello Earth again [mex214]

21. Relativistic Mechanics II
    - Coordinate transformations [mln58]
    - Relative and absolute [mln59]
    - Lorentz transformation I [mex215]
    - Lorentz transformation II [mex216]
    - Observing transverse motion of meter stick [mln60]
    - Skater's paradox [mln61]
    - Skate mail fallacy [mex217]
    - Interstellar speed control [mex218]

- Mass and energy [mln62]
- Relativistic momentum [mln63]
- Momentum conservation [mex221]
- Relativistic mass [mex222]
- Relativistic energy I [mln64]
- Relativistic energy II [mln65]
- Photon rocket [mex223]
- Photon absorption and photon emission [mex224]
- K meson decay [mex225]

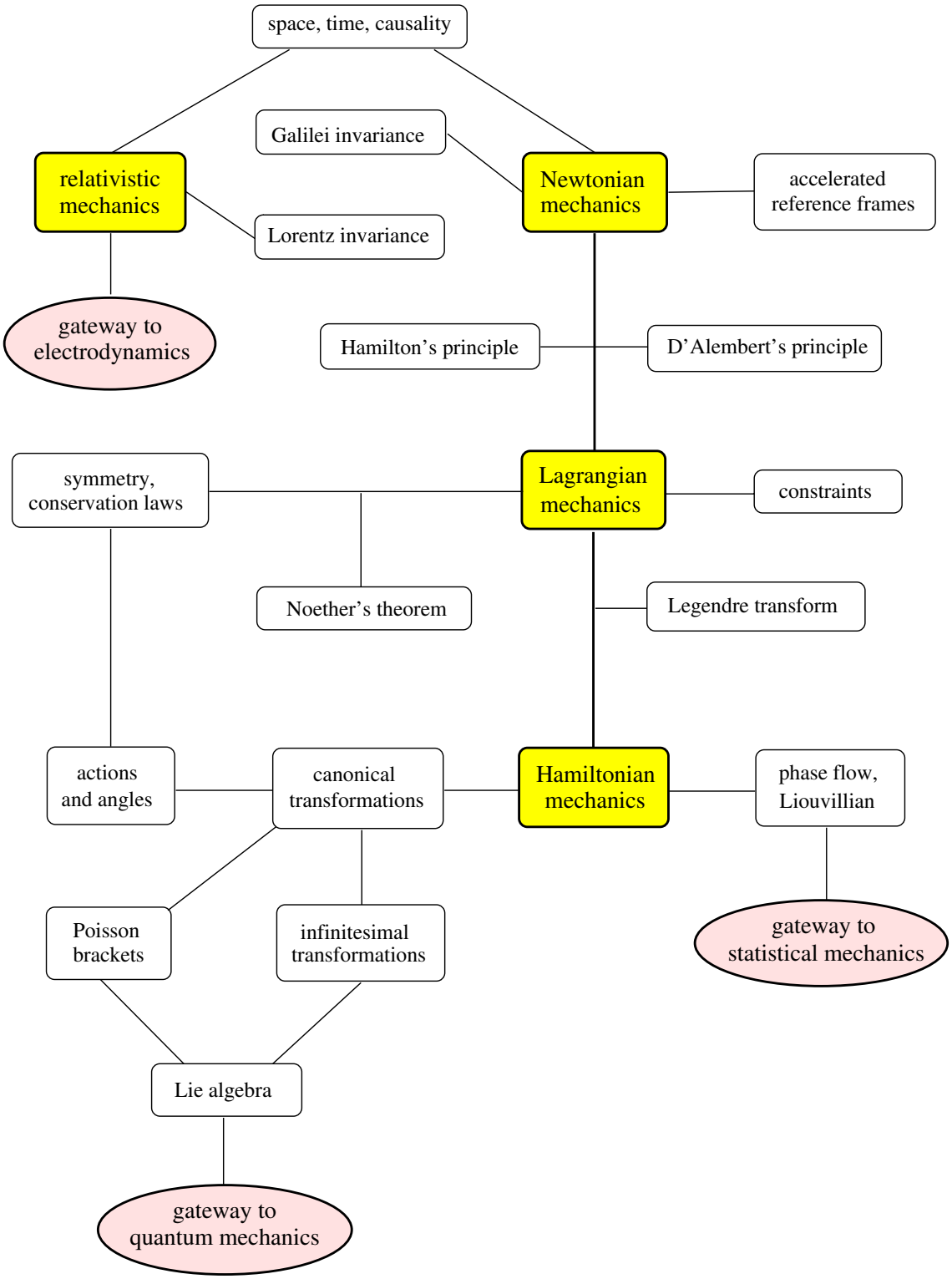
Contents of this Document [mtc1]

1. Newtonian Mechanics

- Table of contents (entire course) [mtc]
- Classical mechanics overview [mln69]
- Space and time – Galilei's principle of relativity – Newton's laws of dynamics [mln1]
- Impact of symmetry [mln70]
- Conservation laws [mln2]
- The shortest path is not the quickest path [mex100]
- Minimizing time of slide when friction is present [mex154]
- Optimized time of travel [mex136]
- Acceleration from clocking consecutive space intervals [mex137]
- Particle sliding down a sphere [mex1]
- Time of slide and time of flight [mex102]
- Atwood machine [mex9]
- On frozen pond [mex204]
- The quick, the short, and the scenic [mex205]
- When push comes to shove [mex206]
- Rubber speed [mex138]
- Water projected into air by wheel rolling on wet road [mex11]
- Design of a lawn sprinkler [mex113]
- Longest shot from the top of a hill [mex139]
- Lowest shot to target across hill [mex140]
- Reel of thread I: statics [mex141]
- Reel of thread II: dynamics [mex142]
- Spherical pendulum of varying length [mex226]
- Dragging block by elastic cord [mex227]

- Centripetal elevator [mex228]
- Lateral force on hanging chain [mex231]
- Let's meet again... and again [mex247]

Classical Mechanics Overview [mln69]



Newtonian Mechanics [mln1]

Space and Time

- Absolute space is 3-dimensional, homogeneous, and isotropic; the (Euclidean) metric is independent of the objects present in space.
- Absolute time is homogeneous.

Relativistic mechanics introduces major modifications regarding the properties of space and time.

Galilei's Principle of Relativity

There exist *inertial* coordinate systems:

- The laws of mechanics are the same in all inertial coordinate systems (invariance under Galilean transfomations).
- All coordinate systems in uniform rectilinear motion with respect to an inertial coordinate system are themselves inertial.

The laws of relativistic mechanics are invariant under Lorentz transformations. In general relativity, the restriction to inertial systems is removed.

Newton's Laws of Dynamics

1. A body remains at rest or in uniform rectilinear motion unless acted upon by a force.
2. A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force exerted on it: $d\mathbf{p}/dt = \mathbf{F}$.
3. If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction (action-reaction pair of forces).

Newton's second law states the (deterministic) relation between *cause* and *effect*. Any deterministic forecast depends on precise knowledge of initial conditions and forces.

The absolute determinism of Newtonian mechanics were subsequently undermined from two sides: by *chaos theory* and by *quantum mechanics*.

Impact of Symmetry [mln70]

Classical mechanics:

- *Continuous* symmetries have major impact on dynamics.
- *Discrete* symmetries have minor impact on dynamics in comparison.

Quantum mechanics:

- *Continuous* and *discrete* symmetries have comparable impact on spectrum and transition rates.

Inertial reference frame:

- Classical mechanics rests on the assumption that it is always possible to find a frame of reference with respect to which space is homogeneous and isotropic and time is homogeneous.

Consequences:

continuous symmetry	conservation law
space is homogeneous	linear momentum
space is isotropic	angular momentum
time is homogeneous	mechanical energy

The general relationship between continuous symmetries and conservation laws in classical mechanics is described by Noether's theorem [mln12], [mln13], [mln42].

Conservation Laws [mln2]

Single Particle

- The component of *linear momentum* $\mathbf{p} = m\mathbf{v}$ in a direction (specified by vector $\mathbf{s}$) in which the applied force $\mathbf{F}$ vanishes is a constant in time:

$$\dot{\mathbf{p}} \cdot \mathbf{s} = \mathbf{F} \cdot \mathbf{s}.$$

- The *angular momentum* $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$ is a constant in time if the applied force $\mathbf{F}$ exerts zero torque $\mathbf{N}$:

$$\dot{\mathbf{L}} \equiv \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F} = \mathbf{N}.$$

- If the applied force $\mathbf{F}$ is *conservative*, then the *total energy* E, which is the sum of the *kinetic energy* T and *potential energy* V, is a constant in time:

$$E = T + V; \qquad T = \frac{1}{2}mv^2, \quad V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{s}, \quad \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}).$$

System of Particles

External force: $\mathbf{F}^{(e)} = \sum_i \mathbf{F}_i^{(e)}$. Internal forces: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ with $\mathbf{F}_{ij} \| \mathbf{r}_{ij}$.

- The component of *total linear momentum* $\mathbf{p}$ in a direction in which the *external force* $\mathbf{F}^{(e)}$ vanishes is a constant in time:

$$\dot{\mathbf{p}} \cdot \mathbf{s} = \mathbf{F}^{(e)} \cdot \mathbf{s}.$$

- The *total angular momentum* $\mathbf{L}$ is a constant in time if the external force $\mathbf{F}^{(e)}$ exerts zero torque $\mathbf{N}^{(e)}$:

$$\dot{\mathbf{L}} \equiv \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F}^{(e)} = \mathbf{N}^{(e)}.$$

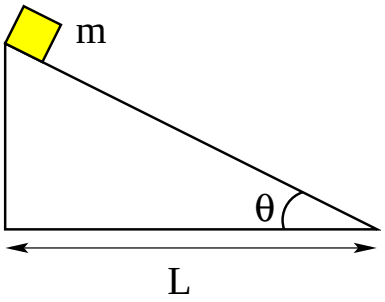
- If the forces $\mathbf{F}^{(e)}$ and $\mathbf{F}_{ij}$ are *conservative*, then the *total (mechanical) energy* E of the system is a constant in time:

$$E = T + V; \qquad T = \sum_i \frac{1}{2}m_i v_i^2, \quad V = \sum_i V_i^{(e)} + \sum_{i<j} V_{ij}.$$

Non-conservative forces (friction, attenuation) imply energy dissipation. Some mechanical energy is then converted into thermal energy or radiation.

[mex100] The shortest path is not the quickest path

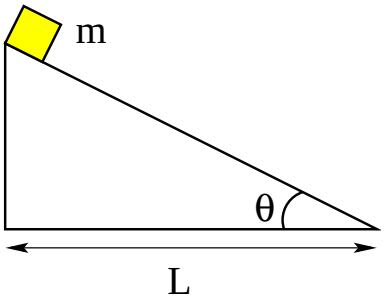
A block of mass m slides from rest down a ramp with base of fixed length L as shown. Friction is assumed to be negligibly small. Find the angle θ for which the block arrives at the bottom of the ramp in the shortest time.



Solution:

[mex154] Minimizing time of slide when friction is present

A block of mass m slides from rest down a ramp with base of fixed length L as shown. The motion is impeded by a frictional force $f = \mu N$, where N is the normal force and μ is the coefficient of kinetic friction. Find the angle θ for which the block arrives at the bottom of the ramp in the shortest time.



Solution:

[mex136] Optimized time of travel

The maximum acceleration of a train is α and its maximum deceleration is β. Show that it cannot run a distance d from rest to rest in a shorter time than

$$t = \sqrt{\frac{2d(\alpha + \beta)}{\alpha\beta}}.$$

Solution:

[mex137] Acceleration from clocking consecutive space intervals

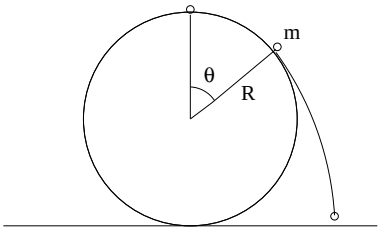
A body moving in a straight line with constant acceleration passes two consecutive equal spaces, each of length d, in times $\Delta t_1, \Delta t_2$. The body is already moving when it enters the first space. Show that the acceleration is

$$a = \frac{2d(\Delta t_1 - \Delta t_2)}{\Delta t_1 \Delta t_2 (\Delta t_1 + \Delta t_2)}.$$

Solution:

[mex1] Particle sliding down sphere

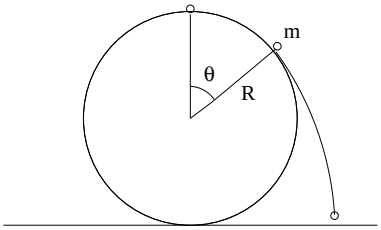
A tiny particle of mass m slides without friction down a spherical surface of radius R. The particle starts at the top with negligible speed.
(a) Identify all forces that act on the particle as it slides along the surface of the sphere.
(b) Calculate the polar angle θ at which the path of the particle leaves the sphere.



Solution:

[mex102] Time of slide and time of flight

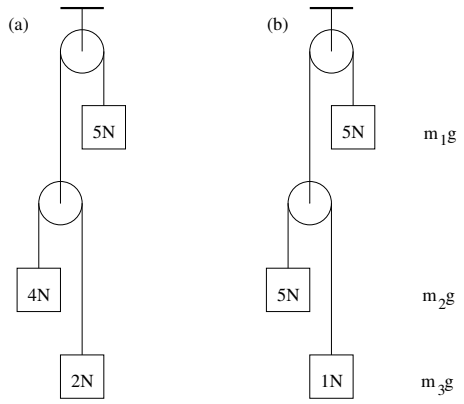
Consider the problem already encountered in [mex1]. The particle starts to slide from rest at $\theta = 0$.
(a) Find the time it takes the particle to slide from angular position $\theta_0 > 0$ to angular position θ_c, where it takes off.
(b) Find (for $R = 10\text{m}$ and $g \simeq 10\text{m/s}^2$) the time between take-off from the sphere and landing on the plane.



Solution:

[mex9] Atwood machine

Consider two Atwood machines with different weights. The pulleys and ropes have negligible mass. All weights and pulleys are initially at rest. Use $g \simeq 9.8\text{ms}^{-2}$. Determine whether the lower pulley will start to move up or down in each case. Calculate the tensions T_1 and T_2 in the upper and lower ropes, respectively. State the final results in SI units with two digits accuracy.



Solution:

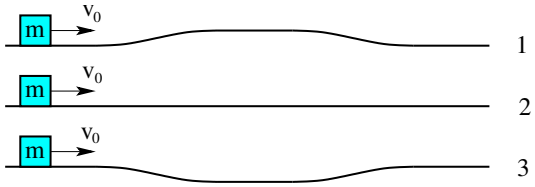
[mex204] On frozen pond

A father ($m_1 = 70$kg) and his son ($m_2 = 35$kg) are standing on the ice in the middle of a pond. The coefficient of static friction between boots and ice is the same for both persons. Who will win the race (from rest) to the edge of the pond...
(a) if both runners move as fast as possible without slipping,
(b) if each runner drags a sled ($m_s = 10$kg) by a cord, where the frictional force acting on the sled is negligibly small?
Justify your answers carefully.

Solution:

[mex205] The quick, the short, and the scenic

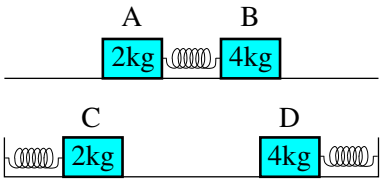
Consider three blocks of equal mass m sliding along frictionless tracks. Track 2 is strictly horizontal. Track 1 leads over a hill and track 3 through a valley. The blocks are simultaneously launched with equal initial velocity v_0 at the positions shown. The curvatures in tracks 1 and 3 are sufficiently gentle for the blocks not to lift off. In what sequence do the blocks arrive at the end of their tracks? Justify your answers carefully.



Solution:

[mex206] When push comes to shove

Consider two situations in which blocks of unequal mass initially at rest on frictionless tracks are accelerated by springs of equal stiffness and with equal initial compression as shown.
(a) Which block, A or B, C or D, has acquired more momentum when the spring in contact with it has become fully relaxed?
(b) Which block, A or B, C or D, has acquired more kinetic energy when the spring in contact with it has become fully relaxed?
Justify your answers carefully.



Solution:

[mex138] Rubber speed

A car is moving with constant velocity v along a level road. Find the instantaneous speed v_T of a point on the tread (perimeter) of the tire, which has radius R. Express v_T as a function of y, where $0 \leq y \leq 2R$ is the distance of the point on the tire from the surface of the road.

Solution:

[mex11] Water projected into air by wheel rolling on wet road

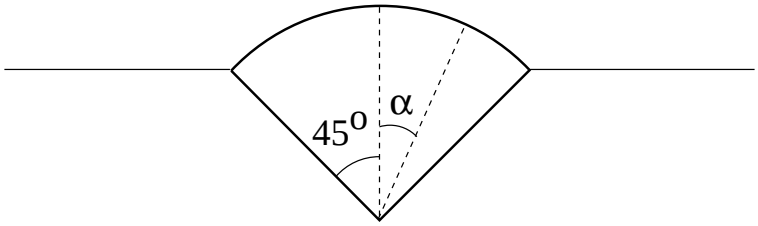
A wheel of radius R, which is part of some vehicle, is rolling along a wet road with speed v. Drops of water are being continuously thrown off the rim. Show that if $v^2 > Rg$ then the maximum height above the road attained by the drops is

$$R + \frac{v^2}{2g} + \frac{R^2 g}{2v^2}.$$

Solution:

[mex113] Design of a lawn sprinkler

The lawn sprinkler is to consist of a spherical cap with cross section as shown. The spherical surface is perforated with a large number of small holes through which water jets are ejected with initial speed v_0. How must the holes be distributed so that a circular area around the sprinkler is watered uniformly? Express this distribution a function $n(\alpha)$ describing the number of holes per unit area on a ring at angle α from the vertical. The size of the cap is small compared to the area being sprinkled and the cap is about level with the lawn. Air resistance is assumed to be negligible.



Solution:

[mex139] Longest shot from the top of a hill

A gun is mounted on a hill of height h above a level plain. Assuming that the muzzle speed is v_0 and that the path of the projectile is parabolic, show that the angle of elevation α for greatest horizontal range depends on h and v_0 as follows:

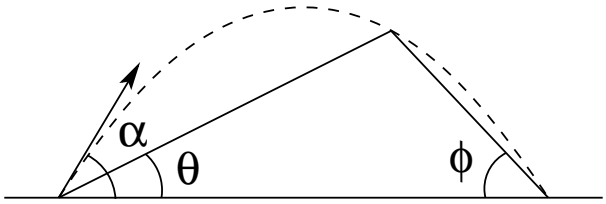
$$\frac{1}{\sin^2 \alpha} = 2\left(1 + \frac{gh}{v_0^2}\right).$$

Solution:

[mex140] Lowest shot to target across hill

Consider a hill consisting of two straight slopes with inclinations θ and ϕ. A projectile is fired along a parabolic trajectory from the foot of the hill on one side to the foot of the hill on the opposite side in such a way that it grazes the summit. Show that the angle of projection must be

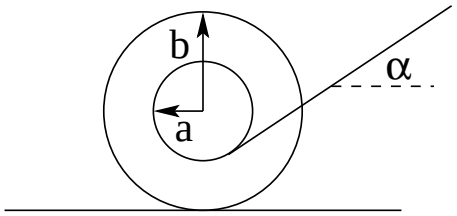
$$\alpha = \arctan(\tan\theta + \tan\phi).$$



Solution:

[mex141] Reel of thread I: statics

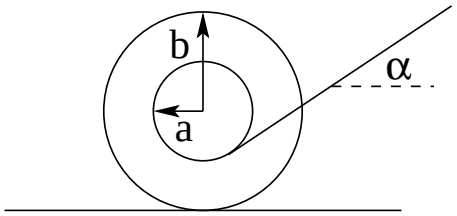
A reel of thread of weight W whose spindle and rim have radii a and b, respectively, rests on a horizontal table. The loose end of the thread passes under the spindle and leads off at an angle α above the horizontal as shown. The static frictional force between the reel and the table is $f \leq \mu_S N$, where N is the normal force and μ_S is a constant.
(a) Find the angle α_c at which a static equilibrium exists for nonzero tension T in the thread.
(b) Find the maximum value T_c of the tension for which the equilibrium holds.



Solution:

[mex142] Reel of thread II: dynamics

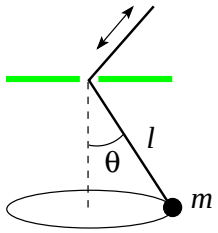
A reel of thread whose spindle and rim have radii a and b, respectively, rests on a horizontal table. The weight of the reel is mg and the moment of inertia for rotations about its axis is I. The loose end of the thread passes under the spindle and leads off at an angle α above the horizontal as shown. The static frictional force between the reel and the table during rolling motion is $f \leq \mu_S N$, where N is the normal force and μ_S is a constant. Consider the range $0 \leq \alpha \leq \pi$ of angles.
(a) For a given tension not too strong to make the reel roll without slipping, find the angular acceleration $\dot{\omega}$, the frictional force f, and the normal force N.
(b) For the three cases $\alpha = 0, \pi/2, \pi$ find the direction (clockwise or counterclockwise) of the angular acceleration $\dot{\omega}$ and the direction (left or right) of the frictional force f.
(c) For the three cases $\alpha = 0, \pi/2, \pi$ find the maximum possible value of $|\dot{\omega}|$ for rolling without slipping.



Solution:

[mex226] Spherical pendulum of slowly varying length

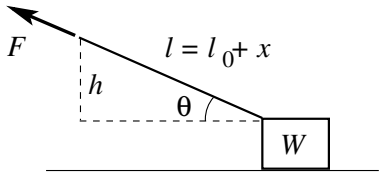
A small mass m rotates in a horizontal circle at the end of a string of length l at constant angle θ from the vertical as shown. When the length l is slowly varied the angle θ changes such that $f(l, \theta) = \text{const}$. Find the function $f(l, \theta)$.



Solution:

[mex227] Dragging block by elastic cord

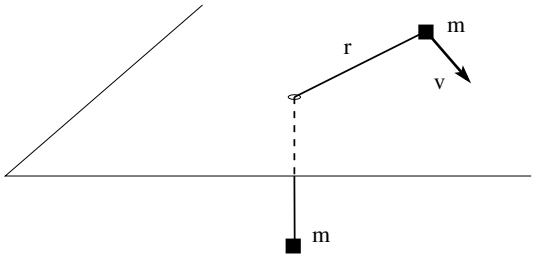
A block of weight $W = 5$N is being dragged along a horizontal surface against kinetic friction with coefficient $\mu = 0.2$. Find the force F (in SI unit) needed to keep the block moving at constant speed if the upper end of the elastic cord is held at constant height $h = 1$m. The length of the cord is $l = l_0 + x$, where $l_0 = 1$m and $F = kx$ with $k = 1$N/m.



Solution:

[mex228] Centripetal elevator

Two small blocks of mass m are connected by a string of length $2l$, which passes through a hole in a horizontal table. One block is free to slide on the table and the other block hangs underneath. In the initial state the block on the table is a distance $r = l$ from the hole and projected with velocity v horizontally and at right angle to the string. The hanging block is initially at rest. Ignore friction. (a) Find v such that the hanging block stays at rest. (b) Find v such that the hanging block moves up but comes to rest at the hole without going through the hole or touching the table.



Solution:

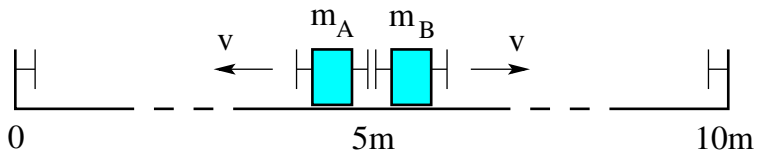
[mex231] Lateral force on hanging chain

Consider a chain of N links in the form of uniform rods of mass m and length $2l$ each, connected by frictionless hinges. At one end the chain is attached to a stationary pivot. The other end is pulled sideways by a constant force F. The links are numbered N through 1 from the pivot outwards. Find the angle θ_n between link n and the horizontal for $n = 1, 2, \ldots, N$ at equilibrium.

Solution:

[mex247] Let's meet again... and again

Two blocks of masses m_A and m_B, positioned in the middle (at $x_0 = 5m$) of a horizontal air track, are launched with equal speed v in opposite directions. They bounce off the ends of the track and off each in a succession of elastic collisions. The track does not recoil significantly and the blocks are of very small size compared to the length of the track. Find the locations x_1 and x_2 on the track where the two blocks collide the first and second times, respectively, if (i) $m_A = 2$kg, $m_B = 4$kg and (ii) $m_A = 2$kg, $m_B = 3$kg.



Solution: