2016

Global Dynamics of Some Quadratic Difference Equations

Mark Dipippo
University of Rhode Island, mdipippo99@gmail.com

GLOBAL DYNAMICS OF SOME QUADRATIC DIFFERENCE EQUATIONS

BY

MARK DIPIPPO

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

UNIVERSITY OF RHODE ISLAND

2016

DOCTOR OF PHILOSOPHY DISSERTATION

OF

MARK DIPIPPO

APPROVED:

Dissertation Committee:

Major Professor Mustafa Kulenovic

Orlando Merino

Koray Ozpolat

Nasser H. Zawia

DEAN OF THE GRADUATE SCHOOL

UNIVERSITY OF RHODE ISLAND

2016

ABSTRACT

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{n-i} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}}, \quad n = 0, 1, \dots$$

where all parameters $\alpha, \beta, a_i, b_i, a_{ij}, b_{ij}, i, j = 0, 1, \dots, k$ and the initial conditions $x_i,\ i \in \{-k, \dots, 0\}$ are nonnegative. We investigate the asymptotic behavior of the solutions of the considered equation. We give simple explicit conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation.

We investigate the global dynamics of several anti-competitive systems of rational difference equations which are special cases of general linear fractional system of the form

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}, \quad n = 0, 1, ...,$$

where all parameters and the initial conditions x_0, y_0 are arbitrary nonnegative numbers such that both denominators are positive. We find the basins of attraction of all attractors of these systems.

We investigate global dynamics of the equation

$$x_{n+1} = \frac{x_{n-1}^2}{a x_n^2 + c x_{n-1}^2 + f}, \quad n = 0, 1, 2, ...,$$

where the parameters a, c and f are nonnegative numbers with condition $a+e+f > 0$ and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$.

ACKNOWLEDGMENTS

I would like to thank my adviser professor M.R.S. Kulenovic who has been and continues to be a great source of inspiration. Professor Kulenovic's knowledge, experience, and support has been vital to my success in the field of difference equations. What I learned from professor Kulenovic transcends mathematics.

I would also like to thank professor Orlando Merino and the difference equations group at URI, their support and feedback has been invaluable.

I would also like to thank collaborators E. J. Janowski and Ann Brett for their support and collaboration on past and future projects.

DEDICATION

I would like to dedicate this dissertation to Lynn, Sophia, and Victoria, without your love and support none of this would be possible. I would also like to dedicate this dissertation to my late mother Shirley DiPippo for giving me the strength to accomplish anything.

PREFACE

This thesis has been prepared using the manuscript format.

Manuscript 1 of this thesis was published as:

M. DiPippo, E. J. Janowski and M. R. S. Kulenović, Global Asymptotic Stability for Quadratic Fractional Difference Equation, *Advances in Difference Equations*, (2015), 2015:179, 13 p.

Manuscript 2 of this thesis was published as:

M. DiPippo and M. R. S. Kulenović, Global Dynamics of Three Anticompetitive Systems of Difference Equations in the Plane, *Discrete Dynamics in Nature and Society*, (2013), 11p. Article ID 751594.

Manuscript 3 of this thesis is the paper in preparation.

TABLE OF CONTENTS

Page

LIST OF FIGURES

Figure **Page**

MANUSCRIPT 1

Global Asymptotic Stability for Quadratic Fractional Difference Equation

Published in *Advances in Difference Equations*, (2015), 2015:179, 13 p.

M. DiPippo, E. J. Janowski and M. R. S. Kulenović

Department of Mathematics

University of Rhode Island

Kingston, RI, 02881, USA

1.1 Introduction

Consider the difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{n-i} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}}, \quad n = 0, 1, \ldots \tag{1}$$

where $k \in \{0, 1, \ldots\}$, the parameters $\alpha, \beta, a_i, b_i, a_{ij}, b_{ij}, i, j = 0, 1, \ldots, k$ and the initial conditions $x_i,\ i \in \{-k, \ldots, 0\}$ are nonnegative and such that the denominator of Eq.(1) is always positive. The important special cases of Eq.(1) are the linear fractional equations such as well-known Riccati equation

$$x_{n+1} = \frac{\alpha + a_0 x_n}{\beta + b_0 x_n}, \quad n = 0, 1, \ldots, \tag{2}$$

the second order linear fractional difference equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{1} a_i x_{n-i}}{\beta + \sum_{i=0}^{1} b_i x_{n-i}}, \quad n = 0, 1, \ldots, \tag{3}$$

and the third order linear fractional difference equation that we get from Eq.(1) for $k = 2$ and $a_{ij} = b_{ij} = 0$ for all i, j. The global behavior and the exact solutions of Eq.(2) even for real parameters have been found in [18]. The global behavior of solutions of Eq.(3), in many subcases when one or more parameters are zero, was established in [18]. There is still one conjecture left whose answer will complete the global picture of the asymptotic behavior of solutions of Eq.(3). As far as the third order linear fractional difference equation is concerned there is a large number of sporadic results that are systemized in a book [6]. The characterization of the global asymptotic behavior of solutions of Eq.(1) for $k = 2$ seems to be much harder than for the second order equation (3). Consequently an attempt at giving the characterization of the global asymptotic behavior of solutions of Eq.(1) seems to be a formidable task at this time. However by using some known global attractivity results we can describe the global asymptotic behavior of solutions of Eq.(1) in some subspaces of the parametric space and the space of initial conditions.

See [4, 5, 6, 21] for a complete description of the behavior of some special cases of Eq.(1), in particular for the cases known as periodic trichotomies. See [14] where the difference in global behavior between the second and third order linear fractional difference equation is emphasized. The results on the global periodicity, that is the results which describe all special cases of Eq.(1) where all solutions are periodic of the same period were obtained in [1, 2]. Most results in [4, 5, 6, 10, 11, 21] are based on known global attractivity or global asymptotic stability results obtained in [6, 17, 18, 19, 20, 22].

The special case of Eq.(1) with quadratic terms such as

$$x_{n+1} = \frac{ax_n^2}{x_n + bx_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{4}$$

where $a, b > 0$ and the initial conditions $x_{-1}, x_0 \geq 0, x_{-1} + x_0 > 0$, has only a negative equilibrium point for $a \leq 1$ and yet all solutions of Eq.(4) satisfy

$$\lim_{n\to\infty} x_n = 0.$$

A k-th order generalization of Eq.(4) with the same property is

$$x_{n+1} = \frac{ax_n^2}{x_n + \sum_{i=1}^{k-1} b_i x_{n-i}^2}, \quad n = 0, 1, \ldots, \tag{5}$$

where $a, b_i > 0$ and the initial conditions $x_{-k+1}, \ldots, x_0 \geq 0, x_{-k+1} + \ldots + x_0 > 0$, when $a \leq 1$.

Another special case of Eq.(1) with quadratic terms is

$$x_{n+1} = \frac{Bx_n x_{n-1}}{dx_n + ex_{n-1}}, \quad n = 0, 1, \ldots, \tag{6}$$

where $B, d, e > 0$ and the initial conditions $x_{-1}, x_0 \geq 0, x_{-1} + x_0 > 0$, does not have an equilibrium point for $B \neq d + e$ and yet all solutions of Eq.(6) satisfy

$$\lim_{n\to\infty} x_n = 0, \quad \text{when } B < d + e,$$

and

$$\lim_{n\to\infty} x_n = \infty, \quad \text{when } B > d+e.$$

Finally, when $B = d + e$, then Eq.(6) has an infinite number of the equilibrium points, each with its basin of attraction, see [13].

Another interesting case of Eq.(1) with quadratic terms is the equation

$$x_{n+1} = \frac{ax_n^2}{1+x_n^2}, \quad n = 0, 1, \ldots, \tag{7}$$

where $a > 0$ and $x_0 \in \mathbb{R}$. When $a > 2$ every solution of Eq.(7) converges either to the zero equilibrium or to the bigger positive equilibrium $x_+ = \frac{a+\sqrt{a^2-4}}{2}$, with basins of attraction being $\mathcal{B}(0) = [0, x_-)$ and $\mathcal{B}(x_+) = (x_-, \infty)$, where $x_- = \frac{a-\sqrt{a^2-4}}{2}$ is the smaller positive equilibrium.

None of these asymptotic behaviors which are present in the cases of Equations (4)-(7) are possible in the case of the linear fractional equation

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i}}{\beta + \sum_{i=0}^{k} b_i x_{n-i}}, \quad n = 0, 1, \ldots$$

and the appearance of these behaviors is caused by the presence of the quadratic terms.

This paper is an attempt at establishing some global stability results for the equilibrium solution(s) of Eq.(1). Our results give effective conditions for global asymptotic stability of the equilibrium solution(s) of Eq.(1) expressed in terms of the inequalities on the parameters.

The following general global results will be applied to Eq.(1), see [12].

1.2 Preliminaries

Consider the difference equation

$$x_{n+1} = f(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{8}$$

where $k \in \{0, 1, \ldots\}$. Sometimes it is more advantageous to investigate Eq.(8) by embedding Eq.(8) into a higher iteration of the form

$$x_{n+l} = F(x_{n+l-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{9}$$

where $l \in \{1, 2, \ldots\}$, see [7, 12] and then linearizing (rearranging) Eq.(9) so that it has the form

$$x_{n+l} = \sum_{i=1-l}^{k} g_i x_{n-i}, \quad n = 0, 1, \ldots \tag{10}$$

where the functions $g_i : \mathbb{R}^{k+l} \to \mathbb{R}$.

Theorem 1 *Let $l \in \{1, 2, \ldots\}$ and let $a \in \mathbb{R}$. Suppose that Eq.(8) has the linearization (10) where the functions $g_i : \mathbb{R}^{k+l} \to \mathbb{R}$ are such that*

$$\sum_{i=1-l}^{k} |g_i| \leq a < 1, \quad n = 0, 1, \ldots \ . \tag{11}$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

As we have observed in [12], condition (11) is actually a contraction condition in the Banach contraction principle.

In addition, we will need the following stability result from [3].

Theorem 2 *Suppose that Eq.(8) can be linearized into the form*

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} g_i (x_{n-i} - \bar{x}), \quad n = 0, 1, \ldots \tag{12}$$

where $\bar{x}$ is an equilibrium of Eq.(8) and the functions $g_i : \mathbb{R}^{k+1} \to \mathbb{R}$. If $\sum_{i=0}^{k} |g_i| \leq 1$, $n \geq 0$, then the equilibrium $\bar{x}$ of Eq.(8) is stable.

The next result follows from Theorems 1 and 2.

Corollary 1 *Let $a \in \mathbb{R}$. Suppose that Eq.(8) has the linearization* (12), *where $\bar{x}$ is a unique nonnegative equilibrium of Eq.(8) on the interval I and the functions $g_i : \mathbb{R}^{k+1} \to \mathbb{R}$ are such that*

$$\sum_{i=0}^{k} |g_i| \leq a < 1 \quad n = 0, 1, \ldots,$$

then the unique nonnegative equilibrium Eq.(8) is globally asymptotically stable on the interval I.

The next result is an analogue of the result obtained in [12].

Lemma 1 *Let $m \leq \bar{x} \leq M$ and $m \leq x_{N-i} \leq M, i = 0, 1, \ldots, k$ for some $N \in \{0, 1, \ldots\}$. Suppose that*

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} h_i (x_{n-i} - \bar{x}), \quad n = 0, 1, \ldots$$

where the nonnegative functions $h_i : [0, \infty)^{k+1} \to [0, \infty)$. Assume that for this N

$$\sum_{i=0}^{k} h_i \leq 1.$$

Then $m \leq x_{N+1} \leq M$.

1.3 Main results

In this section we investigate the stability of the unique positive equilibrium $\bar{x}$ of Eq.(1) by using Theorems 1 and 2. Observe that Eq.(1) has a zero equilibrium if and only if $\alpha = 0$ and $\beta > 0$, in which case Eq.(1) becomes

$$x_{n+1} = \frac{\sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} x_{n-i} x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}}, \quad n = 0, 1, \ldots. \tag{13}$$

Equation (13) yields the nonzero equilibrium points

$$\bar{x} = \frac{-\left(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij}\right) \pm D}{2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}},$$

where

$$D = \sqrt{\left(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}\right)^2 - 4(\beta - \sum_{i=0}^{k} a_i)\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}}.$$

Thus if $\beta \geq \sum_{i=0}^{k} a_i$ and $\sum_{i=0}^{k} b_i > \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}$, then there is no positive equilibrium. The following result shows that there is an interval in which every solution of Eq.(13) converges to the zero equilibrium. For convenience of notation let Q denote the denominator of Eq.(1), that is

$$Q = \beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} x_{n-i} x_{n-j}.$$

Theorem 3 *Let $M \in (0,\infty)$ be such that $M < \frac{\beta - \sum_{i=0}^{k} a_i}{\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}$. Assume that there is no positive equilibrium. Then the zero equilibrium of Eq.(13) is globally asymptotically stable on the interval [0, M).*

Proof. Observe that Eq.(13) can be linearized into the form (10) where $l = 1$ as follows for $n \geq 0$

$$x_{n+1} = \frac{a_0 + \sum_{j=0}^{k} a_{0j} x_{n-j}}{Q} x_n + \frac{a_1 + \sum_{j=1}^{k} a_{1j} x_{n-j}}{Q} x_{n-1} + \ldots + \frac{a_k + a_{kk} x_{n-k}}{Q} x_{n-k}.$$

Then for $i = 0, \ldots, k$

$$|g_i| = \frac{a_i + \sum_{j=i}^{k} a_{ij} x_{n-j}}{Q} \quad n = 0, 1, \ldots.$$

Let $\{x_n\}$ be a solution of Eq.(13) where $\max\{x_0, \ldots, x_{-k}\} \leq M$. Then for $n = 0$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} x_{-j}}{\beta + \sum_{i=0}^{k} b_i x_{-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} x_{-i} x_{-j}}$$

$$\leq \frac{\sum_{i=0}^{k} a_i + M\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}{\beta} < 1.$$

By Lemma 1 with $\bar{x} = 0$, $h_i = |g_i|, i = 0, \ldots, k$ and $N = 0$ we get that $x_1 \leq M$. Hence $x_i \leq M$ for $i = 1, 0, \ldots, -k$. Then for $n = 1$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} x_{1-j}}{\beta + \sum_{i=0}^{k} b_i x_{1-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} x_{1-i} x_{1-j}}$$

$$\leq \frac{\sum_{i=0}^{k} a_i + M\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}{\beta} < 1.$$

Again using Lemma 1 with $\bar{x}=0$, $h_i = |g_i|, i = 0,\ldots,k$ and $N = 1$ we get that $x_2 \leq M$. Hence $x_i \leq M$ for $i = 2, 1, \ldots, -k$. Then for $n = 2$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{2-j}}{\beta + \sum_{i=0}^{k} b_i x_{2-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{2-i}x_{2-j}}$$

$$\leq \frac{\sum_{i=0}^{k} a_i + M\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}{\beta} < 1.$$

By induction we get that for $n \geq 0$

$$\sum_{i=0}^{k} |g_i| = \frac{\sum_{i=0}^{k} a_i + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}}$$

$$\leq \frac{\sum_{i=0}^{k} a_i + M\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}}{\beta} < 1$$

and so the result follows from Corollary 1 where $\bar{x} = 0$ and the interval is $[0, M)$. $\square$

All equilibrium solutions of Eq.(1) satisfy the equilibrium equation

$$\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x^3 + \left(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}\right)x^2 + \left(\beta - \sum_{i=0}^{k} a_i\right)x - \alpha = 0, \qquad (14)$$

which can be rewritten as

$$\alpha - \beta x + x\sum_{i=0}^{k}(a_i - xb_i) + x^2\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - xb_{ij}) = 0. \qquad (15)$$

Equilibrium equation (14) has at least one nonnegative zero and it may have between 0 and 3 positive zeros. When $\alpha > 0$ and either $\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} \geq 0, \beta - \sum_{i=0}^{k} a_i \geq 0$ or $\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} \geq 0, \beta - \sum_{i=0}^{k} a_i \leq 0$ or $\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} \leq 0, \beta - \sum_{i=0}^{k} a_i \leq 0$, Descartes rule of sign implies that there is a unique positive equilibrium of Eq.(1).

If $\bar{x} > 0$ is an equilibrium, then for $n \geq 0$

$$x_{n+1} - \bar{x} = \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{n-i}x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}} - \bar{x}$$

$$= \frac{\alpha - \beta\bar{x} + \sum_{i=0}^{k}(a_i - b_i\bar{x})x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})x_{n-i}x_{n-j}}{Q},$$

$$= \frac{\sum_{i=0}^{k}(a_i - b_i\bar{x})(x_{n-i} - \bar{x}) + \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})(x_{n-i}x_{n-j} - \bar{x}^2) + R}{Q},$$

where in view of Eqs.(14), (15)

$$R = \alpha - \beta\bar{x} + \bar{x}\sum_{i=0}^{k}(a_i - b_i\bar{x}) + \bar{x}^2\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x}) = 0.$$

Now applying the identity

$$x_{n-i}x_{n-j} - \bar{x}^2 = x_{n-i}(x_{n-j} - \bar{x}) + \bar{x}(x_{n-i} - \bar{x}), \quad i, j = 0, 1, \dots$$

we get that for $n \geq 0$

$$x_{n+1} - \bar{x} = \frac{A + B + \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})x_{n-i}(x_{n-j} - \bar{x})}{Q}.$$

Observe that for $n \geq 0$

$$\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})x_{n-i}(x_{n-j} - \bar{x}) = \sum_{i=0}^{k}\sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}(x_{n-i} - \bar{x}).$$

Thus for $n \geq 0$

$$x_{n+1} - \bar{x} = \frac{A + B + \sum_{i=0}^{k}\sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}(x_{n-i} - \bar{x})}{Q}$$

$$= \frac{A + \sum_{i=0}^{k}\left(\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x} + \sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}\right)(x_{n-i} - \bar{x})}{Q}$$

Where

$$A = \sum_{i=0}^{k}(a_i - b_i\bar{x})(x_{n-i} - \bar{x})$$

and

$$B = \sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x}(x_{n-i} - \bar{x}).$$

Therefore for $n \geq 0$

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k}\frac{(a_i - b_i\bar{x}) + \sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x} + \sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}}{\beta + \sum_{i=0}^{k}b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k}b_{ij}x_{n-i}x_{n-j}}(x_{n-i} - \bar{x}). \tag{16}$$

Equation (16) is the linearized equation of Eq.(1) of the form (12) where for $i = 0, \ldots, k$

$$g_i = \frac{(a_i - b_i\bar{x}) + \sum_{j=i}^{k}(a_{ij} - b_{ij}\bar{x})\bar{x} + \sum_{j=0}^{i}(a_{ji} - b_{ji}\bar{x})x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}}, n = 0, 1, \ldots. \quad (17)$$

We can now obtain easy-to-check conditions which show when the positive equilibrium of Eq.(1) is globally asymptotically stable. We will then apply these conditions to varius cases of Eq.(1).

Theorem 4 *Assume that Eq.*(1) *has a unique positive equilibrium* $\bar{x}$ *and there exist* $L \geq 0$ *and* $U, N > 0$ *such that for every solution* $\{x_n\}$ *of Eq.*(1) $L \leq x_n \leq U$ *for all* $n \geq N$ *and*

$$\sum_{i=0}^{k} |a_i - b_i\bar{x}| + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}| < \beta + L\sum_{i=0}^{k} b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}, \quad (18)$$

where $\beta + L > 0$. *Then the unique positive equilibrium* $\bar{x}$ *of Eq.*(1) *is globally asymptotically stable on the interval* $[0, \infty)$.

Proof.

As we have seen Eq.(1) can be written in the form of the linearized equation (16), where the coefficients g_i are given as (17).

We have for $n \geq 0$

$$Q = \beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j} \geq \beta + L\sum_{i=0}^{k} b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} \quad (19)$$

and so for $i = 0, \ldots, k$ and $n \geq 0$

$$|g_i| \leq \frac{|a_i - b_i\bar{x}| + \sum_{j=i}^{k}|a_{ij} - b_{ij}\bar{x}|\bar{x} + \sum_{j=0}^{i}|a_{ji} - b_{ji}\bar{x}|x_{n-j}}{Q}.$$

Thus for $n \geq 0$

$$\sum_{i=0}^{k}|g_i| \leq \frac{\sum_{i=0}^{k}|a_i - b_i\bar{x}| + \sum_{i=0}^{k}\sum_{j=i}^{k}|a_{ij} - b_{ij}\bar{x}|\bar{x} + \sum_{i=0}^{k}\sum_{j=0}^{i}|a_{ji} - b_{ji}\bar{x}|x_{n-j}}{Q}.$$

By rearranging the terms we can show that for $n \geq 0$

$$\sum_{i=0}^{k} \sum_{j=0}^{i} |a_{ji} - b_{ji}\bar{x}| x_{n-j} = \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}| x_{n-i}.$$

Thus for $n \geq 0$

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|\bar{x} + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}| x_{n-i}}{Q}$$

and so

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|(\bar{x} + x_{n-i})}{Q}.$$

In view of (18) and (19), we obtain

$$\sum_{i=0}^{k} |g_i| \leq \frac{\sum_{i=0}^{k} |a_i - b_i\bar{x}| + \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij}\bar{x}|(\bar{x} + U)}{\beta + L\sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}} < 1$$

for $n \geq 0$. So by Corollary 1 $\bar{x}$ is a globally asymptotically stable on the interval $[L, U]$. By assumption every solution of Eq.(1) enters the interval $[L, U]$ and so the result follows.

□

From (18) we see that establishing a lower bound for all the solutions of Eq.(1) will give us a better result. We present some of these cases.

Remark 1 The results on boundedness of all solutions of Eq.(1) are well known, see [6, 7]. For instance, if for every $i, j \in \{0, \ldots, k\}$ such that $b_i > 0, b_{ij} > 0$ we have $a_i > 0, a_{ij} > 0$, then the uniform lower bound L for all solutions $\{x_n\}$ of Eq.(1) for $n \geq 1$, is

$$L = \frac{\min\{\alpha, a_i, a_{ij} | a_i > 0, a_{ij} > 0\}}{\max\{\beta, b_i, b_{ij} > 0 | b_i > 0, b_{ij} > 0\}}.$$

On the other hand, if for every $i, j \in \{0, \ldots, k\}$ such that $a_i, a_{ij} > 0$ we have $b_i, b_{ij} > 0$, then the uniform lower bound L for all solutions of Eq.(1) for $n \geq 1$, is

$$L = \frac{\min\{\alpha, a_i, a_{ij} | a_i, a_{ij} > 0\}}{\max\{U\sum_{j, a_j = 0} b_j, U\sum_{i,j,a_{ij}=0} b_{ij}, \beta, b_i, b_{ij} | b_i > 0\}},$$

where

$$U = \frac{\max\{\alpha, a_i, a_{ij} | a_i, a_{ij} > 0\}}{\min\{\beta, b_i, b_{ij} | b_i, b_{ij} > 0\}}.$$

The next result follows from Lemma 1 and can be used to find the part of the basin of attraction of a positive equilibrium in the case when there are several positive equilibrium points. The proof of this result is similar to the proof of Theorem 3 and it will be omitted.

Theorem 5 *Let $M = \max\{\bar{x}, x_{-i} : i = 0, \dots, k\}$ be such that*

$$\sum_{i=0}^{k} |a_i - b_i \bar{x}| + (M + \bar{x}) \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - b_{ij} \bar{x}| < \beta + m \sum_{i=0}^{k} b_i + m^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}, \quad (20)$$

where $0 \leq m = \min\{\bar{x}, x_{-i} : i = 0, \dots, k\}$ is such that $\beta + m > 0$. Then the equilibrium $\bar{x}$ of Eq.(1) is globally asymptotically stable on the interval $[m, M]$.

The following result is a consequence of Theorem 4 in some special cases when the unique positive equilibrium satisfies some specific conditions.

Theorem 6 *Assume that $\beta > 0$ and $\bar{x}$ is the unique positive equilibrium of Eq.(1). Suppose there exist $L \geq 0$ and $U, N > 0$ such that for every solution $\{x_n\}$ of Eq.(1) $L \leq x_n \leq U$ for all $n \geq N$. Then the positive equilibrium $\bar{x}$ of Eq.(1) is globally asymptotically stable on the interval $[0, \infty)$ provided one of the following holds*

(1) $a_i = \bar{x} b_i, a_{ij} = \bar{x} b_{ij}$ *for all* $i, j \in \{0, \dots, k\}$;

(2) $a_i \geq \bar{x} b_i, a_{ij} \geq \bar{x} b_{ij}$ *for all* $i, j \in \{0, \dots, k\}$ *and* $\alpha \geq 0$ *and*

$$U \sum_{i=0}^{k} \sum_{j=i}^{k} (a_{ij} - \bar{x} b_{ij}) < \frac{\alpha}{\bar{x}} + L \sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}; \quad (21)$$

(3) $a_i \leq \bar{x} b_i, a_{ij} \leq \bar{x} b_{ij}$ *for all* $i, j \in \{0, \dots, k\}$ *and*

$$U \sum_{i=0}^{k} \sum_{j=i}^{k} (\bar{x} b_{ij} - a_{ij}) < 2\beta - \frac{\alpha}{\bar{x}} + L \sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}. \quad (22)$$

Proof.

The positive equilibrium $\bar{x}$ of Eq.(1) satisfies

$$\beta - \frac{\alpha}{\bar{x}} = \sum_{i=0}^{k} a_i - \bar{x}^2 \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} - \bar{x}\left(\sum_{i=0}^{k} b_i - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}\right). \tag{23}$$

(1) Let $a_i = \bar{x}b_i, a_{ij} = \bar{x}b_{ij}$ for all $i, j \in \{0, \ldots, k\}$. Then $\sum_{i=0}^{k} a_i = \bar{x}\sum_{i=0}^{k} b_i$ and $\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} = \bar{x}\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}$, which by (23) implies $\beta = \frac{\alpha}{\bar{x}}$. Then Eq.(1) becomes for $n \geq 0$

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \sum_{i=0}^{k} a_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}x_{n-i}x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}} \\ &= \frac{\beta\bar{x} + \bar{x}\sum_{i=0}^{k} b_i x_{n-i} + \bar{x}\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}}{\beta + \sum_{i=0}^{k} b_i x_{n-i} + \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}x_{n-i}x_{n-j}} = \bar{x}. \end{aligned}$$

(2) In view of our assumption for $i, j \in \{0, \ldots, k\}$,

we have $|a_i - \bar{x}b_i| = a_i - \bar{x}b_i, |a_{ij} - \bar{x}b_{ij}| = a_{ij} - \bar{x}b_{ij}$.

By using (23) we obtain

$$\begin{aligned} &\sum_{i=0}^{k} |a_i - \bar{x}b_i| + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k} |a_{ij} - \bar{x}b_{ij}| \\ &= \sum_{i=0}^{k} a_i - \bar{x}\sum_{i=0}^{k} b_i + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - \bar{x}b_{ij}) \\ &= \beta - \frac{\alpha}{\bar{x}} + U\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - \bar{x}b_{ij}). \end{aligned}$$

Now the condition (18) is simplified to

$$U\sum_{i=0}^{k}\sum_{j=i}^{k}(a_{ij} - \bar{x}b_{ij}) < \frac{\alpha}{\bar{x}} + L\sum_{i=0}^{k} b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}$$

and the result follows from Theorem 4.

(3) In this case we have

$$\sum_{i=0}^{k} |a_i - \bar{x} b_i| + (U + \bar{x}) \sum_{i=0}^{k} \sum_{j=i}^{k} |a_{ij} - \bar{x} b_{ij}|$$

$$= \bar{x} \sum_{i=0}^{k} b_i - \sum_{i=0}^{k} a_i + (U + \bar{x}) \sum_{i=0}^{k} \sum_{j=i}^{k} (\bar{x} b_{ij} - a_{ij})$$

$$= \frac{\alpha}{\bar{x}} - \beta + U \sum_{i=0}^{k} \sum_{j=i}^{k} (\bar{x} b_{ij} - a_{ij}).$$

In view of our assumption

$$\frac{\alpha}{\bar{x}} - \beta + U \sum_{i=0}^{k} \sum_{j=i}^{k} (\bar{x} b_{ij} - a_{ij}) < \beta + L \sum_{i=0}^{k} b_i + L^2 \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij}.$$

and so the result follows from Theorem 4.

$\square$

Many cases of Eq.(1) have some combination of $a_i < \bar{x} b_i, a_i > \bar{x} b_i$ and $a_i = \bar{x} b_i$. In view of this we will adopt the following notations, where $I_> = \{i | \text{such that } a_i > \bar{x} b_i\},\ I_= = \{i | \text{such that } a_i = \bar{x} b_i\},\ I_< = \{i | \text{such that } a_i < \bar{x} b_i\}$:

$$\left\{ \begin{array}{l} A_S = \sum_{i \in I_>} a_i = \text{the sum of all the } a_i\text{'s} \\ B_S = \sum_{i \in I_>} b_i = \text{the sum of all the } b_i\text{'s} \end{array} \right\} \text{ such that } a_i > \bar{x} b_i$$

$$\left\{ \begin{array}{l} A_N = \sum_{i \in I_=} a_i = \text{the sum of all the } a_i\text{'s} \\ B_N = \sum_{i \in I_=} b_i = \text{the sum of all the } b_i\text{'s} \end{array} \right\} \text{ such that } a_i = \bar{x} b_i$$

$$\left\{ \begin{array}{l} A_R = \sum_{i \in I_<} a_i = \text{the sum of all the } a_i\text{'s} \\ B_R = \sum_{i \in I_<} b_i = \text{the sum of all the } b_i\text{'s} \end{array} \right\} \text{ such that } a_i < \bar{x} b_i.$$

Then $A_S + A_N + A_R = \sum_{i=0}^{k} a_i$ and $B_S + B_N + B_R = \sum_{i=0}^{k} b_i$. Also $A_S > \bar{x} B_S, A_N = \bar{x} B_N$ and $A_R < \bar{x} B_R$.

Similarly define

$$\left\{ \begin{array}{l} \bar{A}_S = \sum_{i=0}^{k} \sum_{j=i}^{k} a_{ij} = \text{the sum of all the } a_{ij}\text{'s} \\ \bar{B}_S = \sum_{i=0}^{k} \sum_{j=i}^{k} b_{ij} = \text{the sum of all the } b_{ij}\text{'s} \end{array} \right\} \text{ such that } a_{ij} > \bar{x} b_{ij}$$

$$\left\{\begin{array}{l}\bar{A}_N = \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} = \text{the sum of all the } a_{ij}\text{'s} \\ \bar{B}_N = \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} = \text{the sum of all the } b_{ij}\text{'s}\end{array}\right\} \text{ such that } a_{ij} = \bar{x}b_{ij}$$

$$\left\{\begin{array}{l}\bar{A}_R = \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} = \text{the sum of all the } a_{ij}\text{'s} \\ \bar{B}_R = \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij} = \text{the sum of all the } b_{ij}\text{'s}\end{array}\right\} \text{ such that } a_{ij} < \bar{x}b_{ij}.$$

Then $\bar{A}_S + \bar{A}_N + \bar{A}_R = \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}$ and $\bar{B}_S + \bar{B}_N + \bar{B}_R = \sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}$. Also $\bar{A}_S > \bar{x}\bar{B}_S, \bar{A}_N = \bar{x}\bar{B}_N$, $\bar{A}_R < \bar{x}\bar{B}_R$ and

$$\sum_{i=0}^{k}\sum_{j=i}^{k} |a_{ij} - \bar{x}b_{ij}| = (\bar{A}_S - \bar{x}\bar{B}_S) + (\bar{x}\bar{B}_N - \bar{A}_N) + (\bar{x}\bar{B}_R - \bar{A}_R)$$

$$= \sum_{i=0}^{k}\sum_{j=i}^{k} \bar{x}b_{ij} - \sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij} + 2(\bar{A}_S - \bar{x}\bar{B}_S).$$

Corollary 2 *Suppose that the assumptions of Theorem 6 are satisfied. Let $i, j \in \{0, \ldots, k\}$. Assume that*

(a) For some i, j's $a_{ij} > \bar{x}b_{ij}$ and for other i, j's $a_{ij} < \bar{x}b_{ij}$

(b) For some i's $a_i > \bar{x}b_i$ and for other i's $a_i < \bar{x}b_i$

(c)

$$\frac{\alpha}{\bar{x}} + 2(A_S - \bar{x}B_S) + (U + 2\bar{x})(\bar{A}_S - \bar{x}\bar{B}_S) + U(\bar{x}B_R - \bar{A}_R) < H. \quad (24)$$

Where $H = 2\beta + L\sum_{i=0}^{k} b_i + L^2\sum_{i=0}^{k}\sum_{j=i}^{k} b_{ij}$.

Then the positive equilibrium $\bar{x}$ of Eq.(1) is globally asymptotically stable on the interval $[0, \infty)$.

Proof. In view of the equilibrium equation (15) and by assumption (c), we have that

$$\sum_{i=0}^{k} |a_i - \bar{x}b_i| + (U + \bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k} |a_{ij} - \bar{x}b_{ij}|$$

$$= \bar{x}\sum_{i=0}^{k} b_i - \sum_{i=0}^{k} a_i + 2(A_S - \bar{x}B_S) + 2(U + \bar{x})(\bar{A}_S - \bar{x}\bar{B}_S)$$

$$+(U+\bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}\bar{x}b_{ij}-(U+\bar{x})\sum_{i=0}^{k}\sum_{j=i}^{k}a_{ij}$$

$$=\sum_{i=0}^{k}(\bar{x}b_i-a_i)+\bar{x}\sum_{i=0}^{k}\sum_{j=i}^{k}(\bar{x}b_{ij}-a_{ij})+2(A_S-\bar{x}B_S)$$

$$+(2U+2\bar{x})(\bar{A}_S-\bar{x}\bar{B}_S)+U\sum_{i=0}^{k}\sum_{j=i}^{k}(\bar{x}b_{ij}-a_{ij})$$

$$=\frac{\alpha}{\bar{x}}-\beta+2(A_S-\bar{x}B_S)+(2U+2\bar{x})(\bar{A}_S-\bar{x}\bar{B}_S)+U\bar{x}(\bar{B}_S+\bar{B}_N+\bar{B}_R)-U(\bar{A}_S+\bar{A}_N+\bar{A}_R)$$

$$=\frac{\alpha}{\bar{x}}-\beta+2(A_S-\bar{x}B_S)+(U+2\bar{x})(\bar{A}_S-\bar{x}\bar{B}_S)+U(\bar{x}\bar{B}_R-\bar{A}_R)$$

$$<\beta+L\sum_{i=0}^{k}b_i+L^2\sum_{i=0}^{k}\sum_{j=i}^{k}b_{ij}.$$

Now, the conclusion follows from Theorem 4. □

In the case of general second order quadratic fractional equation of the form

$$x_{n+1}=\frac{Ax_n^2+Bx_nx_{n-1}+Cx_{n-1}^2+Dx_n+Ex_{n-1}+F}{ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n+ex_{n-1}+f},\quad n=0,1,\ldots \tag{25}$$

with nonnegative parameters and initial conditions such that $A+B+C>0$, and $ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n+ex_{n-1}+f>0, n=0,1,\ldots$, the obtained results take the following form.

Corollary 3 *Assume that Eq.*(25) *has the unique positive equilibrium* $\bar{x}$*. If the following condition holds*

$$\frac{(|A-a\bar{x}|+|B-b\bar{x}|+|C-c\bar{x}|)(U+\bar{x})+|D-d\bar{x}|+|E-e\bar{x}|}{(a+b+c)L^2+(d+e)L+f}<1 \tag{26}$$

where L *and* U *are lower and upper bounds of all solutions of Eq.*(25) *and* $f+L>0$*, then* $\bar{x}$ *is globally asymptotically stable on the interval* $[0,\infty)$*.*

In the special case of second order equation with quadratic terms only we obtain the following result.

Corollary 4 *Consider the following equation*

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, \ldots \tag{27}$$

with all positive parameters and nonnegative initial conditions such that $ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 > 0$ *for all* $n \geq 0$. *If the following condition holds*

$$\frac{(|A - a\bar{x}| + |B - b\bar{x}| + |C - c\bar{x}|)(U + \bar{x})}{(a + b + c)L^2} < 1$$

where

$$\bar{x} = \frac{A + B + C}{a + b + c}, \quad L = \frac{\min\{A, B, C\}}{\max\{a, b, c\}}, \quad U = \frac{\max\{A, B, C\}}{\min\{a, b, c\}},$$

then the unique equilibrium $\bar{x}$ *of Eq.*(27) *is globally asymptotically stable on the interval* $[0, \infty)$.

Remark 2 If the strict inequality in the conditions (18), (20), (21), (22), and (24) is replaced by equality, then the conclusions of Theorems 4, 5, 6, and Corollary 2 should be changed from global asymptotic stability to stability.

List of References

[1] E. Bedford and K. Kim, Periodicities in Linear Fractional Recurrences: Degree growth of birational surface maps, *Michigan Math. J.* 54(2006), 647–671.

[2] E. Bedford and K. Kim, Dynamics of Rational Surface Automorphisms: Linear Fractional Recurrences, *J. Geom. Anal.* 19 (2009), no. 3, 553–583.

[3] A. Brett, E. J. Janowski and M. R. S. Kulenović, Global Asymptotic Stability for Linear Fractional Difference Equation, *J. Difference Equations*, 2014 (2014), Article ID 275312, 11 p.

[4] Dž. Burgić, S. Kalabušić and M. R. S. Kulenović, Period-two trichotomies of a difference equation of order higher than two, *Sarajevo J. Math.* 4(16) (2008), 73–90.

[5] E. Camouzis and G. Ladas, When does local asymptotic stability imply global attractivity in rational equations? *J. Difference Equ. Appl.* 12(2006), 863–885.

[6] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations, with Open Problems and Conjectures*, Chapman& Hall/CRC Press, 2007.

[7] E. Camouzis, G. Ladas, F. Palladino and E. Quinn, On the boundedness character of rational equations, part 1, *J. Difference Equ. Appl.* 12(2006), 503–523.

[8] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz and H. Sedaghat, H. Dynamics of rational difference equations containing quadratic terms, *J. Difference Equ. Appl.* 14 (2008), 191–208.

[9] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz and H. Sedaghat, Monotone and oscillatory solutions of a rational difference equation containing quadratic terms, *J. Difference Equ. Appl.*, 14 (2008), 1045–1058.

[10] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, Global attractivity and periodic character of a fractional difference equation of order three, *Yokohama Math. J.* 53 (2007), no. 2, 89–100.

[11] E. M. Elabbasy, H. El-Metwally, and E. M. Elsayed, On the difference equation $x_{n+1} = \frac{a_0x_n+a_1x_{n-1}+\ldots+a_kx_{n-k}}{b_0x_n+b_1x_{n-1}+\ldots+b_kx_{n-k}}$, *Math. Bohem.* 133 (2008), 133–147.

[12] E. J. Janowski and M. R. S. Kulenović, Attractivity and global stability for linearizable difference equations, *Comput. Math. Appl.* 57 (2009), no. 9, 1592–1607.

[13] S. Jašarević and M. R. S. Kulenović, Basins of Attraction of Equilibrium and Boundary Points of Second Order Difference Equation, *J. Difference Equ. Appl.*, 20(2014), 947–959.

[14] S. Kalabušić, M. R. S. Kulenović and C. B. Overdeep, Dynamics of the recursive sequence $x_{n+1} = \frac{\beta x_{n-l}+\delta x_{n-k}}{Bx_{n-l}+Dx_{n-k}}$. *J. Difference Equ. Appl.* 10 (2004), 915–928.

[15] C. M. Kent and H. Sedaghat, Global attractivity in a quadratic-linear rational difference equation with delay. *J. Difference Equ. Appl.* 15 (2009), 913–925.

[16] C. M. Kent and H. Sedaghat, Global attractivity in a rational delay difference equation with quadratic terms, *J. Difference Equ. Appl.*, 17 (2011), 457–466.

[17] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[18] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures*, Chapman& Hall/CRC Press, 2001.

[19] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[20] M. R. S. Kulenović and O. Merino, A global attractivity result for maps with invariant boxes. *Discrete Contin. Dyn. Syst. Ser. B* 6(2006), 97–110.

[21] F. J. Palladino, On the characterization of rational difference equations, *J. Difference Equ. Appl.* 15 (2009), no. 3, 253–260.

[22] H. Sedaghat, *Nonlinear Difference Equations, Theory with applications to social science models,. Mathematical Modelling: Theory and Applications*, 15. Kluwer Academic Publishers, Dordrecht, 2003.

[23] H. Sedaghat, Global behaviours of rational difference equations of orders two and three with quadratic terms. *J. Difference Equ. Appl.* 15 (2009), 215–224.

MANUSCRIPT 2

Global Dynamics of Three Anti-competitive Systems of Difference Equations in the Plane

Published in *Discrete Dynamics in Nature and Society*,

(2013), 11p. Article ID 751594.

M. DiPippo and M. R. S. Kulenović

Department of Mathematics

University of Rhode Island

Kingston, RI, 02881, USA

2.1 Introduction

A first order system of difference equations

$$\begin{cases} x_{n+1} = f\left(x_n, y_n\right) \\ y_{n+1} = g\left(x_n, y_n\right) \end{cases}, \quad n = 0, 1, ..., (x_0, y_0) \in \mathcal{R}, \tag{28}$$

where $\mathcal{R} \subset \mathbb{R}^2, (f, g) : \mathcal{R} \to \mathcal{R}, f, g$ are continuous functions is *competitive* if $f\left(x, y\right)$ is non-decreasing in x and non-increasing in y, and $g\left(x, y\right)$ is non-increasing in x and non-decreasing in y.

System (28) where the functions f and g have monotonic character opposite of the monotonic character in competitive system is called *anti-competitive*, see [11, 18].

In this paper, we consider the following anti-competitive systems of difference equations

$$x_{n+1} = \frac{\gamma_1 y_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\beta_2 x_n}{B_2 x_n + y_n}, \quad n = 0, 1, ..., \tag{29}$$

where all parameters are positive numbers and the initial conditions (x_0, y_0) are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$. In the classification of all linear fractional systems in [3], System (29) was mentioned as system $(18, 18)$. We also consider systems

$$x_{n+1} = \frac{\gamma_1 y_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{y_n}, \quad n = 0, 1, ..., \tag{30}$$

and

$$x_{n+1} = \frac{\gamma_1 y_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n}{y_n}, \quad n = 0, 1, ..., \tag{31}$$

with $x_0 > 0, y_0 > 0$, which were labeled as systems $(18, 23)$ and $(16, 23)$, respectively in [3]. Three systems have interesting and different dynamics. While System (29) has all bounded solutions most of solutions of Systems (30) and (31) are unbounded. Another major difference is the existence of the unique period-two solution for (29) and, in a special case, abundance of such solutions while neither

(30) nor (31) has period-two solutions. We show that every solution of System (30) converge to the unique equilibrium or is approaching $(0, \infty)$ and so System (30) gives an example of a semistable equilibrium point. Finally, we show that all solutions of (31) which start on the stable set converge to the unique equilibrium, while all solutions which start off the stable set are approaching $(0, \infty)$ or $(\infty, 0)$. We also get that for some special values of parameters Systems (31) and (30) can be decoupled and explicitly solved.

Competitive systems of the form (28) were studied by many authors such as Clark and Kulenović [5], Hess [8], Hirsch and Smith [9], Kulenović and Merino [13], Kulenović and Nurkanović [17], Garić-Demirović, Kulenović and Nurkanović [2], [7], Smith [20], [21] and others. Precise results about the basins of attraction of the equilibrium points has been obtained in [1].

The study of anti-competitive systems started in [11] and has advanced since then, see [18]. The principal tool of study of anti-competitive systems is the fact that the second iterate of the map associated with anti-competitive system is competitive map and so elaborate theory for such maps developed recently in [9, 14, 15] can be applied.

The major result on a global behavior of System (29) is the following theorem.

Theorem 7 *(a) Assume that $B_1 < B_2$. Then the unique positive equilibrium point $E(\bar{x}, \bar{y})$ of System (29) is locally asymptotically stable with the basin of attraction $\mathcal{B}(E) = (0, \infty)^2$. The unique period-two solution $\{P_1, P_2\} = \{(\gamma_1, 0), (0, \frac{\beta_2}{B_2})\}$ is a saddle point and its basin of attraction is the union of coordinate axes without the origin, that is, $\mathcal{B}(\{P_1, P_2\}) = ((x, 0)|x > 0) \cup (0, y)|y > 0)$.*

(b) Assume that $B_1 > B_2$. Then every solution of System (29) converges to the period-two solution $\{P_1, P_2\}$ or to the equilibrium E. More precisely, there

exists a set $\mathcal{C} = \{(x, \frac{\bar{y}}{\bar{x}}x) : x > 0\} \subset \mathcal{R} = (0, \infty)^2$ *which is the basin of attraction of* E. *The set* $\mathcal{C}$ *has the property that for*

$$\begin{aligned} \mathcal{W}_- &: = \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\} = \{(x, y) : y > \frac{\bar{y}}{\bar{x}}x \geq 0\}, \\ \mathcal{W}_+ &: = \{x \in \mathcal{R}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\} = \{(x, y) : 0 \leq y < \frac{\bar{y}}{\bar{x}}x\}, \end{aligned}$$

the following hold

i) If $(x_0, y_0) \in \mathcal{W}_+$, *then*

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = P_1 \quad \text{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = P_2.$$

ii) If $(x_0, y_0) \in \mathcal{W}_-$,

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = P_2 \quad \text{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = P_1.$$

(c) *Assume that* $B_1 = B_2$. *Then every solution of System (29) is equal to either the period-two solution* $\left\{\frac{B_1 y_0}{B_1 x_0 + y_0}, \frac{\beta_2 x_0}{B_1 x_0 + y_0}\right\}, \left\{\frac{B_1 D x_0}{B_1 y_0 + D x_0}, \frac{\beta_2 y_0}{B_1 y_0 + D x_0}\right\}$ *or to the equilibrium* $E(\bar{x}, \bar{y})$, *where* $D = \frac{\beta_2}{\gamma_1}$ *and* $\bar{x} = \frac{\sqrt{\beta_2 \gamma_1}}{B_1 + 1}$.

2.2 Preliminaries

We now give some basic notions about competitive systems and maps in the plane of the form (28) where f and g are continuous functions and $f(x, y)$ is non-decreasing in x and non-increasing in y and $g(x, y)$ is non-increasing in x and non-decreasing in y in some domain A with non-empty interior.

Consider a map $T = (f, g)$ on a set $\mathcal{R} \subset \mathbb{R}^2$, and let $E \in \mathcal{R}$. The point $E \in \mathcal{R}$ is called a *fixed point* if $T(E) = E$. An *isolated* fixed point is a fixed point that has a neighborhood with no other fixed points in it. A fixed point $E \in \mathcal{R}$ is an *attractor* if there exists a neighborhood $\mathcal{U}$ of E such that $T^n(\mathbf{x}) \to E$ as $n \to \infty$ for $\mathbf{x} \in \mathcal{U}$; the *basin of attraction* is the set of all $\mathbf{x} \in \mathcal{R}$ such that $T^n(\mathbf{x}) \to E$ as $n \to \infty$. A fixed point E is a global attractor on a set $\mathcal{K}$ if E is an attractor and

$\mathcal{K}$ is a subset of the basin of attraction of E. If T is differentiable at a fixed point E, and if the Jacobian $J_T(E)$ has one eigenvalue with modulus less than one and a second eigenvalue with modulus greater than one, E is said to be a *saddle*. See [19] for additional definitions.

Definition 1 *Let $T = (f, g)$ be a continuously differentiable vector function and let U be a neighborhood of a saddle point $(\bar{x}, \bar{y})$ of (28). The local stable manifold $\mathcal{W}^s_{loc}$ is the set*

$$\mathcal{W}^s_{loc}((\bar{x}, \bar{y})) = \left\{ (x, y) : T^n(x, y) \in U \;\; (\forall n \geq 0) \;\; \wedge \;\; \lim_{n\to\infty} T^n(x, y) = (\bar{x}, \bar{y}) \right\}.$$

The global stable manifold W^s of a saddle point $(\bar{x}, \bar{y})$ is the set

$$\mathcal{W}^s((\bar{x}, \bar{y})) = \left\{ (x, y) : \; \lim_{n\to\infty} T^n(x, y) = (\bar{x}, \bar{y}) \right\}.$$

The map T may be viewed as a monotone map if we define a partial order on $\mathbb{R}^2$ so that the positive cone in this new partial order is the fourth (resp. first) quadrant. Define a South-east (resp. North-east) partial order $\preceq_{se}$ (resp. $\preceq_{ne}$) on $\mathbb{R}^2$ so that the positive cone is the fourth quadrant (resp. first quadrant), that is, $(x^1, y^1) \preceq_{se} (x^2, y^2)$ (resp. $(x^1, y^1) \preceq_{ne} (x^2, y^2)$) if and only if $x^1 \leq x^2$ and $y^1 \geq y^2$ (resp. $x^1 \leq x^2$ and $y^1 \leq y^2$). For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ the *order interval* $[\![\mathbf{x}, \mathbf{y}]\!]$ is the set of all $\mathbf{z}$ such that $\mathbf{x} \preceq \mathbf{z} \preceq \mathbf{y}$. The map T is called competitive (resp. cooperative) on a set S if $\mathbf{v} \preceq_{se} \mathbf{w}$ (resp. $\mathbf{v} \preceq_{ne} \mathbf{w}$) implies $T(\mathbf{v}) \preceq_{se} T(\mathbf{w})$ (resp. $T(\mathbf{v}) \preceq_{ne} T(\mathbf{w})$).

Two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2_+$ are said to be *related* if $\mathbf{v} \preceq \mathbf{w}$ or $\mathbf{w} \preceq \mathbf{v}$. Also, a strict inequality between points may be defined as $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v} \preceq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. A stronger inequality may be defined as $\mathbf{v} \prec\prec \mathbf{w}$ if $v_1 < w_1$ and $w_2 < v_2$. A map $f : \operatorname{Int} \mathbb{R}^2_+ \to \operatorname{Int} \mathbb{R}^2_+$ is *strongly monotone* if $\mathbf{v} \prec \mathbf{w}$ implies that $f(\mathbf{v}) \prec\prec f(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \operatorname{Int} \mathbb{R}^2_+$. Clearly, being related is invariant under iteration of a strongly monotone map. Differentiable strongly monotone maps have Jacobian with constant sign configuration

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}.$$

The mean value theorem and the convexity of $\mathbb{R}^2_+$ may be used to show that T is monotone, as in [6].

For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, define $Q_l(\mathbf{x})$ for $l = 1, ..., 4$ to be the usual four quadrants based at $\mathbf{x}$ and numbered in a counterclockwise direction, for example, $Q_1(\mathbf{x}) = \{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : x_1 \le y_1, x_2 \le y_2\}$.We now state three results for competitive maps in the plane.

The following definition is from [20].

Definition 2 *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition (O+) if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition (O−) if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by de Mottoni-Schiaffino for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [20, 21].

Theorem 8 *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which (O+) holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead (O−) holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [20], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 9 *Let $\mathcal{R} \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$ then T satisfies (O+). If T is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$ then T satisfies (O−).*

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, see [15, 16] and [8], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 5 *If the nonnegative cone with respect to the partial order $\preceq$ is a generalized quadrant in $\mathbb{R}^2$, and if the competitive map $T :\to \mathbb{R}^2$ has no fixed points in $[\![u_1, u_2]\!]$ other than u_1 and u_2, then the interior of $[\![u_1, u_2]\!]$ is either a subset of the basin of attraction of u_1 or a subset of the basin of attraction of u_2.*

Next two results are from [15, 16].

Theorem 10 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or *minimal period-two points. In the latter case, the set of endpoints of* $\mathcal{C}$ *is a minimal period-two orbit of* T*.*

Theorem 11 *Let* $\mathcal{I}_1, \mathcal{I}_2$ *be intervals in* $\mathbb{R}$ *with endpoints* a_1, a_2 *and* $b_{1,} b_2$*with endpoints respectively, with* $a_1 < a_2$ *and* $b_1 < b_2$*, where* $-\infty \leq a_1 < a_2 \leq \infty$ *and* $-\infty \leq b_1 < b_2 \leq \infty$*. Let* T *be a competitive map on an rectangle* $\mathcal{R} = \mathcal{I}_1 \times \mathcal{I}_2$ *and* $\overline{\mathbf{x}} \in int\,(\mathcal{R})$*. Suppose that the following hypotheses are satisfied:*

1. $T\,(int\,(\mathcal{R})) \subset int\,(\mathcal{R})$ *and* T *is strongly competitive on* $int\,(\mathcal{R})$*.*

2. The point $\overline{\mathbf{x}}$ *is the only fixed point of* T *in* $(Q_1\,(\overline{\mathbf{x}}) \cup Q_3\,(\overline{\mathbf{x}})) \cap int\,(\mathcal{R})$*.*

3. The map T *is continuously differentiable in a neighborhood of* $\overline{\mathbf{x}}$*, and* $\overline{\mathbf{x}}$ *is the saddle point.*

4. At least one of the following statemets is true.

a. T *has no minimal period two orbits in* $(Q_1\,(\overline{\mathbf{x}}) \cup Q_3\,(\overline{\mathbf{x}})) \cap int\,(\mathcal{R})$*.*

b. $\det J_T\,(\overline{\mathbf{x}}) > 0$ *and* $T\,(\mathbf{x}) = \overline{\mathbf{x}}$ *only for* $\mathbf{x} = \overline{\mathbf{x}}$*.*

Then the following statements are true.

(i.) The stable manifold $\mathcal{W}^s(\overline{\mathbf{x}})$ *is connected and it is the graph of a continuous increasing curve with endpoints in* $\partial\mathcal{R}$*.* $int\,(\mathcal{R})$ *is divided by the closure of* $\mathcal{W}^s(\overline{\mathbf{x}})$ *into two invariant connected regions* $\mathcal{W}_+$ *("below the stable set"), and* $\mathcal{W}_-$ *("above the stable set"), where*

$$\begin{aligned}
\mathcal{W}_+ &:= \left\{\mathbf{x} \in \mathcal{R}\backslash\mathcal{W}^s(\overline{\mathbf{x}}) : \exists \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ with } \mathbf{x} \preceq_{se} \mathbf{x}'\right\}, \\
\mathcal{W}_- &:= \left\{\mathbf{x} \in \mathcal{R}\backslash\mathcal{W}^s(\overline{\mathbf{x}}) : \exists \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ with } \mathbf{x}' \preceq_{se} \mathbf{x}\right\},.
\end{aligned}$$

(ii.)The unstable manifold $\mathcal{W}^u(\overline{\mathbf{x}})$ *is connected and it is the graph of a continuous decreasing curve.*

(iii.) For every $\mathbf{x} \in \mathcal{W}_+$*,* $T^n\,(\mathbf{x})$ *eventually enters the interior of the invariant set* $Q_4\,(\overline{\mathbf{x}}) \cap \mathcal{R}$*, and for every* $\mathbf{x} \in \mathcal{W}_-$*,* $T^n\,(\mathbf{x})$ *eventually enters the interior of the invariant set* $Q_2\,(\overline{\mathbf{x}}) \cap \mathcal{R}$*.*

(iv.) Let $\mathbf{m} \in Q_2(\overline{\mathbf{x}})$ *and* $\mathbf{M} \in Q_4(\overline{\mathbf{x}})$ *be the endpoints of* $\mathcal{W}^u(\overline{\mathbf{x}})$*, where* $\mathbf{m} \preceq_{se} \overline{\mathbf{x}} \preceq_{se} \mathbf{M}$*. For every* $\mathbf{x} \in \mathcal{W}_-$ *and every* $\mathbf{z} \in \mathcal{R}$ *such that* $\mathbf{m} \preceq_{se} z$*, there exists* $m \in \mathbb{N}$ *such that* $T^m(\mathbf{x}) \preceq_{se} z$*, and for every* $\mathbf{x} \in \mathcal{W}_+$ *and every* $\mathbf{z} \in \mathcal{R}$ *such that* $\mathbf{z} \preceq_{se} \mathbf{M}$*, there exists* $m \in \mathbb{N}$ *such that* $\mathbf{M} \preceq_{se} T^m(\mathbf{x})$*.*

2.3 Global Dynamics of System (29)

The equilibrium point $E(\bar{x}, \bar{y})$ of System (29) satisfies the following system of equations

$$\overline{x} = \frac{\gamma_1 \overline{y}}{B_1 \overline{x} + \overline{y}}, \quad \overline{y} = \frac{\beta_2 \overline{x}}{B_2 \overline{x} + \overline{y}}. \tag{32}$$

It is easy to see that System (32) has unique equilibrium point E in the first quadrant, for all values of the parameters. Indeed, the positive equilibrium point is an intersection of the following two curves

$$y = \frac{B_1 x^2}{\gamma_1 - x}, \tag{33}$$

and

$$x = \frac{y^2}{\beta_2 - B_2 y}. \tag{34}$$

It is clear that at the point of intersection E curve (33) is steeper than curve (34), that is,

$$\left.\frac{dy}{dx}\right|_{(33)} (E) > \left.\frac{dy}{dx}\right|_{(34)} (E)$$

which gives

$$\frac{B_1 \bar{x}(2\gamma_1 - \bar{x})}{(\gamma_1 - \bar{x})^2} > \frac{(\beta_2 - B_2 \bar{y})^2}{\bar{y}(2\beta_2 - B_2 \bar{y})}.$$

This inequality is equivalent to the following inequality

$$\left(\beta_2 + \frac{\bar{y}^2}{\bar{x}}\right)\left(\gamma_1 + B_1 \frac{\bar{x}^2}{\bar{y}}\right) > B_1 \bar{x} \bar{y}$$

which is always satisfied.

2.3.1 Linearized Stability Analysis of System (29)

In this section we present the linearized stability analysis of the equilibrium E of System (29).

Theorem 12 *i) If $B_1 < B_2$, then E is locally asymptotically stable.*

ii) If $B_1 = B_2$, then E is a non-hyperbolic equilibrium point.

iii) If $B_1 > B_2$, then E is a saddle point.

Proof. The map T associated to System (29) is

$$T\left(x,y\right)=\left(\begin{array}{c} f\left(x,y\right) \\ g\left(x,y\right) \end{array}\right)=\left(\begin{array}{c} \frac{\gamma_1 y}{B_1 x+y} \\ \frac{\beta_2 x}{B_2 x+y} \end{array}\right). \tag{35}$$

The Jacobian matrix of map (35) is

$$J_T(x,y)=\begin{pmatrix} -\frac{B_1\gamma_1 y}{(B_1x+y)^2} & \frac{B_1\gamma_1 x}{(B_1x+y)^2} \\ \frac{\beta_2 y}{(B_2x+y)^2} & -\frac{\beta_2 x}{(B_2x+y)^2} \end{pmatrix}, \tag{36}$$

and evaluated at the equilibrium point $E=(\bar{x},\bar{y})$ is

$$J_T\left(\overline{x},\overline{y}\right)=\begin{pmatrix} -\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} & \frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^2} \\ \frac{\bar{y}^3}{\beta_2\bar{x}^2} & -\frac{\bar{y}^2}{\beta_2\bar{x}} \end{pmatrix}. \tag{37}$$

The characteristic equation has the following form

$$\lambda^2+\left(\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}}+\frac{1}{\beta_2}\frac{\bar{y}^2}{\bar{x}}\right)\lambda=0, \tag{38}$$

and the characteristic roots are

$$\lambda_1=0, \quad \lambda_2=-\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}}-\frac{1}{\beta_2}\frac{\bar{y}^2}{\bar{x}}.$$

A straight-forward calculation shows that the conditions $\lambda_2 \in (-1,0), \lambda_2 = -1, \lambda_2 \in (-\infty,-1)$ respectively, are equivalent to the conditions

$$\left(\frac{\bar{x}}{\bar{y}}\right)^2<\frac{\beta_2}{\gamma_1}, \quad \left(\frac{\bar{x}}{\bar{y}}\right)^2=\frac{\beta_2}{\gamma_1}, \quad \left(\frac{\bar{x}}{\bar{y}}\right)^2>\frac{\beta_2}{\gamma_1}. \tag{39}$$

On the other hand, by dividing two equilibrium equations (32) we obtain

$$\left(\frac{\bar{x}}{\bar{y}}\right)^2 = \frac{\beta_2}{\gamma_1}\frac{B_1\bar{x}+\bar{y}}{B_2\bar{x}+\bar{y}},$$

which implies that the condition (39) is equivalent to the condition

$$B_1 < B_2, \quad B_1 = B_2, \quad B_1 > B_2,$$

which completes the proof of theorem. □

2.3.2 Global Results for System (29)

In this section we present the proof of Theorem 7 on the global dynamics of System (29). First we prove the following result on existence and local behavior of the period-two solutions.

Lemma 2 *System* (29) *has the minimal period-two solution*

$$\{P_1, P_2\} = \left\{(\gamma_1, 0), \left(0, \frac{\beta_2}{B_2}\right)\right\}. \tag{40}$$

for all values of parameters. If $B_1 \neq B_2$, then the solution (40) *is the unique period-two solution and when $B_1 = B_2$ there are infinitely many period-two solutions. The set*

$$\mathcal{B} = \{(x,y)|x>0 \text{ and } y=0 \text{ or } x=0 \text{ and } y>0\}$$

is a subset of the basin of attraction of the solution (40). *The period-two solution* (40) *is locally stable if $B_1 > B_2$ and a saddle-point if $B_1 < B_2$. Finally the period-two solution* (40) *is non hyperbolic if $B_1 = B_2$.*

Proof. The second iterate of the map T is given as

$$T^2(x,y) = \left[\begin{array}{cc} x\beta_2\gamma_1\frac{y+xB_1}{xy\beta_2+x^2\beta_2B_1+y^2\gamma_1B_1+xy\gamma_1B_1B_2} & y\beta_2\gamma_1\frac{y+xB_2}{\beta_2B_1x^2+\gamma_1xyB_2^2+\beta_2xy+\gamma_1y^2B_2}\end{array}\right].$$

A period-two solution of System (29) satisfies $T^2(x,y) = (x,y)$, which immediately leads to the following equations

$$(B_2 - B_1)(\beta_2\gamma_1 x - B_2\gamma_1 xy - \gamma_1 y^2) = 0, \tag{41}$$

$$(B_2 - B_1)(\beta_2\gamma_1 y - B_2 xy - \gamma_1 B_1 x^2) = 0, \tag{42}$$

which have either unique solution if $B_1 \neq B_2$ or it has infinitely many solutions if $B_1 = B_2$. In the first case, Eq.(41) gives immediately Eq.(33) and Eq.(42) gives (34), which means that in this case the only minimal period-two solution is (40).

A straight-forward calculation shows that $T(P_1) = P_2, T(P_2) = P_1$, which shows that $\{P_1, P_2\}$ is a minimal period-two solution. Moreover, $T((a, 0) = P_2, T(0, b) = P_1$ for every $a > 0, b > 0$, which shows that the set $\mathcal{B}$ is a subset of the basin of attraction of $\{P_1, P_2\}$. The Jacobian matrix of T^2 is

$$J_{T^2}(x, y) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}.$$

Where,

$$J_{11} = y\beta_2\gamma_1^2 B_1 \frac{B_1 B_2 x^2 + 2B_1 xy + y^2}{(xy\beta_2 + x^2\beta_2 B_1 + y^2\gamma_1 B_1 + xy\gamma_1 B_1 B_2)^2},$$

$$J_{12} = -x\beta_2\gamma_1^2 B_1 \frac{B_1 B_2 x^2 + 2B_1 xy + y^2}{(xy\beta_2 + x^2\beta_2 B_1 + y^2\gamma_1 B_1 + xy\gamma_1 B_1 B_2)^2},$$

$$J_{21} = -y\beta_2^2\gamma_1 \frac{B_1 B_2 x^2 + 2B_1 xy + y^2}{(\beta_2 B_1 x^2 + \gamma_1 xy B_2^2 + \beta_2 xy + \gamma_1 y^2 B_2)^2}, \textit{and}$$

$$J_{22} = x\beta_2^2\gamma_1 \frac{B_1 B_2 x^2 + 2B_1 xy + y^2}{(\beta_2 B_1 x^2 + \gamma_1 xy B_2^2 + \beta_2 xy + \gamma_1 y^2 B_2)^2}.$$

The Jacobian matrix of T^2 evaluated at P_1 is

$$J_{T^2}(P_1) = \begin{bmatrix} 0 & -\frac{1}{\beta_2}\gamma_1 B_2 \\ 0 & \frac{1}{B_1} B_2 \end{bmatrix}$$

and the Jacobian matrix of T^2 evaluated at P_2 is

$$J_{T^2}(P_2) = \begin{bmatrix} \frac{1}{B_1} B_2 & 0 \\ B_2 - \frac{1}{\beta_2\gamma_1}\left(\frac{\beta_2^2}{B_2} + \beta_2\gamma_1 B_2\right) & 0 \end{bmatrix}.$$

In both cases the eigenvalues of the Jacobian matrix of T^2 are $\lambda_1 = 0, \lambda_2 = \frac{B_2}{B_1}$, which implies the result on local stability of the minimal period-two solution $\{P_1, P_2\}$.

$\square$

Proof of Theorem 7. First, observe that the rectangle $\mathcal{R} = [0, \gamma_1] \times \left[0, \frac{\beta_2}{B_2}\right] \setminus \{0, 0\} = [\![P_2, P_1]\!] \setminus \{0, 0\}$ is an invariant and attracting set for the map T and so is for the map T^2. More precisely, $(x_n, y_n) \in \mathcal{R}$ for $n \geq 1$. The map T^2 is competitive map on $\mathcal{R}$.

Case (a). Assume that $B_1 < B_2$. Then in view of Theorem 12 and Lemma 2 the map T^2 has three equilibrium points P_1, P_2 and E where $P_2 \preceq_{se} E \preceq_{se} P_1$.The equilibrium points P_1 and P_2 are saddle points and E is a local attractor. The ordered intervals $[\![P_2, E]\!]$ and $[\![E, P_1]\!]$ are both invariant sets of T^2 and in view of Corollary5 their interiors are attracted to E. If we take the point $(x, y) \in \mathcal{R} \setminus [\![P_2, E]\!] \cup [\![E, P_1]\!]$, we can find the points $(x_l, y_l) \in int[\![P_2, E]\!]$ and $(x_u, y_u) \in int[\![E, P_1]\!]$, such that $(x_l, y_l) \preceq_{se} (x, y) \preceq_{se} (x_u, y_u)$. Consequently, since T^2 is competitive $T^{2n}((x_l, y_l)) \preceq_{se} T^{2n}((x, y)) \preceq_{se} T^{2n}((x_u, y_u))$ for $n \geq 1$ and so $\lim_{n\to\infty} T^{2n}((x, y)) = E$, which by continuity of T implies that

$$\lim_{n\to\infty} T^{2n+1}((x, y)) = \lim_{n\to\infty} T(T^{2n}((x, y))) = T(\lim_{n\to\infty} T^{2n}((x, y))) = T(E) = E,$$

and so $\lim_{n\to\infty} T^n((x, y)) = E$.

Case (b). Assume that $B_1 > B_2$. Then in view of Theorem 12 and Lemma 2 the map T^2 has three equilibrium points P_1 and P_2 which are local attractors and E which is a saddle point. The ordered intervals $[\![P_2, E]\!]$ and $[\![E, P_1]\!]$ are both invariant sets for T^2 and in view of Corollary5 their interiors are attracted to P_2 and P_1 respectively. In view of Theorems 10 and 11 there is the set $\mathcal{C}$ with described properties. Direct calculation shows that the half-line $y = \frac{\bar{y}}{\bar{x}}x, x > 0$ is an invariant set which in view of a uniqueness of stable manifold implies that this half-line is exactly stable manifold mentioned in Theorems 10 and 11. It should be observed that because of the fact that one of the characteristic values at the equilibrium point E is 0, this equilibrium is super-attractive, that is, $T(x_0, y_0) = (\bar{x}, \bar{y})$, for every $(x_0, y_0) \in \mathcal{C}$.

Case (c). Assume that $B_1 = B_2$. Then by dividing two equations of System (29) we obtain that the solution of (29) satisfies

$$\frac{y_{n+1}}{x_{n+1}} = \frac{\gamma_2}{\beta_1}\frac{x_n}{y_n} = \frac{\gamma_2}{\beta_1}\frac{1}{\frac{y_n}{x_n}}.$$

This means that $\frac{y_n}{x_n}$ satisfies first order difference equation $u_{n+1} = \frac{D}{u_n}$, where $D = \frac{\gamma_2}{\beta_1}$. All non-constant solutions of $u_{n+1} = \frac{D}{u_n}$ are period-two solutions $\{u_0, \frac{D}{u_0}\}$. Thus $y_n = \{u_0 x_n, \frac{D}{u_0} x_n\}$. In this case System (29) becomes

$$x_{n+1} = \frac{B_1 u_0}{B_1 + u_0}, \quad y_{n+1} = \frac{\beta_2}{B_1 + u_0}, n = 1, 2, \ldots$$

and

$$x_{n+1} = \frac{B_1 D}{D + B_1 u_0}, \quad y_{n+1} = \frac{\beta_2 u_0}{D + B_1 u_0}, n = 1, 2, \ldots,$$

which completes the proof of Case (c).

Remark 3 System (29) is an example of the homogeneous system which is a special case of a general System (28) where both functions f and g are homogeneous functions of the same degree k, that is $f(tu, tv) = t^k f(u, v), g(tu, tv) = t^k g(u, v)$ for all u, v in intersection of domains of f and g and all $t \neq 0$. In that case, the ratio $z_n = y_n/x_n$ of every solution of (28) satisfies the first order difference equation

$$z_{n+1} = \frac{f(1, z_n)}{g(1, z_n)} = F(z_n), \quad n = 0, 1, \ldots$$

which analysis gives valuable information about the dynamics of System (28), but does not provide the global dynamics. In particular, this approach can not determine precisely the basins of attraction of different types of atractors such as equilibrium points, periodic solutions, and so on. This approach in the case of the system of linear fractional equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n}, \quad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}, \quad n = 0, 1, ..., \tag{43}$$

where all parameters and the initial conditions (x_0, y_0) are arbitrary nonnegative numbers such that $A_i + B_i x_0 + C_i y_0 > 0, i = 1, 2$, was first used in [4] and was systematically developed in the recent paper [10]. In [10], the authors studied all possible homogeneous systems of the form (43) and they proved that every bounded solution converges to either an equilibrium solution or to period-two solution. They were able to find a part of the basin of attraction of the period-two solution but not the complete basin of attraction. In the case of system (29) the auxiliary equation for $z_n = y_n/x_n$ is

$$z_{n+1} = \frac{\beta_2(B_1 + z_n)}{\gamma_1(B_2 + z_n)z_n} = F(z_n), \quad n = 0, 1, \ldots$$

Since F is decreasing every solution of the auxiliary equation is approaching not necessarily minimal period-two solution. Further analysis can be continued either by checking negative feedback condition for F^2 or by using Theorem 3.2 from [10]. In neither case the complete description of the basins of attraction of the equilibrium and the period-two solution is possible. We prefer our approach because it is more precise and also apply equally well to anti-competitive systems which are not homogeneous. The approach which is making use of homogeneous properties of functions is applicable also to the systems which are neither competitive nor anti-competitive.

2.4 Global Dynamics of System (30)

The equilibrium point $E(\bar{x}, \bar{y})$ of System (30) satisfies the following system of equations

$$\overline{x} = \frac{\gamma_1 \overline{y}}{B_1 \overline{x} + \overline{y}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 \overline{x}}{\overline{y}}. \tag{44}$$

It is easy to see that System (44) has the unique equilibrium point E in the first quadrant, for all values of the parameters. Indeed, the positive equilibrium point is an intersection of the following two curves

$$y = \frac{B_1 x^2}{\gamma_1 - x}, \tag{45}$$

and

$$x = \frac{y^2 - \alpha_2}{\beta_2}. \tag{46}$$

It is clear that at the point of intersection E curve (45) is steeper than curve (46), that is,

$$\left.\frac{dy}{dx}\right|_{(45)}(E) > \left.\frac{dy}{dx}\right|_{(46)}(E)$$

which gives

$$\frac{B_1 \bar{x}(2\gamma_1 - \bar{x})}{(\gamma_1 - \bar{x})^2} > \frac{\beta_2}{2\bar{y}}.$$

This inequality is equivalent to the following inequality

$$2\beta_1 \bar{x}\bar{y}(2\gamma_1 - \bar{x}) > \beta_2(\gamma_1 - \bar{x})^2,$$

which in turn is equivalent to

$$2 + 2\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} > \beta_2 \frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^3}. \tag{47}$$

2.4.1 Linearized Stability Analysis of System (30)

In this section we prove the following result

Theorem 13 *The unique equilibrium E of System* (30) *is a saddle point.*

Proof. The map S associated to System (30) is

$$S(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1 y}{B_1 x + y} \\ \frac{\alpha_2 + \beta_2 x}{y} \end{pmatrix}. \tag{48}$$

The Jacobian matrix of map (48) is

$$J_S(x,y) = \begin{pmatrix} -\frac{B_1 \gamma_1 y}{(B_1 x + y)^2} & \frac{B_1 \gamma_1 x}{(B_1 x + y)^2} \\ \frac{\beta_2}{y} & -\frac{\alpha_2 + \beta_2 x}{y^2} \end{pmatrix}, \tag{49}$$

and evaluated at the equilibrium point $E = (\bar{x}, \bar{y})$ is

$$J_S(\overline{x}, \overline{y}) = \begin{pmatrix} -\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} & \frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^2} \\ \frac{\beta_2}{\bar{y}} & -1 \end{pmatrix}. \tag{50}$$

The characteristic equation has the following form

$$\lambda^2 + \left(1 + \frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}}\right)\lambda + \frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} - \beta_2\frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^2} = 0. \tag{51}$$

Set

$$P = 1 + \frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}}, \quad Q = \frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} - \beta_2\frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^2}.$$

Then the necessary and sufficient condition for Eq.(51) to have one root inside the unit circle and one root outside the unit circle is $|P| > |1 + Q|, P^2 - 4Q > 0$, see [12, 13]. The condition $|P| > |1 + Q|$ leads to $P > -1 - Q$ which is equivalent to

$$2 + 2\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} > \beta_2\frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^3},$$

which is condition (47).

The condition $P^2 - 4Q > 0$ becomes

$$\left(1 + \frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}}\right)^2 - 4\left(\frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}} - \beta_2\frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^3}\right) > 0,$$

which is equivalent to

$$\left(1 - \frac{B_1}{\gamma_1}\frac{\bar{x}^2}{\bar{y}}\right)^2 + 4\beta_2\frac{B_1}{\gamma_1}\frac{\bar{x}^3}{\bar{y}^3} > 0,$$

which is clearly satisfied. □

2.4.2 Global Results for System (30)

Lemma 3 *System* (30) *has no minimal period-two solution.*

Proof. The second iterate of the map S is given as $S^2(x, y) = \begin{pmatrix} S_1, & S_2 \end{pmatrix}$.

Where,

$$S_1 = \gamma_1(y + xB_1)\frac{\alpha_2 + x\beta_2}{\beta_2 B_1 x^2 + \beta_2 xy + \alpha_2 B_1 x + \gamma_1 B_1 y^2 + \alpha_2 y},$$

and

$$S_2 = \frac{y}{(y + xB_1)(\alpha_2 + x\beta_2)} (y\alpha_2 + y\beta_2\gamma_1 + x\alpha_2 B_1)$$

Period-two solution satisfies $S^2(x, y) = (x, y)$ which reduces to the following two equations

$$\gamma_1 (y + xB_1)(\alpha_2 + x\beta_2) - x(\beta_2 B_1 x^2 + \beta_2 xy + \alpha_2 B_1 x + \gamma_1 B_1 y^2 + \alpha_2 y) = 0, \quad (52)$$

$$y\alpha_2 + y\beta_2\gamma_1 + x\alpha_2 B_1 - (y + xB_1)(\alpha_2 + x\beta_2) = 0. \quad (53)$$

Equation (53) leads immediately to $y\gamma_1 - yx - x^2 B_1$ which is exactly the equilibrium equation (45). Using (45) in (52) we obtain after some elementary simplification that period-two solution satisfies (46). This shows that System (30) has no minimal period-two solution.

□

Lemma 4 *The maps S and S^2 associated with System* (30) *have the following properties.*

(i) The maps S and S^2 are injective.

(ii) $\det J_{S^2}(x, y) > 0$ *for all* $(x, y), y > 0$.

Consequently, S^2 satisfies (O+) condition and so $\{S^{2n}(x_0, y_0)\}$ is eventually component-wise monotonic.

Proof.

(i) We will prove that S is injective and the injectivity of S^2 will follow immediately. The condition

$$S(x_1, y_1) = S(x_2, y_2)$$

is reduced to the following two conditions

$$x_2 y_1 = x_1 y_2, \quad \alpha_2(y_2 - y_1) = \beta_2(x_2 y_1 - x_1 y_2),$$

which immediately implies $y_1 = y_2$ and so $x_1 = x_2$.

(ii) A direct calculation shows that

$$\det J_{S^2} = \frac{B_1^2 \alpha_2^2 \gamma_1^2 y^2}{(\alpha_2 + x\beta_2)\left(y\alpha_2 + xy\beta_2 + x\alpha_2 B_1 + x^2\beta_2 B_1 + y^2\gamma_1 B_1\right)^2},$$

which implies our statement.

The statement on $(O+)$ condition follows from Theorem 9. □

Theorem 14 *Consider System* (30). *Then there exists a set* $\mathcal{C} \subset \mathcal{D} = [0, \infty) \times (0, \infty)$ *which is invariant subset of the basin of attraction of* E. *The set* $\mathcal{C}$ *is a graph of a strictly increasing continuous function of the first variable on an interval, and separates* $\mathcal{D}$ *into two connected and invariant components, namely*

$$\begin{aligned} \mathcal{W}_- &:= \{x \in \mathcal{D}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \\ \mathcal{W}_+ &:= \{x \in \mathcal{D}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}. \end{aligned}$$

which satisfy:

i) If $(x_0, y_0) \in \mathcal{W}_+$, *then*

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = E \quad \text{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (0, \infty).$$

ii) If $(x_0, y_0) \in \mathcal{W}_-$,

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (0, \infty) \quad \text{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = E.$$

Proof. Clearly the rectangle $[0, \gamma_1] \times (0, \infty)$ is an invariant and attracting set for the map S. In particular, $x_n \le \gamma$ for $n \ge 1$.

Then by Lemma 4 and Theorem 8 every solution $\{(x_n, y_n)\}$ of System (30) has eventually monotone components $\{(x_{2n}, y_{2n})\}$ and $\{(x_{2n+1}, y_{2n+1})\}$, which shows that every bounded solution converges to period-two solution. In view of Lemma 3 there are no minimal period-two solutions and so every bounded solution of System (30) converges to the equilibrium E. In view of Theorems 10 and 11 there is a set $\mathcal{C}$ with described properties. Consequently, every solution with an initial point $(x_0, y_0) \in \mathcal{W}_+$ converges to E, while every solution which starts in $\mathcal{W}_-$ approaches $(0, \infty)$ and is asymptotic to the global unstable manifold $W^u(E)$.

$\square$

2.5 Global Dynamics of System (31)

The equilibrium point $E(\bar{x}, \bar{y})$ of System (31) satisfies the following system of equations

$$\overline{x} = \frac{\gamma_1 \overline{y}}{A_1 + \overline{x}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 \overline{x}}{\overline{y}}. \tag{54}$$

It is easy to see that System (54) has the unique equilibrium point E in the first quadrant, which is an intersection of two parabolas:

$$y = \frac{x(B_1 + x)}{\gamma_1}, \tag{55}$$

and

$$x = \frac{y^2 - \alpha_2}{\beta_2}. \tag{56}$$

It is clear that at the point of intersection E curve (55) is steeper than curve (56), that is,

$$\left.\frac{dy}{dx}\right|_{(55)} (E) > \left.\frac{dy}{dx}\right|_{(56)} (E)$$

which gives

$$2\bar{y}(A_1 + 2\bar{x}) > \beta_2 \gamma_1. \tag{57}$$

2.5.1 Linearized Stability Analysis of System (31)

In this section we prove the following result

Theorem 15 *The unique equilibrium E of System* (31) *is a saddle point.*

Proof. The map S associated to System (31) is

$$U(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1 y}{A_1+x} \\ \frac{\alpha_2+\beta_2 x}{y} \end{pmatrix}. \tag{58}$$

The Jacobian matrix of map (58) has the form

$$J_U(x,y) = \begin{pmatrix} -\frac{\gamma_1 y}{(A_1+x)^2} & \frac{\gamma_1}{A_1+x} \\ \frac{\beta_2}{y} & -\frac{\alpha_2+\beta_2 x}{y^2} \end{pmatrix}, \tag{59}$$

which evaluated at the equilibrium point $E = (\bar{x}, \bar{y})$ is

$$J_U(\overline{x},\overline{y}) = \begin{pmatrix} -\frac{\bar{x}}{A_1+\bar{x}} & \frac{\gamma_1}{A_1+\bar{x}} \\ \frac{\beta_2}{\bar{y}} & -1 \end{pmatrix}. \tag{60}$$

The characteristic equation of System (31) has the following form

$$\lambda^2 + \left(1 + \frac{\bar{x}}{A_1+\bar{x}}\right)\lambda + \frac{\bar{x}}{A_1+\bar{x}} - \frac{\beta_2\gamma_1}{\bar{y}(A_1+\bar{x})} = 0, \tag{61}$$

Set

$$P = 1 + \frac{\bar{x}}{A_1+\bar{x}}, \quad Q = \frac{\bar{x}}{A_1+\bar{x}} - \frac{\beta_2\gamma_1}{\bar{y}(A_1+\bar{x})}.$$

The necessary and sufficient condition for Eq.(61) to have one root inside the unit circle and one root outside the unit circle is $|P| > |1+Q|$, $P^2 - 4Q > 0$, see [12, 13]. In view of the fact that $P > 1+Q$, the condition $|P| > |1+Q|$ leads to $P > -1-Q$ which is equivalent to

$$2 + 2\frac{\bar{x}}{A_1+\bar{x}} > \frac{\beta_2\gamma_1}{\bar{y}(A_1+\bar{x})},$$

which is equivalent to the condition (57).

The condition $P^2 - 4Q > 0$ becomes

$$\left(1 + \frac{\bar{x}}{A_1+\bar{x}}\right)^2 - 4\left(\frac{\bar{x}}{A_1+\bar{x}} - \frac{\beta_2\gamma_1}{\bar{y}(A_1+\bar{x})}\right) > 0,$$

which is equivalent to

$$\left(1 - \frac{\bar{x}}{A_1 + \bar{x}}\right)^2 + 4\frac{\beta_2\gamma_1}{\bar{y}(A_1 + \bar{x})} > 0,$$

which is clearly satisfied. □

2.5.2 Global Results for System (31)

Lemma 5 *System* (31) *has no minimal period-two solution.*

Proof. The second iterate of the map U is given as

$$U^2(x, y) = (S_1, S_2).$$

Where,

$$S_1 = \frac{1}{y}\gamma_1 (x + A_1) \frac{\alpha_2 + x\beta_2}{A_1^2 + xA_1 + y\gamma_1}$$

and

$$S_2 = \frac{y}{(x + A_1)(\alpha_2 + x\beta_2)} (x\alpha_2 + \alpha_2 A_1 + y\beta_2\gamma_1)$$

Period-two solution satisfies $U^2(x, y) = (x, y)$ which reduces to the following two equations

$$xy(A_1x + y) = \gamma_1(\alpha_2 + \beta_2 x), \tag{62}$$

$$\gamma_1 y = x(A_1 + x). \tag{63}$$

Equation (63) is exactly the equilibrium equation (55). Using (55) in (62) we obtain after some elementary simplification that period-two solution satisfies (56). This shows that System (31) has no minimal period-two solution. □

Lemma 6 *The maps U and U^2 associated with System* (31) *have the following properties.*

(i) If $A_1\beta_2 \neq \alpha_2$, then the maps U and U^2 are injective.

(ii) If $A_1\beta_2 \neq \alpha_2$ then $\det J_{U^2}(x, y) > 0$ *for all $(x, y), y > 0$.*

Consequently, U^2 satisfies $(O+)$ condition, when $A_1\beta_2 \neq \alpha_2$.

Proof.

(i) We will prove that U is injective which will imply the injectivity of U^2. The condition

$$U(x_1, y_1) = U(x_2, y_2)$$

is reduced to the following two conditions

$$A_1(y_2 - y_1) = x_2y_1 - x_1y_2, \quad \alpha_2(y_2 - y_1) = \beta_2(x_2y_1 - x_1y_2),$$

which implies that $y_2 - y_1 = \frac{1}{A_1}(x_2y_1 - x_1y_2) = \frac{\beta_2}{\alpha_2}(x_2y_1 - x_1y_2)$ and

$$\left(\frac{1}{A_1} - \frac{\beta_2}{\alpha_2}\right)(x_2y_1 - x_1y_2) = 0$$

and $x_2y_1 - x_1y_2 = 0$, when $A_1\beta_2 \neq \alpha_2$. This implies $y_1 = y_2$ and so $x_1 = x_2$.

(ii) A direct calculation shows that

$$\det J_{U^2} = (\alpha_2 - A_1\beta_2)^2 \frac{\gamma_1}{(\alpha_2 + \beta_2 x)(A_1^2 + A_1 x + \gamma_1 y)^2},$$

which proves our statement.

The statement on $(O+)$ condition follows from Theorem 9. $\square$

Theorem 16 *Consider System* (31)*. Assume that $A_1\beta_2 \neq \alpha_2$. Then there exists a set $\mathcal{C} \subset \mathcal{D} = [0, \infty) \times (0, \infty)$ which is invariant subset of the basin of attraction of E. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval, and separates $\mathcal{D}$ into two connected and invariant components, namely*

$$\mathcal{W}_- \;:\; = \{x \in \mathcal{D}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\},$$

$$\mathcal{W}_+ \;:\; = \{x \in \mathcal{D}\backslash\mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}.$$

which satisfy:

i) If $(x_0, y_0) \in \mathcal{W}_+$, *then*

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (\infty, 0) \quad \textit{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (0, \infty).$$

ii) If $(x_0, y_0) \in \mathcal{W}_-$,

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (0, \infty) \quad \textit{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (\infty, 0).$$

Proof. Assume that $A_1\beta_2 \neq \alpha_2$. Then by Lemma 6 and Theorem 8 every solution $\{(x_n, y_n)\}$ of System (31) has eventually monotone components $\{(x_{2n}, y_{2n})\}$ and $\{(x_{2n+1}, y_{2n+1})\}$, which shows that every bounded solution converges to a period-two solution. In view of Lemma 5 there are no minimal period-two solutions and so every bounded solution of System (31) converges to the equilibrium E. In view of Theorems 10 and 11 there is a set $\mathcal{C}$ with described properties. Consequently, the regions $\mathcal{W}_-$ and $\mathcal{W}_+$ are invariant for U^2 and every solution in $\mathcal{W}_-(\mathcal{W}_+)$ is asymptotic to the unstable manifold $\mathcal{W}^u(E)$ and so the statement of the theorem follows.

□

Theorem 17 *Consider System* (31). *Assume that* $A_1\beta_2 = \alpha_2$. *Then System* (31) *can be decoupled and written as*

$$x_{n+1} = \frac{\beta_2\gamma_1^2}{x_n(A_1 + x_n)}, \quad y_{n+1} = \beta_1\frac{\beta_2\gamma_1 + A_1 y_n}{y_n^2}, \quad n = 0, 1, \dots . \tag{64}$$

Every solution $\{(x_n, y_n)\}$ *of System* (64) *has eventually monotone subsequences* $\{(x_{2n}, y_{2n})\}$ *and* $\{(x_{2n+1}, y_{2n+1})\}$. *Every bounded solution converges to the unique positive equilibrium. Every unbounded solution* $\{(x_n, y_n)\}$ *approaches either* $(\infty, 0)$ *or* $(0, \infty)$.

Proof. Using the condition $A_1\beta_2 = \alpha_2$ in second equation of System (31) gives

$$y_{n+1} = \beta_2\frac{A_1 + x_n}{y_n},$$

and so $x_{n+1}y_{n+1} = \beta_2\gamma_1$ which shows that System (31) has an invariant of the form

$$x_n y_n = \beta_2\gamma_1, \quad n = 1, 2, \dots . \tag{65}$$

Using (65) System (31) is reduced to System (64). Both equations of System (64) are first order difference equations with decreasing functions and so by Theorem 1.19 [13] the subsequences of even and odd indexes are eventually monotonic and so every bounded solution converges to a period-two solution. An immediate checking show that neither one of two equations of System (64) has period-two solutions. For example, the unique equilibrium $\bar{x}$ of first equation of System (64) satisfies equation

$$x^3 + A_1 x^2 - \beta_2\gamma_1^2 = 0, \tag{66}$$

while period-two solution satisfies $f^2(x) = x$ which becomes $x^2 \frac{(A+x)^2}{A^2x+Ax^2+C} = x$ and so is reduced to Eq.(66). Thus every bounded solution converges to the unique equilibrium.

The result for unbounded solutions follows immediately from (65).

□

List of References

[1] A. Brett and M. R. S. Kulenović, Basins of Attraction of Equilibrium Points of Monotone Difference Equations, *Sarajevo J. Math.*, 5(2009), 211–233.

[2] A. Brett, M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, Global Behavior of Two Competitive Rational Systems of Difference Equations in the Plane, *Comm. Appl. Nonlinear Anal.*, 16(2009), 1-18.

[3] E. Camouzis, M. R. S. Kulenović, G. Ladas and O. Merino, Rational Systems in the Plane - Open Problems and Conjectures, *J. Differ. Equations Appl.*, 15(2009), 303–323.

[4] E. Camouzis, G. Ladas and L. Wu, On the global character of the system $x_{n+1} = \frac{\alpha_1+\gamma_1 y_n}{x_n}, y_{n+1} = \frac{\beta_2 x_n+\gamma_2 y_n}{B_2 x_n+C_2 y_n}$, *Int. J. Pure Appl. Math.*, 53(2009), 21–36.

[5] D. Clark and M. R. S. Kulenović, On a Coupled System of Rational Difference Equations, *Comput. Math. Appl.* 43 (2002), 849-867.

[6] C. A. Clark, M. R. S. Kulenović, and J.F. Selgrade, On a system of rational difference equations, *J. Differ. Equations Appl.* 11 (2005), 565 - 580.

[7] M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, Global Behavior of Four Competitive Rational Systems of Difference Equations in the Plane, *Discrete Dyn. Nat. Soc.* (2009), Article ID 153058, 34 pages.

[8] P. Hess, *Periodic-parabolic boundary value problems and positivity*, Pitman Research Notes in Mathematics Series, 247. Longman Scientific Technical, Harlow; copublished in the United States with John Wiley Sons, Inc., New York, 1991

[9] M. Hirsch and H. Smith, *Monotone Dynamical Systems, Handbook of Differential Equations, Ordinary Differential Equations* (second volume), 239-357, Elsevier B. V., Amsterdam, 2005.

[10] Y. S. Huang and P. M. Knopf, Global convergence properties of first-order homogeneous systems of rational difference equations, *J. Differ. Equations Appl.*, 18(2012), 1683–1707.

[11] S. Kalabušić and M.R.S. Kulenović, Dynamics of Certain Anti-competitive Systems of Rational Difference Equations in the Plane, *J. Differ. Equations Appl.*, 17(2011), 1599–1615.

[12] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC, Boca Raton, London, 2001.

[13] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall/CRC, Boca Raton, London, 2002.

[14] M. R. S. Kulenović and O. Merino, Competitive-Exclusionversus Competitive-Coexistence for Systems in the Plane*, Discrete Cont. Dyn. Syst. Ser. B*, 6 (2006), 1141–1156.

[15] M. R. S. Kulenović and O. Merino, Global Bifurcation for Discrete Competitive Systems in the Plane, *Discrete Cont. Dyn. Syst. Ser. B*, 12 (2009), 133–149.

[16] M. R. S. Kulenović and O. Merino, Invariant Manifolds for Competitive Discrete Systems in the Plane, *Int. J. of Bifurcations and Chaos*, 20(2010), 2471–2486.

[17] M. R. S. Kulenović and M. Nurkanović, Asymptotic behavior of a system of linear fractional difference equations, *Advances in Difference Equations,.*(2006), 1-13.

[18] M. R. S. Kulenović and M. Nurkanović, Basins of Attraction of an Anti-competitive System of Difference Equations in the Plane, *Comm. Appl. Nonlinear Anal.*, 19(2012), 41–53.

[19] C. Robinson, *Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1995.

[20] H. L. Smith, Planar Competitive and Cooperative Difference Equations, *J. Differ. Equations Appl.* 3(1998), 335-357.

[21] H. L. Smith, The discrete dynamics of monotonically decomposable maps, *J. Math. Biology* 53(2006), 747-758.

[22] H. L. Smith, Invariant curves for mappings, *SIAM J. Math. Anal.* 17(1986), 1053-1067.

MANUSCRIPT 3

Global Dynamics of Some Second Order Fractional Difference Equations in the Plane

In preparation for submition in *Discrete Dynamics in Nature and Society* .

M. DiPippo and M. R. S. Kulenović

Department of Mathematics

University of Rhode Island

Kingston, RI, 02881, USA

3.1 Introduction

In this paper, we investigate global behavior of the equation

$$x_{n+1} = \frac{x_{n-1}^2}{ax_n^2 + cx_{n-1}^2 + f}, \quad n = 0, 1, 2, ..., \tag{67}$$

where the parameters a, c and f are nonnegative numbers with condition $a + c > 0, f \neq 0$ and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. Equation (67) is a special case of equations

$$x_{n+1} = \frac{Ax_n^2 + Cx_{n-1}^2 + F}{ax_n^2 + cx_{n-1}^2 + f}, \quad n = 0, 1, 2, ... \tag{68}$$

and

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{69}$$

Some special cases of Eq.(69) have been considered in the series of papers [4, 5, 10, 12, 13, 24]. Some special second order quadratic fractional difference equations have appeared in analysis of competitive and anti-competitive systems of linear fractional difference equations in the plane, see [7, 11, 21]. In this paper we take an approach based on the theory of monotone maps developed in [16, 17] and use it to describe precisely the basins of attraction of all attractors of this equation as well as all undergoing bifurcations. The special case of Eq.(69) for $a = 0$ is well known Thomson equation [2] used in the modeling of fish population [25].

The presence of quadratic terms in Eq.(67) effects the dynamic behavior of the corresponding linear fractional equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{70}$$

in an interesting way by introducing new dynamic phenomena such as the coexistence of two locally stable equilibrium points, the coexistence of locally stable equilibrium point and minimal period-two solutions and existence of so called Allee effect. The global dynamics of Eq.(67), in the case when $f > \frac{1}{4c}$, is quite simple as the zero equilibrium is globally asymptotically stable and interesting dynamics happens when $f \leq \frac{1}{4c}$.

The special case of Eq.(67) which plays an important role in mathematical biology is the following equation

$$x_{n+1} = \frac{x_{n-1}^2}{cx_{n-1}^2 + f} \quad n = 0, 1, \ldots \tag{71}$$

where c and f are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. The dynamics of (71), described in [18], is very interesting and follows from the dynamics of related equation first considered by Thomson [25]

$$x_{n+1} = \frac{x_n^2}{cx_n^2 + f} \quad n = 0, 1, \ldots \tag{72}$$

and completely described in [2]. Equation (71) has either one or three period-two solutions and all its solutions are bounded and so its dynamics is considerably richer and more interesting from the modeling point of view.

Theorem 18 *1. If $f > \frac{1}{4c}$, then (71) has one equilibrium E_0 which is globally asymptotically stable with the basins of attraction*

$$\mathcal{B}(E_0) = \{(u, v) : 0 \leq u, 0 \leq v\}.$$

2. If $f = \frac{1}{4c}$, then (71) has two equilibrium points $E_0, E\left(\frac{1}{2c}, \frac{1}{2c}\right)$ and the minimal period-two solution $\{P_x, P_y\} = \left\{\left(\frac{1}{2c}, 0\right), \left(0, \frac{1}{2c}\right)\right\}$.

The basins of attraction of the equilibrium points and the period-two solution are given as

$$\mathcal{B}(E_0) = \left\{(u, v) : 0 \leq u < \tfrac{1}{2c}, 0 \leq v < \tfrac{1}{2c}\right\},$$

$$\mathcal{B}(E) = \left\{(u, v) : \tfrac{1}{2c} \leq u, \tfrac{1}{2c} \leq v\right\},$$

$$\mathcal{B}(P_x)=\left\{(u,v):\tfrac{1}{2c}\le u<\infty,0\le v<\tfrac{1}{2c}\right\},$$

$$\mathcal{B}(P_y)=\left\{(u,v):0\le u<\tfrac{1}{2c},\tfrac{1}{2c}\le v<\infty\right\}.$$

3. *If $f<\frac{1}{4c}$, then (71) has three equilibrium points*

$$E_0,E_-\left(\tfrac{1-\sqrt{1-4cf}}{2c},\tfrac{1-\sqrt{1-4cf}}{2c}\right),\ \textit{and}\ E_+\left(\tfrac{1+\sqrt{1-4cf}}{2c},\tfrac{1+\sqrt{1-4cf}}{2c}\right)$$

and three minimal period-two solutions

$$\{P_x^1,P_y^1\}=\left\{\left(\tfrac{1-\sqrt{1-4cf}}{2c},0\right),\left(0,\tfrac{1-\sqrt{1-4cf}}{2c}\right)\right\},$$

$$\{P_x^2,P_y^2\}=\left\{\left(\tfrac{1+\sqrt{1-4cf}}{2c},0\right),\left(0,\tfrac{1+\sqrt{1-4cf}}{2c}\right)\right\}$$

and

$$\{P_\mp^3,P_\pm^3\}=\left\{\left(\tfrac{1-\sqrt{1-4cf}}{2c},\tfrac{1+\sqrt{1-4cf}}{2c}\right),\left(\tfrac{1+\sqrt{1-4c}}{2c},\tfrac{1-\sqrt{1-4cf}}{2c}\right)\right\}.$$

The corresponding basins of attraction are given as

$$\mathcal{B}(E_0)=\left\{(u,v):0\le u<\tfrac{1-\sqrt{1-4cf}}{2c},0\le v<\tfrac{1-\sqrt{1-4cf}}{2c}\right\},\quad \mathcal{B}(E_-)=\{E_-\},$$

$$\mathcal{B}(E_+)=\left\{(u,v):\tfrac{1-\sqrt{1-4cf}}{2c}<u,\tfrac{1-\sqrt{1-4cf}}{2c}<v\right\},$$

$$\mathcal{B}(P_\pm^3)=\left\{\left(u,\tfrac{1-\sqrt{1-4cf}}{2c}\right):\tfrac{1-\sqrt{1-4cf}}{2c}<u<\infty\right\},$$

$$\mathcal{B}(P_\mp^3)=\left\{\left(\tfrac{1-\sqrt{1-4cf}}{2c},v\right):\tfrac{1-\sqrt{1-4cf}}{2c}<v<\infty\right\},$$

$$\mathcal{B}(P_x^1)=\left\{\left(\tfrac{1-\sqrt{1-4cf}}{2c},v\right):0\le v<\tfrac{1-\sqrt{1-4cf}}{2c}\right\},$$

$$\mathcal{B}(P_y^1)=\left\{\left(u,\tfrac{1-\sqrt{1-4cf}}{2c}\right):0\le u<\tfrac{1-\sqrt{1-4cf}}{2c}\right\},$$

$$\mathcal{B}(P_x^2)=\left\{(u,v):\tfrac{1-\sqrt{1-4cf}}{2c}<u<\infty,0\le v<\tfrac{1-\sqrt{1-4cf}}{2c}\right\},$$

$$\mathcal{B}(P_y^2)=\left\{(u,v):0\le u<\tfrac{1-\sqrt{1-4cf}}{2c},\tfrac{1-\sqrt{1-4cf}}{2c}<v<\infty\right\}.$$

Thus Eq.(67) has some potential applications in mathematical modeling.

The local and global dynamics of Eq.(67) depends on the location of the parameter f. We obtain seven different dynamics scenarios for local and global dynamics of Eq.(67) vs. the parameters $\frac{c-3a}{4(a-c)^2}$, $\frac{1}{4(a+c)}$ and $\frac{1}{4c}$. The large role in dynamical scenarios is played by the equilibrium solutions and period-two solutions. The global dynamics can be explained in terms of bifurcation theory for the second iterate of the corresponding map T^2. Since both the equilibrium solutions and the period-two solutions are the equilibrium solutions of the second iterate of the corresponding map, we can express our results as a change of stability bifurcations for T^2. Figure 1 gives a visual representation of local dynamics of Eq.(67).

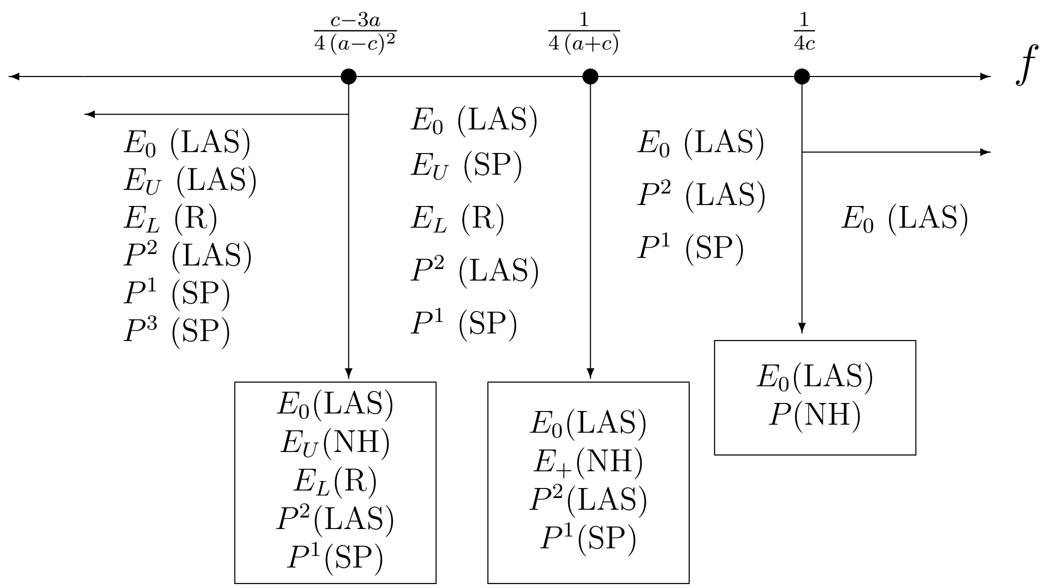

Figure 1. Visual representation of LAS of Eq. (67). Here LAS=locally asymptotically stable; R=repeller; SP=saddle point; NH= non-hyperbolic.

The dynamics of Eq.(67) undergoes three bifurcations as the parameter f passes through three critical values $\frac{c-3a}{4(a-c)^2}$, $\frac{1}{4(a+c)}$ and $\frac{1}{4c}$. As f passes through $\frac{c-3a}{4(a-c)^2}$ the equilibrium E_U loses its stability and the period-two solution p^3 disappears, which affects the basins of attraction in such a way that the basin of attraction of the equilibrium E_U shrinks to its global stable manifold. The second bifurcation occurs when f passes through $\frac{1}{4(a+c)}$ in which case two equilibrium solutions E_U and E_L disappear and the basins of attraction of the remaining equilibrium and one of the period-two solution remain. Finally, the third bifurcation occurs when f passes through $\frac{1}{4c}$ in which case two period-two solutions disappear and E_0 becomes globally asymptotically stable. If we consider these bifurcations when parameter f decreases through three critical values then the first bifurcation as f decreases through $\frac{1}{4c}$ and third bifurcation as f decreases through $\frac{c-3a}{4(a-c)^2}$ are period doubling bifurcations. The result of second bifurcation as f decreases through $\frac{1}{4(a+c)}$ is appearance of two new equilibrium points, which only effects the boundary of the basins of attraction of existing attractors.

3.2 Preliminaries

We use the following theorem for a general second order difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}\right), \qquad x_{-1}, x_0 \in I,\ n = 0, 1, 2, ... \tag{73}$$

the following result is from [3].

Theorem 19 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is non-increasing in the first variable and non-decreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation (73) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually both monotonic.*

Theorem 19 tells us that every bounded solution of (73) converges to either an equilibrium or a period-two solution or to the point on the boundary where equation is not defined, see [1, 9]. To determine the basins of attraction of these solutions, we will use theory of monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

In this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East (NE) ordering is called *cooperative* and a map monotone with respect to the South-East (SE) ordering is called *competitive*. A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which second iterate T^2 is monotone with respect to the North-East (resp. South-East) ordering is called *anti-cooperative* (resp. *anti-competitive*), see [11].

If T is differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone (resp. anti-monotone) with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \quad \left(\text{resp. } \operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} - & + \\ + & - \end{bmatrix}\right), \tag{74}$$

provided that $\mathcal{R}$ is open and convex.

For $(x_1, x_2) \in \mathbb{R}^2$, define $Q_\ell(x_1, x_2)$ for $\ell = 1, 2, 3, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x_1, x_2) = \{y = (y_1, y_2) \in R^2 : \; x_1 \leq y_1, \; x_2 \leq y_2\}$. Basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p.

The following results, from [17, 16], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [23, 24].

Theorem 20 *Let $\mathcal{R}$ be a rectangular subset of $\mathbb{R}^2$ and let T be a competitive map on $\mathcal{R}$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $(\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}$ has nonempty interior (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$).*

Suppose that the following statements are true.

a. *The map T is strongly competitive on* $\operatorname{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R})$.

b. *T is C^2 on a relative neighborhood of $\overline{x}$.*

c. *The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $|\lambda| < \mu$, where λ is stable and the eigenspace E^λ associated with λ is not a coordinate axis.*

d. *Either $\lambda \geq 0$ and*

$$T(x) \neq \overline{x} \quad \textit{and} \quad T(x) \neq x \quad \textit{for all } x \in \operatorname{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R}),$$

or $\lambda < 0$ and

$$T^2(x) \neq x \quad \textit{for all } x \in \operatorname{int}((\mathcal{Q}_1(\overline{x}) \cup \mathcal{Q}_3(\overline{x})) \cap \mathcal{R})$$

Then there exists a curve $\mathcal{C}$ in $\mathcal{R}$ such that

(i) *$\mathcal{C}$ is invariant and a subset of $\mathcal{W}^s(\overline{x})$.*

(ii) *the endpoints of $\mathcal{C}$ lie on $\partial\mathcal{R}$.*

(iii) *$\overline{x} \in \mathcal{C}$.*

(iv) *$\mathcal{C}$ the graph of a strictly increasing continuous function of the first variable,*

(v) $\mathcal{C}$ *is differentiable at* $\overline{x}$ *if* $\overline{x} \in \operatorname{int}(\mathcal{R})$ *or one sided differentiable if* $\overline{x} \in \partial\mathcal{R}$, *and in all cases* $\mathcal{C}$ *is tangential to* E^{λ} *at* $\overline{x}$,

(vi) $\mathcal{C}$ *separates* $\mathcal{R}$ *into two connected components, namely*

$$\mathcal{W}_{-} := \{x \in \mathcal{R} : \exists y \in \mathcal{C} \textit{ with } x \preceq y\}$$

and

$$\mathcal{W}_{+} := \{x \in \mathcal{R} : \exists y \in \mathcal{C} \textit{ with } y \preceq x\}$$

(vii) $\mathcal{W}_{-}$ *is invariant, and* $\operatorname{dist}(T^n(x), \mathcal{Q}_2(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_{-}$.

(viii) $\mathcal{W}_{+}$ *is invariant, and* $\operatorname{dist}(T^n(x), \mathcal{Q}_4(\overline{x})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_{+}$.

Theorem 21 *For the curve* $\mathcal{C}$ *of Theorem 20 to have endpoints in* $\partial\mathcal{R}$, *it is sufficient that at least one of the following conditions is satisfied.*

i. The map T *has no fixed points nor periodic points of minimal period two in* Δ.

ii. The map T *has no fixed points in* Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, *and* $T(x) = \overline{\mathrm{x}}$ *has no solutions* $x \in \Delta$.

iii. The map T *has no points of minimal period-two in* Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, *and* $T(x) = \overline{\mathrm{x}}$ *has no solutions* $x \in \Delta$.

Theorem 22 *(A) Assume the hypotheses of Theorem 20, and let* $\mathcal{C}$ *be the curve whose existence is guaranteed by Theorem 20. If the endpoints of* $\mathcal{C}$ *belong to* $\partial\mathcal{R}$, *then* $\mathcal{C}$ *separates* $\mathcal{R}$ *into two connected components, namely*

$$\mathcal{W}_{-} := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \textit{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \tag{75}$$

$$\mathcal{W}_{+} := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \textit{ with } y \preceq_{se} x\}, \tag{76}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ *is invariant, and* $\operatorname{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ *is invariant, and* $\operatorname{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ *is an interior point of* $\mathcal{R}$ *and* T *is* C^2 *and strongly competitive in a neighborhood of* $\overline{\mathrm{x}}$*, then* T *has no periodic points in the boundary of* $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ *except for* $\overline{\mathrm{x}}$*, and the following statements are true.*

(iii) For every $x \in \mathcal{W}_-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_2(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_4(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\Big\{\ x \in \mathcal{R} : \exists \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x,\ \text{and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are actually the global stable and unstable manifolds of $\overline{x}$.

Theorem 23 *In addition to the hypotheses of part (B) of Theorem 22, suppose that* $\mu > 1$ *and that the eigenspace* E^μ *associated with* μ *is not a coordinate axis. If the curve* $\mathcal{C}$ *of Theorem 20 has endpoints in* $\partial\mathcal{R}$*, then* $\mathcal{C}$ *is the stable set* $\mathcal{W}^s(\overline{\mathrm{x}})$ *of* $\overline{\mathrm{x}}$*, and the unstable set* $\mathcal{W}^u(\overline{\mathrm{x}})$ *of* $\overline{x}$ *is a curve in* $\mathcal{R}$ *that is tangential to* E^μ *at* $\overline{\mathrm{x}}$ *and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of* $\mathcal{W}^u(\overline{\mathrm{x}})$ *in* $\mathcal{R}$ *are fixed points of* T.

Remark 4 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Eq.(73) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Eq.(73) is a strictly competitive map on $I \times I$, see [17].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Eq.(73) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array} \quad , \qquad n = 0, 1, \ldots .$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [17].

Remark 5 The characteristic equation of Eq.(73) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{77}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 20-23 depends on the existence of minimal period-two solution.

A *non-hyperbolic point* of Eq.(73), where both characteristic roots are real, is said to be of *stable type* (resp. *unstable type*) if the second characteristic root is in the interval $(-1, 1)$ (resp. outside the interval $[-1, 1]$). A non-hyperbolic point of Eq.(73) is said to be of the *resonance type* $(i, j), i, j \in \{-1, 1\}$ if one characteristic root has value i and the other has value j. See [20] for definitions of non-hyperbolic points of resonance types for general difference equations.

There are several global attractivity results for Eq.(73). Some of these results give the sufficient conditions for all solutions to approach a unique equilibrium and they were used efficiently in [14]. The next result is from [6].

Theorem 24 *(See [6]) Consider Eq.*(73) *where* $f : I \times I \to I$ *is a continuous function and* f *is decreasing in the first argument and increasing in the second argument. Assume that* $\overline{x}$ *is a unique equilibrium point which is locally asymptotically stable and assume that* (φ, ψ) *and* (ψ, φ) *are minimal period-two solutions which are saddle points such that*

$$(\varphi, \psi) \preceq_{se} (\overline{x}, \overline{x}) \preceq_{se} (\psi, \varphi).$$

Then, the basin of attraction $\mathcal{B}((\overline{x}, \overline{x}))$ *of* $(\overline{x}, \overline{x})$ *is the region between the global stable sets* $\mathcal{W}^s((\varphi, \psi))$ *and* $\mathcal{W}^s((\psi, \varphi))$. *More precisely*

$$\mathcal{B}((\overline{x}, \overline{x})) =$$

$\{(x, y) : \exists y_u, y_l : y_u < y < y_l, (x, y_l) \in \mathcal{W}^s((\varphi, \psi)), (x, y_u) \in \mathcal{W}^s((\psi, \varphi))\}$. *The basins of attraction* $\mathcal{B}((\varphi, \psi)) = \mathcal{W}^s((\varphi, \psi))$ *and* $\mathcal{B}((\psi, \varphi)) = \mathcal{W}^s((\psi, \varphi))$ *are exactly the global stable sets of* (φ, ψ) *and* (ψ, φ).

If $(x_{-1}, x_0) \in \mathcal{W}_+((\psi, \varphi))$ *or* $(x_{-1}, x_0) \in \mathcal{W}_-((\varphi, \psi))$, *then* $T^n((x_{-1}, x_0))$ *converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region* $I \times I$.

Here $\mathcal{W}_+$ *(resp.* $\mathcal{W}_-$*) denotes the region below (resp. above) the stable manifold* $\mathcal{W}^s$ *in the North-east ordering.*

The next result provides better understanding of the behavior of orbits in some cases (see [11]).

Lemma 7 *Let* T *be an anti-competitive map on a rectangular region* $\mathcal{R} \subset \mathbb{R}^2$ *and let* $\overline{\mathbf{x}} \in \mathcal{R}$ *be a fixed point of* T. *The stable set* $\mathcal{W}^s(\overline{\mathbf{x}})$ *of* $\overline{\mathbf{x}}$ *in Theorem 23 satisfies:*

i) if $\mathbf{x} \in \mathcal{W}_+$*, then* $T(\mathbf{x}) \in \mathcal{W}_-$,

ii) if $\mathbf{x} \in \mathcal{W}_-$*, then* $T(\mathbf{x}) \in \mathcal{W}_+$.

3.3 Local stability analysis

The equilibrium points $\overline{x}$ of Eq.(67) satisfy

$$\overline{x} = \frac{\overline{x}^2}{(a+c)\overline{x}^2 + f}, \tag{78}$$

i.e.

$$\bar{x} = 0 \quad \text{and/or} \quad (a+c)\overline{x}^2 - \overline{x} + f = 0.$$

Consequently,

i) The zero equilibrium $\overline{x} = 0$ exists for all values of the parameters.

ii) If $f > \frac{1}{4(a+c)}$, the zero equilibrium, E_0, is the only solution to the equilibrium equation.

iii) If $f = \frac{1}{4(a+c)}$, there exists the positive equilibrium point $E_+ = \frac{1}{2(a+c)}$.

iv) If $f < \frac{1}{4(a+c)}$, there exists two positive equilibrium points

$$E_U = \frac{1+\sqrt{1-4f(a+c)}}{2(a+c)} \text{ and } E_L = \frac{1-\sqrt{1-4f(a+c)}}{2(a+c)}.$$

If we denote

$$h(u,v) = \frac{v^2}{au^2 + cv^2 + f},$$

then Eq.(67) has a linearized equation:

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = \frac{\partial h}{\partial u}(\overline{x}, \overline{x}) = \frac{-2a\overline{x}^3}{((a+c)\overline{x}^2 + f)^2}, \text{and} \quad q = \frac{\partial h}{\partial v}(\overline{x}, \overline{x}) = \frac{2\overline{x}(a\overline{x}^2 + f)}{((a+c)\overline{x}^2 + f)^2}.$$

If $\overline{x} \neq 0$, we obtain by (78), that

$$p = -2a\overline{x} \quad \text{and} \quad q = 2(1 - c\overline{x}).$$

Proposition 1 *a) If* $f = \frac{1}{4(b+c)}$, *then the equilibrium point* E_+ *of Eq.*(67) *is non-hyperbolic of unstable type.*

b) If $f < \frac{1}{4(b+c)}$, *then the equilibrium point* E_L *of Eq.*(67) *is repeller and the equilibrium point* E_U *is:*

i) a saddle point when $c \leq 3a$ *or* $f > \frac{c-3a}{4(a-c)^2} > 0$,

ii) a non-hyperbolic point of stable type when $f = \frac{c-3a}{4(a-c)^2} > 0$ *and*

iii) locally asymptotically stable when $f < \frac{c-3a}{4(a-c)^2}$ *and* $c > 3a$.

c) Equilibrium E_0 *of Eq.*(67) *is locally asymptotically stable for all value of parameters.*

Proof.

a) We have that

$$p = -2a\overline{x} = -\frac{a}{(a+c)}, \quad q = 2(1 - c\overline{x}) = \frac{2a+c}{(a+c)}$$

so $\lambda_1 = -1, \lambda_2 = -\frac{2a+c}{(a+c)} < -1$ therefore the equilibrium point E_+ is non-hyperbolic of unstable type.

b) In this case we have that $|q| > 1 \Leftrightarrow |2 - 2c\overline{x}| > 1 \Leftrightarrow 2 - 2c\overline{x} > 1$ or $2 - 2c\overline{x} < -1$. Now, $2 - 2c\overline{x} > 1 \Leftrightarrow 1 > 2c\overline{x} \Leftrightarrow a + c > c - c\sqrt{1 - 4f(a+c)}$ which is always true therefore $|q| > 1$.

Now observe that $|p| < |1 - q| \Leftrightarrow |-2a\overline{x}| < |1 - (2 - 2c\overline{x})| \Leftrightarrow 2a\overline{x} < |2c\overline{x} - 1| \Leftrightarrow 2c\overline{x} - 1 > 2a\overline{x}$ or $2c\overline{x} - 1 < -2a\overline{x}$.

Since $2c\overline{x} - 1 < -2a\overline{x} \Leftrightarrow 2\overline{x} < \frac{1}{a+c} \Leftrightarrow \frac{1-\sqrt{1-4f(a+c)}}{a+c} < \frac{1}{a+c}$, which is always true, it follows that $|p| < |1 - q|$, therefore $E_L = \frac{1-\sqrt{1-4f(a+c)}}{2(a+c)}$ is a repeller.

bi) In this case,

$|p| > |1-q| \Leftrightarrow 2a\overline{x} > |2c\overline{x}-1|$. Now, $2a\overline{x} > |2c\overline{x}-1| \Leftrightarrow -2a\overline{x} < 2c\overline{x}-1 < 2a\overline{x}$.

In regards to the left hand side, we have

$-2a\overline{x} < 2c\overline{x}-1 \Leftrightarrow 1 < 2\overline{x}(a+c) \Leftrightarrow 1 < 1 + \frac{1+\sqrt{1-4f(a+c)}}{(a+c)}$. Which is always true.

In regards to the right hand side, we have

$2c\overline{x}-1 < 2a\overline{x} \Leftrightarrow 2c\overline{x}-2a\overline{x} < 1 \Leftrightarrow 1+\sqrt{1-4f(a+c)} < \frac{a+c}{c-a}$. Which is always true.

Therefore $E_U = \frac{1+\sqrt{1-4f(a+c)}}{2(a+c)}$ is a saddle.

bii) In this case $|p| = |1-q| \Leftrightarrow 2a\overline{x} = |2c\overline{x}-1|$.

Observe that $2a\overline{x} = |2c\overline{x}-1| \Leftrightarrow 2c\overline{x}-1 = 2a\overline{x}$ or $2c\overline{x}-1 = -2a\overline{x}$.

Now $2c\overline{x}-1 = 2a\overline{x} \Leftrightarrow f = \frac{(a-c)^2-4a^2}{4(a-c)^2(a+c)} = \frac{c-3a}{4(a-c)^2}$, which is true in this case. Therefore $|p| = |1-q|$, and we have that $E_U = \frac{1+\sqrt{1-4f(a+c)}}{2(a+c)}$ is a nonhyporbolic point.

biii) Observe that $|p| < 1-q < 2 \Rightarrow 2a\overline{x} < 2c\overline{x}-1 < 2$.

In regards to the left hand side,

$2a\overline{x} < 2c\overline{x}-1 \Leftrightarrow f < \frac{4a^2-(a-c)^2}{(-4(a+c))(a-c)^2} = \frac{c-3a}{4(a-c)^2}$, which is true in this case.

In regards to the right hand side,

$2c\overline{x}-1 < 2 \Leftrightarrow 1-4f(a+c) < (\frac{3a+2c}{c})^2 \Leftrightarrow f > \frac{9a^2+3c^2+12ac}{-4(a+c)c^2}$, which is always true. Therefore the equilibrium point E_U is locally asymptotically stable.

c) Since, $p = q = 0$ for $\overline{x} = 0$, we have $\lambda^2 = 0$ which means that E_0 is locally asymptotically stable for all value of parameters.

$\square$

3.4 Period-two solutions of Eq. (67)

Now, we present results about existence and local stability of minimal period-two solutions of Eq.(67).

Theorem 25 *Assume that $a > 0$ and $c > 0$.*

i) If $f > \frac{1}{4c}$, then Eq. (67) has no minimal period-two solutions.

ii) If $f = \frac{1}{4c}$, then Eq. (67) has a minimal period-two solutions $P_x\left(\frac{1}{2c}, 0\right), P_y\left(0, \frac{1}{2c}\right)$.

iii) If $f < \frac{1}{4c}$, then Eq. (67) has two minimal period-two solutions $P_x^1\left(\frac{1-\sqrt{1-4cf}}{2c}, 0\right), P_y^1\left(0, \frac{1-\sqrt{1-4cf}}{2c}\right)$ and $P_x^2\left(\frac{1+\sqrt{1-4cf}}{2c}, 0\right), P_y^2\left(0, \frac{1+\sqrt{1-4cf}}{2c}\right)$.

iv) If $0 < f < \frac{c-3a}{4(a-c)^2}$, then Eq. (67) has three minimal period-two solutions $\left\{P_x^1, P_y^1\right\}, \left\{P_x^2, P_y^2\right\}$ and
$P_{\pm}^3\left(\frac{(a+c)+\sqrt{-(a+c)((3a-c)+4(a-c)^2 f)}}{2(c-a)(a+c)}, \frac{(a+c)-\sqrt{-(a+c)((3a-c)+4(a-c)^2 f)}}{2(c-a)(a+c)}\right)$,
$P_{\mp}^3\left(\frac{(a+c)-\sqrt{-(a+c)((3a-c)+4(a-c)^2 f)}}{2(c-a)(a+c)}, \frac{(a+c)+\sqrt{-(a+c)((3a-c)+4(a-c)^2 f)}}{2(c-a)(a+c)}\right)$.

Proof. Suppose that there exists a minimal period-two solution $\{\phi, \psi, \phi, \psi, ...\}$ of Eq.(67), where ϕ and ψ are distinct nonnegative real numbers such that $\phi^2 + \psi^2 \neq 0$. Then ϕ, ψ satisfy the following system:

$$\phi = \frac{\phi^2}{a\psi^2 + c\phi^2 + f}, \psi = \frac{\psi^2}{a\phi^2 + c\psi^2 + f}, \tag{79}$$

which is equivalent to the system

$$\phi(a\psi^2 + c\phi^2 + f) = \phi^2, \quad \psi(a\phi^2 + c\psi^2 + f) = \psi^2.$$

Thus we obtain that if $\phi = 0$, then the second equation in (79) is of the form $c\psi^2 - \psi + f = 0$, so if

i) $D=1-4cf<0$ i.e. $f>\frac{1}{4c}$, there is no minimal period-two solutions,

ii) $D=1-4cf=0$ i.e. $f=\frac{1}{4c}$, then $\psi=\frac{1}{2c}$ and there is a minimal period-two solutions $(\phi,\psi)=(0,\frac{1}{2c})$,

iii) $D=1-4cf>0$ i.e. $f<\frac{1}{4c}$, then $\psi_{\pm}=\frac{1\pm\sqrt{1-4cf}}{2c}$ and there is two minimal period-two solutions $(\phi,\psi)=\left(0,\frac{1-\sqrt{1-4cf}}{2c}\right),(\phi,\psi)=\left(0,\frac{1+\sqrt{1-4cf}}{2c}\right)$.

If $\psi=0$, we get analogous conclusion for ϕ.

iv) If $\phi\neq 0$ and $\psi\neq 0$, we have the following system

$$a\psi^2+c\phi^2+f=\phi,\quad a\phi^2+c\psi^2+f=\psi, \tag{80}$$

which implies

$$\phi+\psi=\frac{1}{c-a}. \tag{81}$$

and

$$\phi\psi=\frac{1-(a+c)(c-a)}{2(c-a)^2}-f. \tag{82}$$

In light of (81) and (82) we have no interior period two solutions if $a\geq c$. Assume that $a<c$, then we have the solution

$$(\phi,\psi)\pm=\frac{(a+c)\pm\sqrt{-(a+c)((3a-c)+4(a-c)^2f)}}{2(c-a)(a+c)}.$$

By substitution $x_{n-1}=u_n, x_n=v_n$ Eq.(67) becomes the system of equations

$$\begin{cases} u_{n+1}=v_n \\ v_{n+1}=\dfrac{u_n^2}{av_n^2+cu_n^2+f} \end{cases}. \tag{83}$$

The map T corresponding to (83) is of the form

$$T\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} v \\ h(u,v) \end{pmatrix}=\begin{pmatrix} v \\ \dfrac{u^2}{av^2+cu^2+f} \end{pmatrix}. \tag{84}$$

The second iteration of the map T is

$$T^2\begin{pmatrix} u \\ v \end{pmatrix}=T\begin{pmatrix} v \\ h(u,v) \end{pmatrix}=\begin{pmatrix} h(u,v) \\ h(v,h(u,v)) \end{pmatrix}=\begin{pmatrix} G(u,v) \\ H(u,v) \end{pmatrix},$$

where

$$H(u,v) = \frac{v^2}{ah(u,v)^2 + cv^2 + f},$$

and the map T^2 is competitive by Remark 5. Now we obtain that the Jacobian matrix of the map T^2 at the point (ϕ, ψ) is of the form

$$J_{T^2}(\phi,\psi) = \begin{pmatrix} \frac{\partial G(\phi,\psi)}{\partial u} & \frac{\partial G(\phi,\psi)}{\partial v} \\ \frac{\partial H(\phi,\psi)}{\partial u} & \frac{\partial H(\phi,\psi)}{\partial v} \end{pmatrix},$$

where

$$\frac{\partial G(\phi,\psi)}{\partial u} = \frac{2a\psi^2+2f}{\phi}, \tag{85}$$

$$\frac{\partial G(\phi,\psi)}{\partial v} = -2a\psi, \tag{86}$$

$$\frac{\partial H(\phi,\psi)}{\partial u} = -4a(f + a\psi^2), \tag{87}$$

$$\frac{\partial H(\phi,\psi)}{\partial v} = 2 + 4a\phi\psi. \tag{88}$$

□

Theorem 26 *The local character of the period-two solutions is as follows:*

i) The minimal period-two solutions $\{P_x, P_y\}$ are non-hyperbolic points of stable type.

ii) The minimal period-two solutions $\{P_x^1, P_y^1\}$ are saddle points, $\{P_x^2, P_y^2\}$ are locally asymptotically stable.

iii) The minimal period-two solutions $\{P_{\mp}^3, P_{\pm}^3\}$ are saddle points.

Proof.

i) The minimal period-two solutions $\{P_x, P_y\}$ we have for $f = \frac{1}{4c}$. In that case, the Jacobian matrix of the map T^2 at the points P_x and P_y is of the form

$$J_{T^2}(P_x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{T^2}(P_y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$, which means that the periodic solutions $\{P_x, P_y\}$ are non-hyperbolic points of stable type.

ii) The Jacobian matrix of the map T^2 at the points $\left\{P_x^1, P_y^1\right\}$ is of the form

$$J_{T^2}\left(P_x^1\right) = \begin{pmatrix} 1+\sqrt{1-4cf} & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{T^2}\left(P_y^1\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1+\sqrt{1-4cf} \end{pmatrix}.$$

Since $\lambda_1 = 0 < 1$ and $\lambda_2 = 1+\sqrt{1-4cf} > 1$, it means that periodic solutions $\left\{P_x^1, P_y^1\right\}$ are saddle points. The Jacobian matrix of the map T^2 at the points $\left\{P_x^2, P_y^2\right\}$ is of the form

$$J_{T^2}\left(P_x^2\right) = \begin{pmatrix} 1-\sqrt{1-4cf} & 0 \\ 0 & 0 \end{pmatrix}, \quad J_{T^2}\left(P_y^2\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1-\sqrt{1-4cf} \end{pmatrix}.$$

Since $\lambda_1 = 0$ and $\lambda_2 = 1 - \sqrt{1-4cf} \in (0,1)$ it follows that the periodic solutions $\left\{P_x^2, P_y^2\right\}$ are locally asymptotically stable.

iii) The Jacobian matrix of the map T^2 at the points $\left\{P_{\mp}^3, P_{\pm}^3\right\}$ using (85)-(88) is of the form

$$J_{T^2}(\phi,\psi) = \begin{pmatrix} \frac{2a\psi^2+2f}{\phi} & -2a\psi \\ -4a(f+a\psi^2) & 2+4a\phi\psi \end{pmatrix}.$$

Now, by (81) and (82) we have

$p = TrJ_{T^2}(\phi,\psi) = \frac{2a\psi^2+2f}{\phi} + 2 + 4a\psi\phi > 0$

$q = -DetJ_{T^2}(\phi,\psi) = -(-4a(f+a\psi^2))(-2a\psi) - (\frac{2a\psi^2+2f}{\phi})(2+4a\psi\phi) < 0$

so it holds

$$|p| = p \quad \text{and} \quad |1-q| = 1-q.$$

It follows that

$$|p| > |1-q| \Leftrightarrow p > 1-q \Leftrightarrow f < \frac{c-3a}{4(a-c)^2},$$

which means that $\left\{P_{\mp}^3, P_{\pm}^3\right\}$ are saddle points.

$\square$

3.5 Global dynamics of Eq.(67) in hyperbolic case

In this section, we present global dynamics results for Eq.(67).

Equation (67) is equivalent to the system of difference equations (83), which can be decomposed into the system of the even-indexed and odd-indexed terms as follows:

$$\left\{\begin{array}{c} u_{2n}=v_{2n-1}, \\ u_{2n+1}=v_{2n}, \\ v_{2n}=\frac{u_{2n-1}^2}{av_{2n-1}^2+cu_{2n-1}^2+f}, \\ v_{2n+1}=\frac{u_{2n}^2}{av_{2n}^2+cu_{2n}^2+f}. \end{array}\right. \tag{89}$$

Every solution of Eq.(67) satisfies

$$x_n \leq \frac{1}{c}, n=1,2,\dots \,. \tag{90}$$

Theorem 27 *If $f>\frac{1}{4c}$, then the unique zero equilibrium solution of Eq.(67) is globally asymptotically stable.*

Proof. By Theorem 19 subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$ are eventually monotonic. Since Eq.(67) has no minimal period-two solutions and $\{x_n\}_{n=-1}^{\infty}$ is bounded by (90), i.e. $x_n \leq \frac{1}{c}$, $n=1,2,\dots$ every solution of Eq.(67) converges to $\overline{x}=0$ and by Proposition 1 zero equilibrium $\overline{x}=0$ is globally asymptotically stable. □

Theorem 28 *If $\frac{1}{4(b+c)}<f<\frac{1}{4c}$ then Eq.(67) has one equilibrium point $E_0(0,0)$ which is locally asymptotically stable and two minimal period-two solutions: $\left\{P_x^1,P_y^1\right\}$ which are a saddle points and $\left\{P_x^2,P_y^2\right\}$ which are locally asymptotically stable. There exist global stable manifolds $W^s\left(P_x^1\right)$ and $W^s\left(P_y^1\right)$ which are basins of attraction of the periodic solutions $\left\{P_x^1,P_y^1\right\}$, and the unstable manifolds have the following form*

$$\mathcal{W}^u\left(P_x^1\right)=\{(x,0): x\in I_1\cup I_2\}, \mathcal{W}^u\left(P_y^1\right)=\{(0,y): y\in I_1\cup I_2\}.$$

The basin of attraction of the equilibrium point $E_0 = (0,0)$ *is the region between the global stable sets*

$$\mathcal{B}(E_0) = \mathcal{W}^{-}\left(P_x^1\right) \cap \mathcal{W}^{+}\left(P_y^1\right).$$

The basin of attraction of the minimal period-two solutions $\left\{P_x^2, P_y^2\right\}$ *is given with the following*

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^{+}\left(P_x^1\right),\ \mathcal{B}\left(P_y^2\right) = \mathcal{W}^{-}\left(P_y^1\right).$$

Proof. The proof follows from Theorems 4 and 5 in [17], Theorem 24. □

The next two results have the same proof as Theorems 8 and 9 in [18] and so it will be skipped.

Theorem 29 *If* $\frac{c-b}{4c^2} < f < \frac{1}{4(b+c)}$, *then Eq.*(67) *has three equilibrium points:*

- E_0 *is locally asymptotically stable,*
- E_- *is repeller,*
- E_+ *is a saddle point,*

  and two minimal period-two solutions:

- $\left\{P_x^1, P_y^1\right\}$ *are saddle points,*
- $\left\{P_x^2, P_y^2\right\}$ *are locally asymptotically stable.*

There exist the global stable manifolds $\mathcal{W}^s\left(P_x^1\right)$ *and* $\mathcal{W}^s\left(P_y^1\right)$ *which are also basins of attraction of the minimal period-two solutions* $\left\{P_x^1, P_y^1\right\}$, *while the unstable manifolds have the following form*

$$\mathcal{W}^u\left(P_x^1\right) = \left\{(x,0) : x \in I_1 \cup I_2\right\}, \mathcal{W}^u\left(P_y^1\right) = \left\{(0,y) : y \in I_1 \cup I_2\right\}.$$

The basin of attraction of the equilibrium point E_0 *is the region between those stable manifolds i.e.*

$$\mathcal{B}\left(E_0\right) = \mathcal{W}^{-}\left(P_x^1\right) \cap \mathcal{W}^{+}\left(P_y^1\right),$$

while the basin of attraction of the minimal period-two solutions $\left\{P_x^2, P_y^2\right\}$ *is given with*

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^+\left(P_x^1\right),\ \mathcal{B}\left(P_y^2\right) = \mathcal{W}^-\left(P_y^1\right).$$

Theorem 30 *If* $0 < f < \frac{c-b}{4c^2}$*, then Eq.*(67) *has three equilibrium points:*

- E_0 *is locally asymptotically stable,*
- E_- *is repeller,*
- E_+ *is locally asymptotically stable,*

  and three minimal period-two solutions:

- $\left\{P_x^1, P_y^1\right\}$ *are saddle points,*
- $\left\{P_x^2, P_y^2\right\}$ *are locally asymptotically stable,*
- $\left\{P_\mp^3, P_\pm^3\right\}$ *are saddle points.*

There exist global stable manifolds $\mathcal{W}^s\left(P_x^1\right)$, $\mathcal{W}^s\left(P_y^1\right)$ *and* $\mathcal{W}^s\left(P_\mp^3\right)$, $\mathcal{W}^s\left(P_\pm^3\right)$ *which are also basins of attraction of the minimal period-two solutions* $\left\{P_x^1, P_y^1\right\}$ *and* $\left\{P_\mp^3, P_\pm^3\right\}$ *respectively. The basin of attraction of the equilibrium point* E_0 *is the region between the stable manifolds of minimal period-two solutions* $\left\{P_x^1, P_y^1\right\}$*, while the basin of attraction of the equilibrium point* E_+ *is the region between the stable manifolds of the minimal period-two solutions* $\left\{P_\mp^3, P_\pm^3\right\}$ *i.e.*

$$\mathcal{B}\left(E_+\right) = \mathcal{W}^-\left(P_\pm^3\right) \cap \mathcal{W}^+\left(P_\mp^3\right).$$

The basin of attraction of the minimal period-two solutions $\left\{P_x^2, P_y^2\right\}$ *is given with*

$$\mathcal{B}\left(P_x^2\right) = \mathcal{W}^+\left(S_x\right),\ \text{where } S_x = \mathcal{W}^s\left(P_x^1\right) \cup \mathcal{W}^s\left(P_\pm^3\right),$$

$$\mathcal{B}\left(P_y^2\right) = \mathcal{W}^-\left(S_y\right),\ \text{where } S_y = \mathcal{W}^s\left(P_y^1\right) \cup \mathcal{W}^s\left(P_\mp^3\right).$$

List of References

[1] A. Brett and M. R. S. Kulenović, Basins of attraction of equlilibrium points of monotone difference equations, *Sarajevo J. Math.*, Vol. 5 (18) (2009), 211-233.

[2] A. Brett and M. R. S. Kulenović, Two Species Competitive Model with the Allee Effect, *Adv. Difference Equ.*, Volume 2014 (2014):307, 28 p.

[3] E. Camouzis and G. Ladas, When does local asymptotic stability imply global attractivity in rational equations? *J. Difference Equ. Appl.*, 12(2006), 863-885.

[4] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz and H. Sedaghat, H. Dynamics of rational difference equations containing quadratic terms, *J. Difference Equ. Appl.*, 14 (2008), 191-208.

[5] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz and H. Sedaghat, Monotone and oscillatory solutions of a rational difference equation containing quadratic terms, *J. Difference Equ. Appl.*, 14 (2008), 1045-1058.

[6] M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, Basins of attraction of equilibrium points of second order difference equations, *Appl. Math. Lett.*, 25 (2012) 2110-2115.

[7] M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, Global Dynamics of Certain Homogeneous Second Order Quadratic Fractional Difference Equation, *The Scientific World Journal, Mathematical Analysis* Volume 2013, Article ID 210846, 10 pages.

[8] M. Garić-Demirović and M. Nurkanović, Dynamics of an anti-competitive two dimensional rational system of difference equations, *Sarajevo J. Math.*, 7 (19) (2011), 39-56.

[9] S. Jašarević and M. R. S. Kulenović, Basins of Attraction of Equilibrium and Boundary Points of Second Order Difference Equation, *J. Difference Equ. Appl.*, 20(2014), 947–959.

[10] S. Jašarević Hrustić, M. R. S. Kulenović and M. Nurkanović, Global Dynamics and Bifurcations of Certain Second Order Rational Difference Equation with Quadratic Terms, submitted.

[11] S. Kalabušić, M. R. S. Kulenović, E. Pilav Global Dynamics of Anti-Competitive Systems in the Plane, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 20(2013), 477–505.

[12] C. M. Kent and H. Sedaghat, Global attractivity in a quadratic-linear rational difference equation with delay. *J. Difference Equ. Appl.*, 15 (2009), 913–925.

[13] C. M. Kent and H. Sedaghat, Global attractivity in a rational delay difference equation with quadratic terms, *J. Difference Equ. Appl.*, 17 (2011), 457–466.

[14] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures,* Chapman and Hall/CRC, Boca Raton, London, 2001.

[15] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[16] M. R. S. Kulenović and O. Merino, Global bifurcations for competitivesystem in the plane, *Discrete Contin. Dyn. Syst. Ser. B* ,12 (2009), 133-149.

[17] M. R. S. Kulenović and O. Merino, Invariant manifolds for competitive discrete systems in the plane, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 20 (2010),2471-2486.

[18] M. R. S. Kulenović, S. Moranjkić and Z. Nurkanović, Global Dynamics and Bifurcation of a Perturbed Sigmoid Beverton-Holt Difference Equation, *Mathematical Methods in Applied Sciences*, (2016),20p.

[19] M. R. S. Kulenović, E. Pilav and E. Silić, Local Dynamics and Global Attractivity of a Certain Second Order Quadratic Fractional Difference Equation, *Adv. Difference Equ.*, 2014 (2014):68, 32p.

[20] Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, Second Edition, Springer, NewYork, 1998.

[21] S. Moranjkić and Z. Nurkanović, Basins of attraction of certain rational anti-competitive system of difference equations in the plane, *Adv. Difference Equ.*, 2012, 2012:153.

[22] H. Sedaghat, Global behaviours of rational difference equations of orders two and three with quadratic terms. *J. Difference Equ. Appl.*, 15 (2009), 215–224.

[23] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations*, 64 (1986), 163-194.

[24] H. L. Smith, Planar competitive and cooperative difference equations. *J. Difference Equ. Appl.*, 3 (1998), 335-357.

[25] G. G.Thomson, A proposal for a threshold stock size and maximum fishing mortality rate. In: Smith, SJ, Hunt, JJ, Rivard, D (eds.) *Risk Evaluation and Biological Reference Points for Fisheries Management. Canad. Spec. Publl. Fish. Aquat. Sci.*, vol. 120, pp. 303-320 (1993).