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Degree-Limited Defective Three Colorings of Planar Graphs Containing No 4-Cycles or 5-Cycles

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DEGREE-LIMITED DEFECTIVE THREE COLORINGS OF PLANAR GRAPHS CONTAINING NO 4-CYCLES OR 5-CYCLES

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ABSTRACT

This dissertation explores and advances results for several variants on a long-open problem in graph coloring. Steinberg's conjecture states that any planar graph containing no 4-cycles or 5-cycles is 3-colorable. The conjecture has remained open for more than forty years and brought a great deal of interest to coloring planar graphs with certain structural restrictions. In this dissertation, we present a new type of defective graph coloring that allows us to prove two main results advancing the state of Steinberg's conjecture.

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CHAPTER 1

Introduction

The study of graphs has been an important branch of combinatorics for over two centuries, dating back to Leonhard Euler's resolution of the Königsberg Bridge Problem in 1736 [1]. Graph theory has applications in numerous areas, including computer and data science, business, and the social sciences. We begin this section with a number of necessary definitions and useful facts.

A graph G is a set V of vertices together with a multi-set E of edges consisting of one- or two-element subsets of V. The number of vertices in a graph is denoted $|V|$ and the number of edges is denoted $|E|$. Two vertices that have an edge between them, i.e. are part of the same two-element subset in E, are called *adjacent* or *neighbors*. A vertex v is of *degree* s if it is part of exactly s edges; the notation $d(v) = s$ will be used to indicate that vertex v is of degree s. The *minimum degree* of a graph G is the smallest degree of any vertex in the graph; this is denoted $\delta(G)$. A *simple* graph is one in which no vertex has an edge to itself (i.e. there are no one-element subsets in E) and in which there is at most one edge between any two vertices (i.e. every two-element subset in E is unique). We will consider only simple graphs in this work.

Many families of graphs have been studied for their structural properties; our results center around several specific families. A *drawing* of a graph is a mapping of the vertices and edges onto a surface. A graph is said to be *planar* if it can be drawn in the plane so that none of the edges cross, and edges only meet at vertices; such a drawing is called a *plane drawing* of the graph. In a plane drawing of a planar graph, a *face* is a region of the plane bounded by edges and containing no edges; a face will be referred to as a *k-face* if there are k vertices on its boundary.

Figure 1 illustrates a planar graph containing the labeled 6-face F, with boundary formed by the vertices $a, b, c, d, e,$ and f.

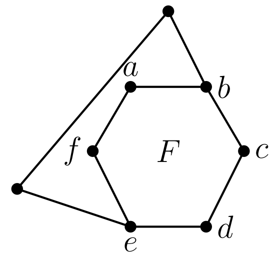

Figure 1. A planar graph containing a 6-face F

A *chain* of vertices is a sequence of vertices connected by edges. A *path* on n vertices, denoted P_n, is a set of n vertices, $v_1, v_2, \ldots v_n$ such that $v_i v_{i+1}$ is an edge for each $i \in 1, 2, \ldots n-1$. A *complete graph on n vertices*, denoted K_n, is a graph on n vertices in which every vertex is connected to every other vertex. A *cycle* on n vertices, denoted C_n and referred to as an *n-cycle*, is a graph that consists of n vertices connected in a closed chain. Figure 1 contains a 6-cycle along the vertices a, b, c, d, e, f. The 3-cycle C_3 is also known as a *triangle*. Two triangles are said to be *adjacent* if they share an edge and *chained* if they share a single vertex. We say that two triangles are *next to each other* if there is only one edge between them. Figure 2 illustrates all of these configurations of triangles. We define the *distance* between two structures in a graph as the minimum number of edges contained in any path between any pair of vertices in which one vertex comes from each structure. In Figure 2 the triangles next to each other are at distance 1 from each other.

Several useful results concerning the structure of graphs containing no cycles of certain sizes, or containing no chained or adjacent triangles, were given by Lam, Liu, Shiu, and Wu [2]. These lemmas offer valuable insight into the properties of graphs with forbidden substructures, allowing us to utilize such properties in other

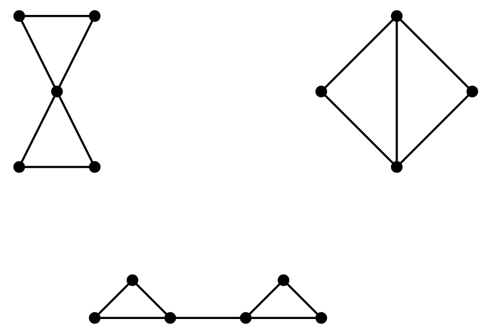

Figure 2. Chained triangles (left), adjacent triangles (right), and triangles next to each other (bottom)

proofs.

Lemma 1. *(Lam, Liu, Shiu, Wu [2])*

If $k \geq 4$ and G is a connected planar graph containing no adjacent triangles and no cycles C_i for any i with $3 < i < k$, then the maximum number of edges in G is:

$$|E| \leq \frac{3k}{2k-3}\left(|V|-2\right).$$

Lemma 2. *(Lam, Liu, Shiu, Wu [2])*

If $k \geq 4$ and G is a connected planar graph containing no chained triangles and no cycles C_i for any i with $3 < i < k$, then the maximum number of edges in G is:

$$|E| \leq \frac{4k-3}{3(k-2)}|V| - \frac{2k}{k-2}.$$

Using Lemma 1, it is straightforward to prove the following useful lemma concerning planar graphs containing no 4-cycles or 5-cycles.

Lemma 3. *Let G be a connected planar graph with no 4-cycles or 5-cycles. Then $\delta(G) \leq 3$.*

Proof. By taking $k = 6$ in Lemma 1, we have that the number of edges in G is

$|E| \leq 2|V| - 4$. Hence the average degree is

$$\begin{aligned} d_{\text{avg}} &= \frac{2|E|}{|V|} \\ &\leq \frac{2\,(2|V| - 4)}{|V|} \\ &= 4 - \frac{8}{|V|} \\ &< 4. \end{aligned}$$

Since the average degree is less than 4, there must exist some vertex of degree no larger than 3. $\square$

1.1 Colorings

Within the field of graph theory, graph colorings have been a major area of exploration for nearly as long as the field has existed. While many important advancements in the theory have been made during this time, numerous open questions remain, some of which have turned out to be quite challenging to solve. In this dissertation, we prove several results that advance the state of Steinberg's Conjecture, one of these long-open problems.

A *vertex coloring* of a graph is an assignment of colors to the vertices of the graph. Such a coloring is *proper* if no two adjacent vertices share a color. A graph that can be properly colored with k colors is called *k-colorable*. The minimum k such that a graph G can be k-colored is called the *chromatic number of G*, denoted $\chi(G)$. It is easy to see that $\chi(K_n) = n$ for any complete graph. The *chromatic number of a family of graphs F*, denoted $\chi(F)$, is the maximum of the chromatic numbers of every member of the family. Figure 3 shows a properly colored 5-cycle using 3 colors, c_1, c_2, and c_3.

One of the natural questions that arises when considering graph colorings is that of the minimum number of colors needed to properly color a given graph or family of graphs, i.e. determining the chromatic number of a graph or family of

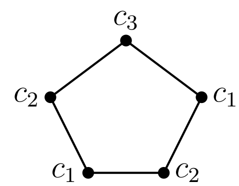

Figure 3. A proper 3-coloring of C_5

graphs. Such coloring problems have been significant elements of graph theory for many years. The study of graph colorings has applications in multiple areas including scheduling problems, assignment of frequencies to radio and cell towers, and assignments of security codes to both digital and physical networks. Additionally, several graph coloring problems, including determining if a graph is k-colorable for any $k \geq 3$, are part of the family of NP-Complete problems [3]. This implies that there are no known efficient computer algorithms for determining if a graph is k-colorable and gives results in coloring theory particular importance to practical problems.

A famous result in the field of graph coloring is the Four Color Theorem which states that any planar graph may be properly colored using only four colors. This problem was first posed in the early nineteenth century and remained open for more than a century until Appel and Haken finally proved the Four Color Theorem in 1977 [4]. A shorter proof was given by Robertson, Sanders, Seymour, and Thomas in 1996 [5]. While the Four Color Problem was still unsolved, Grötzsch proved that all planar graphs containing no triangles are 3-colorable [6, 7]. This result raised substantial interest in determining which structural properties allow planar graphs to be 3-colored. In 1976, Richard Steinberg [8, 9] posed the following conjecture concerning planar graphs containing no 4-cycles or 5-cycles.

Conjecture 1. *(Steinberg [8]) Every planar graph without 4-cycles and 5-cycles is 3-colorable.*

Steinberg's conjecture has remained open for the past four decades, though a great deal of research has been directed toward proving it. In 1991, Paul Erdös posed a wider version of the problem asking for the smallest integer l so that any planar graph containing no cycles of lengths 4 through l is 3-colorable. Borodin, Glebov, Raspaud, and Salavatipour [10] provided the best result to date when they proved that $l \leq 7$.

In light of the Grötzsch theorem, several authors attacked the conjecture from the standpoint of distance between triangles or between triangles and other faces. Borodin and Raspaud showed in [11] that any planar graph containing no 5-cycles or triangles closer than 4 vertices apart is 3-colorable. In 2006, Baogang Xu proved that planar graphs containing no 5-cycles, 7-cycles, or adjacent triangles are 3-colorable [12]. Another 2006 paper [13] by Borodin, Glebov, Jensen, and Raspaud showed that every planar graph without triangles adjacent to cycles of lengths 3 through 9 is 3-colorable.

While work continued on the more direct path toward Steinberg's conjecture, an alternative method of studying coloring problems was also in development.

1.2 Defective Colorings

Instead of changing the structure of the family of graphs under consideration, several authors relaxed the restrictions of the coloring. The idea of weakening the demands on a coloring was first introduced by Cowen, Cowen, and Woodall in 1986 [14]. A *defective coloring* or *improper coloring* is a graph coloring in which some adjacent vertices are allowed to have the same color. We define a *flaw* to be a pair of adjacent vertices that share the same color. We say a vertex *has a flaw* if it shares a color with one of its neighbors.

It is common to specify which colors are allowed to exhibit such flaws and to place a limit on the number of flaws that a vertex may have. We use the notation

(i_1, i_2, i_3)-coloring with $i_1, i_2, i_3 \in \mathbb{N}$ to denote a defective 3-coloring with i_1 flaws allowed for vertices colored with color c_1, i_2 flaws allowed for a vertex colored with c_2, and i_3 flaws allowed for a vertex colored with c_3. For example, a $(3, 1, 0)$-coloring allows any vertex colored with c_1 to be adjacent to at most three other vertices colored with c_1, any vertex colored with c_2 to be adjacent to at most one other vertex colored c_2, and any vertex colored with c_3 to be adjacent to no vertices of the same color. Under this convention, a proper 3-coloring is a $(0, 0, 0)$-coloring. We say a vertex in a (i_1, i_2, i_3)-coloring is *fully-flawed* if it is adjacent to the maximum number of flaws in its color. Figure 4 shows a $(1, 0, 0)$-coloring in which vertices v, w, x, and y are fully-flawed.

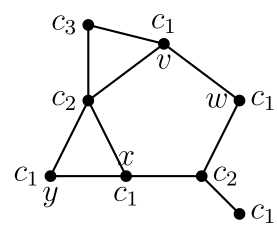

Figure 4. A $(1, 0, 0)$-coloring of a planar graph.

Using the defective coloring techniques, Smith et al. [15] proved that all planar graphs without 4-cycles or 5-cycles are $(3, 0, 0)$-colorable. A 2014 paper of Xu et al. [16] proved that graphs in this family are $(1, 1, 0)$-colorable and raised the question of whether or not all planar graphs of this type are $(2, 0, 0)$-colorable. Combining the defective colorings with the restrictions on triangles, Liu, Li, and Yu [17] showed in 2014 that planar graphs without 4-cycles, 5-cycles or chained triangles are $(2, 0, 0)$-colorable, leading to the conjecture that they are $(1, 0, 0)$-colorable. Recently, in [18], Eaton and Smith provided a proof that planar graphs containing no 4-cycles, 5-cycles, or certain types of 8-cycles or 9-cycles are $(1, 0, 0)$ colorable.

We define a new type of defective coloring in which every flaw must involve a vertex of no more than a certain degree. A *degree-limited defective coloring*, denoted a $(i_1, i_2, i_3)^{(d_1,d_2,d_3)}$-coloring, is one in which every vertex colored with c_1 may have at most i_1 flaws, each involving a vertex of degree no more than d_1, every vertex colored with c_2 may have at most i_2 flaws, each involving a vertex of degree no more than d_2, and every vertex colored with c_3 may have at most i_3 flaws, each involving a vertex of degree no more than d_3. If any of i_1, i_2 or i_3 are zero, we will leave out the corresponding degree of the flaw; for example, a $(1,0,0)^{(3)}$-coloring is a three coloring in which any vertex colored with c_1 may have at most one flaw and that flaw must involve a vertex of degree 3 or less.

The center of this dissertation is two results on degree-limited defective colorings of planar graphs that advance the current state of Steinberg's Conjecture and taking steps to address the questions that Xu and Liu, Li, and Wu posed in 2014. In proving these results, we also pioneer a new technique for dealing with graph colorings by introducing a stronger form of the defective coloring.

Theorem 1. *A planar graph containing no 4-cycles, no 5-cycles, no chained triangles, and none of the faces listed in Appendix B at distance 2 or less can be* $(1,0,0)^{(3)}$*-colored.*

Theorem 2. *A planar graph containing no 4-cycles, no 5-cycles, and none of the faces listed in Appendix D at distance 2 or less can be* $(2,0,0)^{(4)}$*-colored.*

In addition to these theorems, preliminary Steinberg type results are obtained for a family of non-planar graphs.

Chapter 2 focuses on proving Theorem 1 using an alternative take on the classic discharging method and several lemmas that allow the extension of an existing coloring. Chapter 3 utilizes the discharging method and a similar variety of lemmas to prove Theorem 2. Finally,Chapter 4 extends the ideas presented in the study of Steinberg's Conjecture to the class of Earth-Moon graphs and presents several conjectures for future work.

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CHAPTER 2

Coloring Planar Graphs Containing No Chained Triangles

This chapter centers around proving Theorem 1 using a combination of a discharging method, coloring extension lemmas, and direct coloring. Let $\mathcal{F}_{\mathcal{N}}$ be the family of planar graphs containing no chained triangles, no 4-cycles, no 5-cycles, and containing no two of the faces listed in Appendix B at distance less than 3. In order to show that the members of $\mathcal{F}_{\mathcal{N}}$ are $(1,0,0)^{(3)}$-colorable, we begin by proving several useful lemmas that allow for the extension of a coloring while preserving a particular property. Next, we use a discharging procedure to prove the existence of certain substructures in the graphs in $\mathcal{F}_{\mathcal{N}}$, namely those listed in Appendix A. Finally, we will show that every graph in $\mathcal{F}_{\mathcal{N}}$ can be $(1,0,0)^{(3)}$-colored.

We use the colors c_1, c_2, and c_3. We define a c_2-c_3 *switch* to be a recoloring scheme that simply swaps the colors c_2 and c_3 in some specified region of the graph. These switches will be performed only when the region specified is surrounded by vertices colored with c_1, so that they have no impact on the rest of the colors used in the graph.

2.1 Coloring Lemmas

In this section we prove several lemmas that allow the extension of a $(1,0,0)^{(3)}$ coloring on a given graph $G \in \mathcal{F}_{\mathcal{N}}$. Each lemma begins with a $(1,0,0)^{(3)}$-coloring of $G \setminus v$ for some particular v and then extends the coloring to vertex v.

Lemma 4. *Suppose G contains a vertex v such that v has a neighbor u_1 with $d(u_1) \leq 3$ and the $(1,0,0)^{(3)}$-coloring on $G \setminus v$ returns u_1 colored with c_1 and no other neighbors of v colored with c_1. Then the $(1,0,0)^{(3)}$-coloring of $G \setminus v$ can be extended to G.*

Proof. If u_1 is not fully-flawed, color v with c_1 and the result follows. Suppose instead that u_1 is fully-flawed. Let u_1' be the neighbor of u_1 that is colored with c_1 and u_1'' be the neighbor of u_1 that is not colored with c_1, if it exists. Recolor u_1 with c_2 or c_3, whichever is not used on its neighbors, and color v with c_1. The result follows. $\square$

Corollary 1. *Suppose G contains a vertex v such that $d(v) = 3$ and each of the neighbors of v also has degree 3. A $(1,0,0)^{(3)}$-coloring of $G \setminus v$ can be extended to a $(1,0,0)^{(3)}$-coloring of G.*

Proof. Assume exactly one neighbor of v is colored with c_1; otherwise v can be colored with whichever color was not used on its neighbors. The result now follows from Lemma 4. $\square$

Lemma 5. *Suppose G contains a vertex v with $d(v) = 3$ and with neighbors u_1, u_2, and u_3 such that u_1u_2v forms a triangle and $d(u_1) = 4$, $d(u_2) = 3$, and $d(u_3) = 3$; this configuration is illustrated in Figure 5. A $(1,0,0)^{(3)}$-coloring of $G \setminus v$ can be extended to a $(1,0,0)^{(3)}$-coloring of G.*

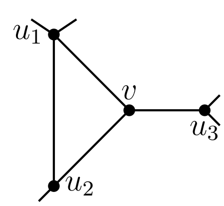

Figure 5. Configuration of v's neighbors in Lemma 5

Proof. Assume that each of the neighbors of v has a different color; if they did not, we could color v with the unused color. Lemma 4 applies if either u_2 or u_3 is returned with c_1, so we consider only the case in which u_1 has c_1 and is fully flawed.

Suppose that u_1 is colored with c_1 and fully-flawed, and without loss of generality, suppose u_2 is colored with c_2 and u_3 is colored with c_3. If the neighbor of u_1 not colored with c_1 is also colored with c_2, then recolor u_1 with c_3 and use c_1 on v. So assume that one neighbor of u_1 is colored with c_3.

Recolor u_1 with c_2, u_2 with c_1 or c_3, whichever is not used on the neighbor of u_2 that is not u_1. Color v with c_1. Since any flaw introduced involves v, it would involve a vertex of degree 3.

Hence we have extended the $(1,0,0)^{(3)}$-coloring to G in the needed way and the result follows. □

Lemma 6. *Let G contain an even face C such that all vertices on C have degree 3, except possibly one, called y which may have degree 4. Let v be of degree 3, on C, and not adjacent to y. Then a $(1,0,0)^{(3)}$-coloring of $G \setminus v$ can be extended to a $(1,0,0)^{(3)}$-coloring of G.*

Proof. Let u_1 be the neighbor of v not on C and let u_2 and u_3 be the neighbors of v on C. If u_1 has degree 3 then the result follows from Corollary 1. Assume instead that $d(u_1) \geq 4$. Assume also that each of the neighbors has a different color; if they did not, we color v with the unused color. Lemma 4 applies if either u_2 or u_3 is returned with c_1, so we consider only the case in which u_1 has c_1 and is fully-flawed. Without loss of generality, suppose u_2 has c_2 and u_3 has c_3.

Since C is an even cycle, there must exist some vertex w on C that has c_1. Choose w in such a way that if it is of degree 4, then it is the only vertex on C colored with c_1. If more than one vertex on C has c_1, then let w be the vertex of degree 3 that is colored with c_1 and such that it gives the shortest path along the cycle between w and v consisting entirely of vertices of degree 3 and colored with c_2 and c_3. Let $P = vx_1x_2x_3 \ldots x_kw$ be the path between v and w along C and let x_i' be the neighbor of x_i not on C. Here, x_1 is either u_2 or u_3. Suppose without loss of generality that x_1 is u_2.

Note that for each x_i on the path P, x_i is colored with either c_2 or c_3. If some x'_i is colored the same as x_{i+1} for any $i < k-1$, then let j be the smallest index such that x'_j is colored the same as x_{j+1}. Recolor x_j with c_1, perform a c_2-c_3 switch from x_1 to x_j along P, and color v with c_2. Assume now that each x_i has color c_1. If x'_k is colored with c_1, a c_2-c_3 switch may be performed along P and v may be colored with c_2. In the remaining situation each x'_i, $i < k$, must have c_1 and x'_k must be colored with either c_2 or c_3. We now split the proof into three cases depending on the degree of w.

Case 1: If w is not fully-flawed, then recolor x_k with c_1 and perform a c_2-c_3 switch along P. Color v with c_2.

Case 2: If w is of degree 3 and fully-flawed, then recolor w with whatever color is not used on the neighbor not colored with c_1 and not x_k. Recolor x_k with c_1 and perform a c_2-c_3 switch along P. Color v with c_2.

Case 3: The vertex w is of degree 4 and fully-flawed. By selection of w, the flaw occurs on some vertex off of C. Let $Q = vz_1z_2\ldots z_mw$ be the other path between v and w along C; without loss of generality, we assume that $z_1 = u_3$. By a symmetric argument to that concerning the neighbors of P, observe that each z'_i must be colored with c_1, except for z'_m.

Note that only one neighbor of w not on C has a color other than c_1; call this neighbor w''. If the color of w'' is the same as the color of z_m, recolor w with the color used on x_k, recolor x_k with c_1 and perform a c_2-c_3 switch along P. Finally color v with c_2.

If instead, the color of w'' is the same as the color of x_k, then recolor w with the color used on z_m, recolor z_m with c_1 and perform a c_2-c_3 switch along Q. Finally color v with c_3.

Thus we have provided a $(1,0,0)^{(3)}$-coloring for G. The only potential flaw introduced involves a vertex of degree 3, namely x_k. Hence the coloring satisfies the necessary property and the proof is complete. $\square$

Each of the above lemmas will allow us to extend colorings using an induction argument on graphs G in $\mathcal{F}_{\mathcal{N}}$.

2.2 Discharging Procedure and Forced Substructures in $G \in \mathcal{F}_{\mathcal{N}}$

This section centers around proving several structural properties of $G \in \mathcal{F}_{\mathcal{N}}$ and that any $G \in \mathcal{F}_{\mathcal{N}}$ with minimum degree 3 contains at least one of the configurations listed in Appendix A. The proof of this fact relies on a traditional discharging procedure.

Theorem 3. *Let G be a member of $\mathcal{F}_{\mathcal{N}}$ with minimum degree 3. Then G contains at least one of the faces listed in Appendix A.*

Proof. Assign weights to the vertices and faces of G as follows: weight each vertex v_i with $d(v_i) - 4$ and each face f_i with $|f_i| - 4$, where $|f_i|$ is the number of vertices on the boundary of f_i. The total weight on the graph after assignment is:

$$\begin{aligned}\sum_{v_i \in V(G)} w(v_i) + \sum_{f_i \in F(G)} w(f_i) &= \sum_{v_i \in V(G)} (d(v_i) - 4) + \sum_{f_i \in F(G)} (|f_i| - 4) \\ &= 2e - 4n + 2e - 4f \\ &= -4(n + f - e) \\ &= -8.\end{aligned}$$

Note that every triangle has weight -1 and every other face has weight at least 2 since there are no 4-faces or 5-faces. Furthermore, only vertices that have negative weight are of degree 3. We will proceed to redistribute the weights in such a way as to make every triangle and every vertex have non-negative weight.

In the discharging rules, we classify triangles by the degree of each of the vertices. For example, a $(3,3,4)$-triangle is one in which two of the vertices are of degree 3 and one is of degree 4. We will also refer to a face based on its position relative to the vertices of an attached triangle. For example, we may refer to "the face across from the degree 4 vertex of a $(3,3,4)$-triangle" to mean the face which shares the $3,3$ edge with the triangle. We also say a *vertex is of degree* $k+$ to indicate that it is of degree k or more.

The weight discharged from a face to a triangle will be referred to as a *contribution*.

With this terminology in place, redistribute the weights according to the following discharging rules:

- Each triangle gives 1 to every vertex of degree 3 on its boundary. This move concentrates most of the negative weight on the triangles.

- Each vertex of degree 5 that is part of a triangle gives 1 to the triangle, each vertex of degree 6 gives 2 to the triangle, and each vertex of degree 7+ gives 3 to the triangle.

- Each face of size 6 or more gives $\frac{1}{3}$ to any degree 3 vertex on its boundary that is not part of a triangle.

- Each face adjacent to a $(3,3,3)$-triangle gives $\frac{4}{3}$ to the triangle.

- Each face adjacent to a $(3,3,4)$-triangle gives $\frac{3}{3}$ to the triangle.

- Each face adjacent to a $(3,3,5)$-triangle gives $\frac{2}{3}$ to the triangle.

- Each face adjacent to a $(3,3,6)$-triangle gives $\frac{1}{3}$ to the triangle.

- Each face adjacent to a $(3,4,4)$-triangle gives $\frac{2}{3}$ to the triangle.

- Each face adjacent to a $(3,4,5)$-triangle gives $\frac{1}{3}$ to the triangle.

- Each face adjacent to a $(4,4,4)$-triangle gives $\frac{1}{3}$ to the triangle.

Notice that for each type of triangle, the positive contribution to the triangle from the adjacent faces cancels out the negative weight on the triangle. For example, a $(3,3,3)$-triangle has weight -4 after the first step in the redistribution of weights, and each adjacent face contributes $\frac{4}{3}$, totaling 0.

After redistributing the weights, the weight on each triangle is non-negative and the weight on each vertex of degree 3 is 0. Hence some of the faces of size 6 or more must have negative weight.

Observe that no face of size 11 or more has negative weight. Indeed, the initial weight on any k-face is $k-4$; the maximum weight discharged from such a face would be $\frac{4}{3}$ to each of $\lfloor \frac{k}{2} \rfloor$ $(3,3,3)$-triangles and possibly $\frac{1}{3}$ to a degree 3 vertex not part of a triangle. For odd k, the maximum weight discharged is $\frac{2k-1}{3}$ and for even k it is $\frac{2k}{3}$. For odd $k \geq 11$, $k-4 \geq \frac{2k-1}{3}$ and for even $k \geq 12$, $k-4 \geq \frac{2k}{3}$; hence the minimum weight on a face of size at least 11 is 0 after discharging.

Thus all faces with negative weight must be of size 10 or less. Appendix A describes all of the faces of size 10 or less that have negative weight. Since the initial weight on G was negative, G must contain at least one of these faces and the result follows. $\square$

Theorem 3 gives us a list of possible structural characteristics of a given graph $G \in \mathcal{F}_{\mathcal{N}}$. Furthermore, any graph containing none of these faces has at least one vertex of degree 2. These guaranteed faces or vertices of degree 2 allow us to apply an induction argument to coloring G.

2.3 Coloring Members of $\mathcal{F}_{\mathcal{N}}$

With a number of coloring lemmas and substantial knowledge of the faces contained in graphs of the family $\mathcal{F}_{\mathcal{N}}$, we now proceed to show that members of this family are $(1,0,0)^{(3)}$-colorable. The proof will proceed by induction on the number of vertices of G. During the inductive step, the lemmas from Section 2.1 will be frequently used to do the bulk of the coloring.

Define the set $\mathcal{S}_{\mathcal{NC}}$ to be the set of faces listed in Appendix B and the set $\mathcal{G}_{\mathcal{NC}}$ to be the set of faces listed in Appendix A minus $\mathcal{S}_{\mathcal{NC}}$.

We begin by coloring a subfamily, $\mathcal{F}'_{\mathcal{N}}$ consisting of all planar graphs containing no C_4, no C_5, no chained triangles, and in which the only faces of negative weight are those in $\mathcal{G}_{\mathcal{NC}}$. Throughout the proof, we list a number of faces and label the vertices on those faces. We adopt the convention that, if x is a vertex on a face, then x' and x'' refer to neighbors of x not part of the face, not part of a triangle in the configuration, and not necessarily drawn in the figure.

Theorem 4. *Let $G \in \mathcal{F}'_{\mathcal{N}}$. Then there is a $(1,0,0)^{(3)}$-coloring of G.*

Proof. We proceed by induction on $|V(G)|$.

For the base case, observe that every graph on 3 vertices may be $(1,0,0)^{(3)}$-colored.

Let $G \in \mathcal{F}_{\mathcal{N}}$ and assume that $G \setminus v$ has a $(1,0,0)^{(3)}$-coloring for any vertex v in G. We split the proof into two cases depending on the minimum degree of G.

Case 1: $\delta(G) \leq 2$

Let v be a vertex of degree 2 or less in G. Remove v. Color $G \setminus v$ using induction. Replace v. Since $d(v) \leq 2$, at most two colors can be used on the neighbors of v in G. Color v with the third color. This coloring does not induce any flaws and the result follows.

Case 2: $\delta(G) = 3$

We split the proof into subcases reflecting each of the possible faces that G could contain. In each subcase, we detail and color configurations from Appendix A and note those that we cannot as exceptions in Appendix B.

Subcase: 10-faces

A 10-face with negative weight must be adjacent to five triangles at least four of which are $(3,3,3)$-triangles. Hence at least two of these triangles must be next to each other on the 10-face. Thus there must exist some v on the 10-face which is of degree 3 and has neighbors of degree 3. Corollary 1 applies to this v and the result follows.

Subcase: 9-faces

A 9-face with negative weight must be adjacent to four triangles, at least three of which are $(3,3,3)$-triangles. Hence at least two of these triangles must be next to each other on the 9-face. Thus there must exist some v on the 9-face which is of degree 3 and has neighbors of degree 3. Corollary 1 to this v and the result follows.

Subcase: 8-faces

An 8-face with negative weight could be adjacent to three or four triangles. We deal with each option separately.

(i) Adjacent to three triangles: For the weight on the 8-face to be negative either all three triangles require a contribution $\frac{4}{3}$ or two of the triangles require a contribution of $\frac{4}{3}$, one requires a contribution of $\frac{3}{3}$ and the two vertices not part of a triangle are of degree 3, configurations [8.1]-[8.3] in Appendix A.

If all three triangles require a contribution of $\frac{4}{3}$, then two of them are next to each other and Corollary 1 applies to a vertex on one of these triangles.

Suppose instead that only two triangles require a contribution of $\frac{4}{3}$. If they are next to each other, Corollary 1 applies to a vertex on one. If they are not next to each other, then one of them is next to a vertex of degree 3 on the 8-face and Corollary 1 applies to this vertex.

(ii) Adjacent to four triangles: For the weight to be negative, at least one of the triangles must be a $(3,3,3)$-triangle, these are configurations [8.4]-[8.11] in Appendix A. Arbitrarily assume it is triangle A in Figure 6. Let the face be labeled as it is in the figure. If Corollary 1 does not apply, then $d(v_1) \geq 4$ and $d(v_6) \geq 4$, and at least one of triangles B or D is a $(3,3,4)$-triangle, suppose it is triangle B, then $d(v_1) = 4$, $d(\beta) = 3$, and $d(v_2) = 3$.

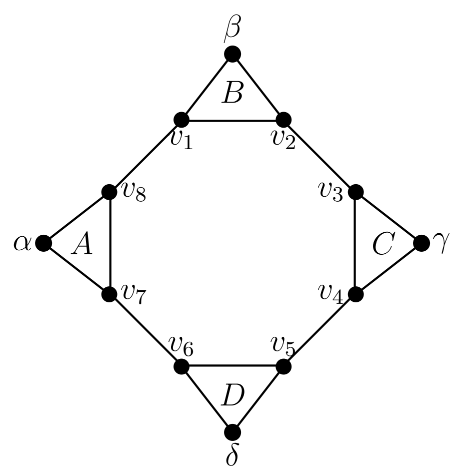

Figure 6. An 8-face adjacent to four triangles.

In order for the weight on the 8-face to be negative, either both triangle C and triangle D require a contribution of at least $\frac{3}{3}$ or triangle C requires a contribution of $\frac{4}{3}$ and triangle D requires a contribution of $\frac{2}{3}$ in order for the weight on the 8-face to be negative. If D requires $\frac{2}{3}$ and C requires $\frac{4}{3}$, then $d(v_3) = 3$ and Lemma 5 applies to vertex v_2. If both C and D require a contribution of $\frac{3}{3}$, then $d(v_6) = 4$ and one of v_3 or v_4 is of degree 3. Here Lemma 5 will apply to either v_2 or v_5 and the result follows.

Subcase: 7-faces

A 7-face with negative weight could be adjacent to two or three triangles. We deal with each option separately below.

(i) Adjacent to two triangles: For the weight on the 7-face to be negative at least one of the triangles must require a contribution of $\frac{4}{3}$. If both triangles requires a contribution of $\frac{4}{3}$, then at least two of the vertices on the 7-face that are not on triangles are of degree 3. Hence one of the triangles has a vertex v which is of degree 3 and has neighbors of degree 3. Corollary 1 applies to vertex v and the result follows.

If only one triangle requires a contribution of $\frac{4}{3}$ then the other triangle requires a contribution of $\frac{3}{3}$ and all three of the vertices not on triangles must be of degree 3. Hence one of the vertices, v, on the 7-face and the $(3,3,3)$-triangle is of degree 3 and has neighbors of degree 3. As before, Corollary 1 applies to v and the result follows.

(ii) Adjacent to three triangles: Negative weight could come about in several ways. Let the 7-face be labeled as in Figure 7. We discuss each possible scenario.

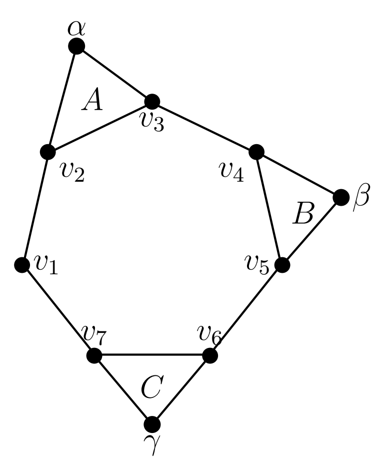

Figure 7. A 7-face adjacent to three triangles.

(a) If two triangles next to each other are $(3,3,3)$-triangles, then there is some vertex v on a triangle and the 7-face that is of degree 3 and adjacent to neighbors of degree 3. Corollary 1 applies to v and the result follows.

(b) If triangles A and C require a contribution of $\frac{4}{3}$, and triangle B requires a contribution of $\frac{3}{3}$ or $\frac{2}{3}$, and at least one vertex on the third triangle and the 7-face is of degree three, then v_4 or v_5 is of degree 3 on the 7-face. Hence one of the vertices of degree 3 on triangle A or C has only neighbors of degree 3. Corollary 1 applies to this vertex and the result follows.

(c) If triangles A and C require a contribution of $\frac{4}{3}$, and triangle B requires a contribution of $\frac{1}{3}$, then the vertex not on a triangle is of degree 3. Hence Corollary 1 will apply to a v_2 or v_7 and the result follows.

(d) If all three triangles require contributions of $\frac{3}{3}$, then the vertex not on a triangle is of degree 3. Since the only type of triangle that requires a contribution of $\frac{3}{3}$ is one containing two vertices of degree 3 and one vertex of degree 4, there must be at least four vertices of degree 3 on the 7-face; hence two of them must be adjacent and not part of the same triangle. Call the vertex on the triangle v. Lemma 5 applies to v and the result follows.

Note that the configuration in which triangles A and C require a contribution of $\frac{4}{3}$ and triangle B requires a contribution of $\frac{2}{3}$, and the configurations in which only one of triangles A, B, or C requires a contribution of $\frac{4}{3}$ are configurations [7.1], [7.2], and [7.3] in Appendix B and need not be considered.

Subcase: 6-faces

A 6-face adjacent to one, two, or three triangles may have negative weight. We deal with each case separately below.

(i) Adjacent to one triangle: For the weight on the 6-face to be negative, the contribution required by vertices and triangles must total $\frac{7}{3}$. Hence the triangle requires a contribution of at least $\frac{3}{3}$. If the triangle requires a contribution of $\frac{4}{3}$, then three of the other vertices on the 6-face are of degree 3 and one may be of degree 4. If the triangle requires a contribution of $\frac{3}{3}$, then all four of the other vertices on the 6-face are of degree 3. In each scenario, there are at least five vertices of degree 3 on the 6-face. Since it is an even face, Lemma 6 applies and the result follows.

(ii) Adjacent to two triangles, next to each other: Let each 6-face be labeled as in Figure 8

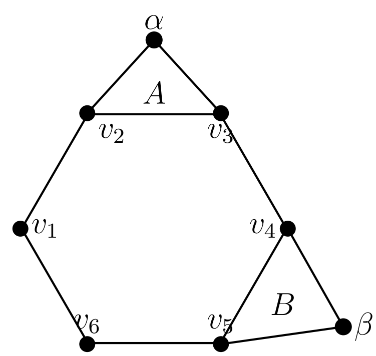

Figure 8. A 6-face adjacent to two triangles, next to each other.

(a) If both triangles require a contribution of $\frac{4}{3}$, then Corollary 1 applies to either v_3 or v_4 and the result follows.

(b) If one triangle requires a contribution of $\frac{4}{3}$ and the other requires a contribution of $\frac{3}{3}$, then the vertices not on triangles may be of any degree. Suppose that A is the triangle that requires the $\frac{4}{3}$ contribution. If v_4 is of degree 3 then Corollary 1 applies. Suppose that v_4 is instead of degree 4; this forces v_5 and β to be of degree 3. If v_1 is of degree 3, then Corollary 1 applies to v_2 and the result follows. If v_6 is of degree 3, then Lemma 5 applies to v_5. Assume now that v_1 and v_6 are of degree at least 4. Let v_2 be v; remove v, color the remainder using induction, and replace v. If v_3 or α is colored with c_1, then Lemma 4 applies, and the result follows. Suppose that v_1 is colored with c_1, and without loss of generality that v_3 is colored with c_3 and α is colored with c_2. Note that v_6 cannot have c_1 since any flaw must involve a vertex of degree 3. Furthermore, if α' is c_3, α may be recolored with c_1 and c_2 may be used on v. Similarly, if v_4 is c_2, then v_3 may be recolored with c_1 and c_3 may be used on v. Hence assume α' and v_4 are colored with c_1 and must be fully-flawed, since if they were not, we could recolor v_3 or α with c_1 anyway. Suppose first that v_5 is colored with c_1, i.e. is the flaw: If the color of v_6 matches the color of β, then recolor v_5 with the remaining color, recolor v_3 with c_1 and use c_3 on v. If v_6 and β have different colors, then consider the color of β'. If β' has c_1, then recolor β to match v_6, and revert to the case above. If β' is not colored with c_1, then recolor β with c_1, v_4 with the color not used on v_4', recolor v_3 with c_1, and use c_3 on v.

Now suppose that β has c_1 i.e. is the flaw: If v_5 and v_4' have the same color, recolor v_4 with the unused color, recolor v_3 with c_1 and use c_3 on v. If v_5 and v_4' do not have the same color, recolor v_5 with c_1, use the now unusued color on v_4, recolor v_3 with c_1, and use c_3 on v.

Suppose now that v_4' has c_1 i.e. is the flaw: Recolor v_4 with the color of v_5, recolor v_5 with c_1, recolor v_3 with c_1 and use c_3 on v. A coloring with the needed property has been produced and the result follows.

(c) If one triangle requires a contribution of $\frac{4}{3}$ and the other requires a contribution of $\frac{2}{3}$, then at least one of the vertices not on a triangle is of degree 3. Let A be the triangle requiring a contribution of $\frac{4}{3}$ and B be the triangle requiring a contribution of $\frac{2}{3}$. If v_1 or v_4 is of degree 3, then Corollary 1 applies to v_2, hence v_1 is of degree 4 and v_6 is of degree 3.

(c.1) Suppose first that v_4 is of degree 4, β is of degree 4, and v_5 is of degree 3, i.e. B is a $(3, 4, 4)$-triangle. Let v_3 be v, remove v, color the remainder of the graph using induction, and replace v. If α or v_2 is c_1, then Lemma 4 applies to v, so assume that v_4 must have c_1 and be fully-flawed. Without loss of generality, suppose that v_2 is c_2 and α is c_3. If either v_1 or α' is not colored with c_1, then recolor either v_2 or α with c_1 and use the unused color on v. So suppose that both v_1 and α' have c_1 and are fully-flawed. We consider the location of v_4's flaw; note that it will never be β since β is of degree 4. If v_5 is c_1, then v_6 must have some other color and v_1's flaw must be off the 6-face. If the color of β matches the color of v_4', change v_4 to the other color and use c_1 on v. If the color of β matches the color of v_6, then recolor v_5 with the unused color and use c_1 on v. Assume then, that v_6 and v_4' have the same color and β has the other color. If v_6' has c_1, then recolor v_6 to match β and proceed as above; so assume that v_6' has a color other than c_1. Observe that v_1 has neighbors off the 6-face colored with c_1 and one other color, call the one not colored with c_1, v_1'. If the color of v_1' matches the color of v_6, then recolor v_1 with the unused color, recolor v_2 with c_1 and use c_2 on v. If v_1' and v_6 have different colors, recolor v_5 with the color of v_6, recolor v_6 with c_1, and recolor v_1 with the color previously used on v_6. Recolor v_2 with c_1 and use c_1 on v. Note that in this scenario, we have removed two flaws.

Now suppose that v_4' is c_1. If v_6 is not c_1, then recolor v_5 with c_1, recolor v_4 with the color previously used on v_5, and use c_1 on v. If v_6 is colored with c_1 and v_6' has the same color as v_5, recolor v_6 with the unused color, recolor v_5 with c_1, v_4 with the color previously used on v_5, and use c_1 on v. If v_6 is colored with c_1 and v_6' and v_5 have different colors, recolor v_6 and v_4 with the color used on v_5, recolor v_5 with c_1, and use c_1 on v.

(c.2) Suppose now that v_4 and v_5 are of degree 4 and β is of degree 3 to generate the triangle requiring a contribution of $\frac{2}{3}$. Let v_3 be v, remove v, color the remainder of the graph using induction, and replace v. If either α or v_2 are colored with c_1, then Lemma 4 applies to v, so assume that v_4 is colored with c_1 and, without loss of generality that v_2 has c_2 and α has c_3. If either v_1 or α' is not colored with c_1, recolor v_2 or α with c_1 respectively, and use the unused color on

v. Hence assume that v_1 and α' are colored with c_1 and fully-flawed. As above, we consider the location of v_4's flaw.

Suppose v_4' is colored with c_1, i.e. is the flaw. If β' is not colored with c_1 or is not fully-flawed, then recolor v_4 with the color used on β and β with c_1; use c_1 on v. Assume now that β' is colored with c_1 and fully-flawed; without loss of generality, also assume that β is colored with c_2 and v_5 is colored with c_3. If v_6 is colored with c_1 then v_6' is colored with some other color. If the color of v_6' matches v_5, recolor v_6 with the unused color, recolor v_2 with c_1, and use c_2 on v. If v_6' is colored with c_2 (i.e. does not match v_5), then recolor v_6 with c_3. If v_5' is colored with c_1, recolor v_5 with c_2, β with c_3, v_2 with c_1, and use c_2 on v. If v_5' is colored with c_2, recolor v_5 with c_1, v_4 with c_3 and use c_1 on v. If v_6 is colored with c_2, then v_1's flaw must be off the 6-face. If v_6' is colored with c_3, then recolor v_1 with the color not used in its neighbor off the 6-face not colored with c_1, recolor v_2 with c_1, and use c_2 on v, and recolor v_6 with c_1. If v_6' is colored with c_1 and v_5' is colored with c_1 then perform a c_2-c_3 switch between β and v_6, if necessary so that v_6 matches the color used on the neighbor of v_1 not colored c_1. Recolor v_1 with the color unused on its neighbors off the 6-face, recolor v_2 with c_1, and use c_2 on v. If v_6' is colored with c_1 and v_5' is colored with c_2, then recolor v_4 with c_3, v_5 with c_1, and use c_1 on v.

Suppose now that β is colored with c_1, i.e. is the flaw of v_4. Suppose also, without loss of generality that v_5 is colored with c_3 and v_4' is colored with c_2. If β' has c_3, then recolor β with c_2 and use c_1 on v; assume that β' has c_2. If v_6 and v_5' have the same color, recolor v_5 with either c_2 or c_1, recolor v_4 with c_3, and use c_1 on v; so assume instead that v_5' and v_6 have different colors. If v_6 has c_1 and v_6' has c_3, recolor v_6 with c_2, recolor v_2 with c_1 and use c_2 on v. If v_6 has c_1 and v_6' has c_2, recolor v_6 with c_3, recolor v_5 with c_1, recolor v_4 with c_3, and use c_1 on v. Now, if v_6 has c_2 and v_5' has c_1 and is fully flawed, then assume v_6' has c_3, otherwise recolor v_6 with c_3, v_5 with c_2 and v_4 with c_3 to use c_1 on v. This forces the flaw on v_1 to be off the 6-face, and furthermore, only two colors are used by neighbors of v_1 off the 6-face, c_1, and one other color. Recolor v_1 with the color unused by its neighbors off the 6-face. Recolor v_2 with c_1 and use c_2 on v; if necessary, recolor v_6 with c_1 also.

(c.3) Suppose now that v_4 is of degree 5 and v_5 and β are both of degree 3, i.e. B is a $(3, 3, 5)$-triangle. Let v_5 be v; remove v, color the remainder of the graph using induction, and replace v. If either v_6 or β are colored with c_1, then Lemma 4 applies to v; so assume that v_4 is colored with c_1 and is fully flawed, and assume without loss of generality that v_6 has c_3 and β has c_2. If β' has c_1, then recolor β with c_3 and use c_2 on v. Similarly, if v_1 has the same color as v_6' then recolor v_6 with the unused color and use c_3 on v. We now consider the location of the flaw on v_4.

If v_3 is colored with c_1, i.e. is the flaw of v_4, then α an v_2 are colored with c_2 and c_3 in some order. If v_1 is not colored with c_1, recolor v_2 with c_1, v_3 with the color previously used on v_2, and use c_1 on v. Similarly, if α' has a color other than c_1, recolor α with c_1, and v_3 with the color previously used on α, then use c_1 on v. If both v_1 and α are colored with c_1 and fully flawed, then v_6' is colored with c_2, since it does not have the same color as v_1, and the flaw of v_1 is not on the 6-face so only c_1 and one other color are used on the neighbors of v_1 off the 6-face. Recolor v_1 with the color not used off the 6-face; recolor v_6 with c_1, v_2 with c_1, and v_3 with the color formerly used on v_2. Use c_3 on v. Observe that, in this scenario, we have removed two flaws.

Suppose v_4' is colored with c_1, i.e. the flaw of v_4 is off the 6-face. If v_4'' shares the same color as v_3, then recolor v_4 with the unused color, β with c_1 if necessary since β' is not colored with c_1, and use c_1 on v. If v_4'' does not share the same color as v_3, without loss of generality, suppose v_3 has c_2 and v_4'' has c_3, then v_2 or α is c_1. Recolor v_4 with c_2 (not used on v_4''); recolor v_3 with c_1, and α or v_2 with c_2 if necessary to remove a flaw. If needed, recolor β with c_1 and use c_1 on v. A flaw will be generated with v, but v is of degree 3 and the property is preserved and the result follows.

In each scenario, a $(1,0,0)^{(3)}$-coloring has been obtained that satisfies the needed property and the result follows.

(d) If one triangle requires a contribution of $\frac{4}{3}$ and the other requires a contribution of $\frac{1}{3}$, then both vertices not part of the triangles must be of degree 3 and Corollary 1 applies to a vertex on the triangle requiring a contribution of $\frac{4}{3}$.

(e) If both triangles require a contribution of $\frac{3}{3}$, then at least one vertex not part of a triangle is of degree 3. Let the vertices of the 6-face be labeled as in Figure 8. We consider the configuration depending on triangles A and B.

(e.1) If A is a $(3,3,4)$-triangle in which v_2 and v_3 are of degree 3, then either Lemma 5 applies to v_2 or v_3, or both v_1 and v_4 are of degree at least 4 and therefore v_6 is of degree 3. Furthermore, B must be a $(4,3,3)$-triangle with v_5 and β being of degree 3. Hence Lemma 5 applies to v_5 and the result follows.

(e.2) If A is a $(3,3,4)$-triangle in which v_3 and α are of degree 3, then v_4 must be of degree 4 otherwise Lemma 5 applies to v_3. This forces B to be a $(3,3,4)$-triangle in which v_5 and β are of degree 3. Furthermore, if v_6 is of degree 3, then Lemma 5 applies to v_5 and the result follows; so assume that v_6 is of degree 4 and v_1 is of degree 3. Let v_3 be v; remove v, color the remainder of the graph, and replace v.

If α is colored with c_1, then Lemma 4 applies to v, so assume that α has c_3. Suppose v_2 has c_1 and is fully-flawed, and v_4 is colored with c_2, and α is colored with c_3. If α' has c_1, recolor α with c_2 and use c_3 on v; so α' has c_2. If v_1 has c_1, i.e. is the flaw for v_2, then recolor v_2 with whatever color is not used on v_2', recolor α with c_1 if necessary, and use c_1 on v. If v_2' has c_1 instead, i.e. v_2' is the flaw for v_2, then recolor v_2 with the color not used on v_1, recolor α with c_1 if necessary, and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2, and α is colored with c_3. If α' is colored with c_2, then recolor α with c_1 and use c_3 on v; so assume that α' has c_1. We consider the location of v_4's flaws. If v_5 is colored with c_1, i.e. is the flaw for v_4, then if necessary recolor β with either c_1 (if β' is not colored with c_1) or the color used on v_4', recolor v_4 with the color not used on v_4' and use c_1 on v. Similarly, if β is colored with c_1, i.e. is the flaw for v_4, then recolor v_5 if necessary, and recolor v_4 with the color not used on v_4', use c_1 on v.

If v_4' is c_1, i.e. is the flaw of v_4, then suppose without loss of generality that v_5 has c_2 and β has c_3. If either v_6 or β' is not colored with c_1, then recolor v_5 or β respectively with c_1, recolor v_4 with the unused color, and use c_1 on v; so assume that both v_6 and β' have c_1 and are fully-flawed. If v_1 is colored with c_1, i.e. is the flaw of v_6, then recolor v_1 with whatever color is not used on v_1', recolor v_2 with c_1 or c_3 as necessary depending on the color of v_2', and recolor α if necessary. Use c_1 on v_5, c_2 on v_4 and c_1 on v. If v_1 has c_3 and the flaw of v_6 is on some vertex not on the 6-face, then recolor v_6 with the color not used on its other neighbor off the 6-face; recolor v_5 with c_1, and use c_2 on v_4. If necessary, recolor v_1 with either c_1 or c_2 depending on the color of v_1', v_2 with c_3 (note that v_1' must have been colored with c_1), and α with c_2. Use c_1 on v. This produces a coloring in which the only flaws involve vertices of degree 3 and the result follows.

(e.3) If A is a $(3,3,4)$-triangle in which v_2 and α are of degree 3, then Lemma 5 applies to v_2 unless v_1 is of degree 4, so assume v_1 is of degree 4. This forces v_6 to be of degree 3. If B is a $(3,3,4)$-triangle in which v_5 is of degree 3, then

Lemma 5 applies to v_5; so assume that v_5 is of degree 4 and v_4 and α are of degree 3. Let v_2 be v; remove v, color the remainder of the graph, and replace v. If α is colored with c_1, then Lemma 4 applies and the result follows; instead, assume without loss of generality that α has c_2. If v_3 has c_1 and is fully-flawed and v_1 has c_3, then α' has either c_1, in which case recolor α and use c_2 on v, or α' has c_3. In this case, recolor v_3 with whatever color is not used on its neighbor not colored with c_1, recolor α with c_1 if necessary, and use c_1 on v.

If v_1 has c_1 and is fully-flawed, then assume without loss of generality that v_3 has c_3 and α has c_2. If α' has c_3, then recolor α with c_1, and use c_2 on v; assume that α' has c_1 and is fully-flawed. We consider the location of the flaw on v_1.

If v_6 has c_1, i.e. is the flaw for v_1, then v_5 must be colored with either 2 or 3. Recolor v_6 with the color not used on v_6'. If necessary, recolor v_5 with either c_1 or the color not used on v_5'. If v_5 now has the same color as β, recolor β with either c_1 or the color unused on β'. If instead v_5 now has the same color as v_4, recolor v_4 with c_1, switching, if β is c_1 and fully-flawed, the color of β to that formerly on v_5. Finally color v with c_1.

If v_1' has c_1, i.e. the flaw for v_1 is off the 6-face, then recolor v_1 with the color not used on v_1''. Recolor v_6 with either c_1, or the the color used on v_1''. As in the previous case, recolor v_5 with the color not used on v_5'; recolor β with c_1 or the color unused on β' or recolor v_4 with c_1 and switch, if necessary the color of β to that formerly used on v_5. Finally, color v with c_1. No flaws have been introduced that do not involve a vertex of degree more than 3 as every vertex that received c_1 was of degree 3 and the result follows.

(f) If one triangle requires a contribution of $\frac{3}{3}$ and the other requires a contribution of $\frac{2}{3}$, then both v_1 and v_6 are of degree 3. If A is a $(3,3,4)$-triangle in which v_2 is of degree 3, then Lemma 5 applies to v_2 and the result follows; so assume that A is a $(3,3,4)$-triangle in which v_2 is of degree 4. If v_4 is of degree 3, then Lemma 5 applies to v_3 and the result follows. Hence B may be a $(3,4,4)$-triangle in which v_4 has degree 4 or a $(3,3,5)$-triangle in which v_4 has degree 5.

(f.1) Suppose first that B is a $(3,4,4)$-triangle in which v_4 and v_5 are of degree 4. Let v_6 be v; remove v, color the remainder, and replace v. If v_1 has c_1, then Lemma 4 applies to v_6; so assume without loss of generality that v_1 has c_2, and v_5 and v_6' have colors c_1 and c_3 in some order.

Suppose first that v_4 is colored with any color except c_1, or is colored with c_1 but is not fully-flawed. If v_2 has c_1 and is fully-flawed, then v_1' has c_3, or we can recolor v_1 with c_3 and use c_2 on v. If v_2' is colored with c_1, then recolor v_3 with c_1, recolor v_2 with the color previously used on v_3, recolor v_1 with c_1, and use c_2 on v. If α or v_3 has c_1, i.e. is the flaw for v_2, then recolor v_2 with the color not used on v_2', recolor α or v_3 if necessary with c_1 or the color not used on their neighbors, recolor v_1 with c_1, and use c_2 on v.

So instead, assume that v_2 has c_3 and v_1' has c_1 and is fully-flawed. If v_2' has c_2, recolor v_2 with c_1, recolor α or v_3 with c_3 if necessary, recolor v_1 with c_3, and use c_2 on v. If v_2' has c_1, then recolor v_2 with c_2, recolor v_3 and/or α with c_1 or c_3 as necessary, recolor v_1 with c_3, and use c_2 on v.

Suppose now that v_4 has c_1 and is fully-flawed, then v_5 must have c_3, since both are of degree 4. If v_2 and α are colored with c_1, then recolor v_3 as necessary so that it has the same color as v_2', recolor v_2 with the color not used on v_2', recolor v_1 with c_1, and use c_2 on v. If v_2 and v_2' are colored with c_1, then v_3 has a color other than c_1. If v_4' is colored with c_1, then β has c_2. Recolor v_5 with either c_2 or c_1 whichever is not used on v_5', recolor β with c_3 or c_1 as necessary, and recolor v_4 with c_3 as needed, recolor v_3 with c_1 and v_2 with c_3 if necessary. Use c_3 on v. If β has c_1, then recolor v_4 with the color not used on v_4', recolor v_3 and α by switching colors around triangle A as needed. Recolor v_5 with c_1, or the color not used on v_5', and use c_3 on v. A coloring satisfying the needed condition has been provided and the result follows.

(f.2) Second, suppose that B is a $(3, 4, 4)$-triangle in which v_5 is of degree 3. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If α has c_1, then Lemma 4 applies to v and the result follows, so assume without loss of generality that α has c_3. If v_2 and v_2' have c_1, then recolor v_2 with c_3 or 2 (if v_1 has c_2, recolor α with either c_1 or c_2, whichever is not used on α'), and use c_1 on v. If v_2 and v_1 have c_1, then recolor v_2 with the color not used on v_2', recolor α with c_1 or c_2 as needed, and use c_1 on v and the result follows.

Assume instead that v_2 has c_2 and v_4 has c_1 and is fully-flawed. If α' has c_2, then recolor α with c_1 and use c_3 on v; so assume that α' has c_1.

Suppose that v_4 and v_5 are colored with c_1, i.e. v_5 forms the flaw for v_4. If v_4' has the same color as β recolor v_4 with the unused color and use c_1 on v. If v_1 is colored with c_1, recolor v_6 with c_1 or the color not used on v_6', if needed recolor v_1 with c_3, recolor v_5 with the color not used on β and use c_1 on v. If v_1 is colored with c_3, then recolor v_5 with the color not used on β, if needed; recolor v_1 with c_1 or c_2, recolor v_2 with c_1 or c_3 and α with c_2, recolor v_6 with the same color as that used on β or with c_1, whichever is not used on v_6'. Finally, color v with c_1. Suppose instead that v_4 and v_4' are colored with c_1, i.e. v_4' forms the flaw for v_4. If v_6 is not colored with c_1 or is colored with c_1 and is not fully-flawed, then recolor v_5 with c_1, recolor v_4 with the color previously used on v_5, and use c_1 on v. If v_6 is colored with c_1 and is fully-flawed, then recolor v_6 with the color is not used on v_1 or v_6', whichever is not colored with c_1 and repeat the scenario with v_6 not colored with c_1. A coloring satisfying the needed properties has been produced and the result follows.

(f.3) Finally, suppose that B is a $(3, 3, 5)$-triangle in which v_4 has degree 5. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If α has c_1, then Lemma 4 applies to v and the result follows, so assume without loss of generality that α has c_3. If v_2 and v_2' have c_1, then recolor v_2 with c_3 or c_2 (if v_1 has c_2, recolor α with either c_1 or c_2, whichever is not used on α'), and use c_1 on v. If v_2 and v_1 have c_1, then recolor v_2 with the color not used on v_2', recolor α with c_1 or c_2 as needed, and use c_1 on v and the result follows. Assume instead that v_2 has c_2 and v_4 has c_1 and is fully-flawed. If α' has c_2, then recolor α with c_1 and use c_3 on v; so assume that α' has c_1. If v_2' and v_1 are colored with c_3, then recolor v_2 with c_1 and use c_2 to color v. We consider the position of the flaw of v_4.

If v_5 has c_1, i.e. is the flaw for v_4, then recolor v_5 with the color not used on β. If v_6', v_1', and v_2' have c_1, perform a c_2-c_3 color switch on v_6, v_1, v_2, and α. If one of v_6', or v_1' is not colored with c_1, recolor the corresponding vertex on the 6-face with c_1, and perform the c_2-c_3 switch between v_6 and that vertex. Finally color v with c_1.

If β has c_1, i.e. is the flaw for v_4, then either recolor β with the unused color if β' and v_5 share a color or follow the above recoloring scheme, using β instead of v_5 and changing the color of v_5 to that matching β'. Use c_1 on v.

If the flaw for v_5 is on a vertex off the 6-face, then recolor v_5 with whatever color is not used on the neighbor off the 6-face not colored with c_1. If β is now the same color as v_4', recolor β with c_1 or the color unused on v_4 in the case that β' is colored with c_1. Recolor v_5 with c_1, if needed. If v_6 has c_1 and is fully-flawed, change v_6 to the color not used on v_1 or v_6', whichever not colored with c_1; otherwise, there is no need to change the color of v_6. Use c_1 on v. This produces a coloring in which every flaw involves a vertex of degree 3 and the result follows.

(iii) Adjacent to two triangles, on opposite sides of the 6-face: We now consider the case in which the two triangles adjacent to the 6-face are not next to each other. Let each 6-face be labeled as in Figure 9. Note that v_1 and v_4 are of degree no more than 4, otherwise the face is one of the forbidden configurations.

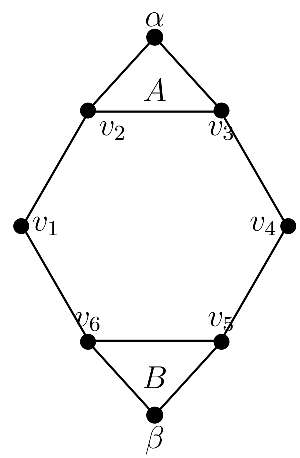

Figure 9. A 6-face adjacent to two triangles, on opposite sides.

(a) If one of the triangles requires a contribution of $\frac{4}{3}$ then Corollary 1 applies to a corner of that triangle unless both v_1 and v_4 are of degree 4. Suppose that A is the $(3,3,3)$-triangle requiring a contribution of $\frac{4}{3}$. Note that triangle B must require a contribution of at least $\frac{3}{3}$ and, therefore, one of v_5 or v_6 has degree 3. Assume without loss of generality that v_6 has degree 3. Let v_2 be v; remove v, color the remainder of the graph, and replace v. If v_3 or α has c_1 then Lemma 4 applies to v, so assume without loss of generality that v_3 has c_3, α has c_2 and v_1 has c_1 and is fully-flawed. If α' has c_3 or has c_1 but is not fully-flawed, then recolor α with c_1 and use c_2 on v; hence assume α' has c_1 and is fully-flawed. Similarly, if v_4 has c_2 or has c_1 but is not fully-flawed, then recolor v_3 with c_1 and use c_3 on v; so assume also that v_4 has c_1 and is fully-flawed. If v_5 is of degree 4, then v_5 does not have c_1.

If the flaw on v_1 is on a vertex off the 6-face, then recolor v_1 with the color not used on its neighbor off the 6-face not colored with c_1, unless the color of v_6 now matches the color of v_1, use c_1 on v and the needed coloring has been produced. Suppose now that v_1 has the same color as v_6. If v_5 is colored with c_1, recolor v_6 with the c_1, and recolor v_5 with the color previously used on v_6. If v_5 is not colored with c_1, then β must have c_1. If β has degree 3, recolor v_6 with c_1 and β with the color formerly used on v_6 if needed. If β has degree 4, then v_5 has degree 3; switch the colors of v_5 and v_6. Finally color v with c_1.

If v_6 is colored with c_1, i.e. is the flaw for v_1, then the flaw on v_4 is off the 6-face. Recolor v_4 with the color not used on its neighbor off the 6-face that does not have c_1; if necessary, switch the colors of v_5 and v_6 or on v_5, v_6 and β. Recolor v_3 with c_1 and use c_3 on v. The coloring has been extended in the needed way and the result follows.

(b) If both triangles require a contribution of $\frac{3}{3}$, then one of v_1 or v_4 must have degree 3. If both v_1 and v_4 are of degree 3, then Lemma 5 will apply to at least one of v_1, v_3, v_5 or v_6 and the result follows; so assume without loss of generality that v_1 has degree 4 and v_4 has degree 3. Furthermore, if either v_3 or v_5 are of degree 3 then Lemma 5 will apply. Any other configuration is forbidden and the result follows.

(iv) Adjacent to three triangles: As before, the weight contribution required by the triangles and the vertices must total at least $\frac{7}{3}$ in order for the weight on the 6-face to be negative. Among the allowed configurations, this weight contribution requirement may be accomplished in several ways; we will deal with each of them below. Let the 6-face be labeled as in Figure 10.

First we consider the cases in which A is a $(3, 3, 3)$-triangle requiring a contribution of $\frac{4}{3}$. If either v_1 or v_4 is of degree 3, then Corollary 1 applies to v_2 or v_3 respectively and the result follows. Assume now that both v_1 and v_4 are of degree at least 4.

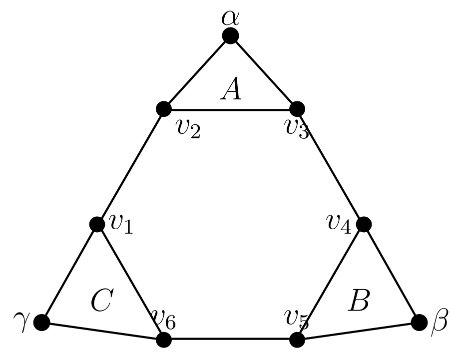

Figure 10. A 6-face adjacent to three triangles.

(a) If B is a $(3, 3, 4)$ triangle in which v_4 is of degree 4, then Lemma 5 applies unless v_6 is of degree at least 4; so assume that v_6 is of degree at least 4. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 or α is colored with c_1, then Lemma 4 applies to v, so assume without loss of generality that v_2 has c_2, α has c_3, and v_4 has c_1 and is fully-flawed. If α' or v_1 is not colored with c_1, or is colored with c_1 but not fully-flawed, recolor α or v_2 respectively with c_1 and use the unused color on v. Hence we proceed with the assumption that α' and v_1 are colored with c_1 and are fully-flawed. This forces v_6 to have a color other than c_1, since it is of degree at least 4. If the flaw of v_4 is on either v_5 or β, recolor v_4 with the color not used on v_4', recolor v_5 or β if necessary with either c_1 or the color not used on v_4 (since v_6 is does not have c_1, the v_5 may always be colored with c_1), and use c_1 on v. If the flaw of v_4 is v_4' then switch the colors on v_5 and v_4 since v_6 must not have c_1, and use c_1 on v. This provides the needed coloring an the result follows.

(b) If B is a $(3, 4, 4)$-triangle in which v_5 is of degree 3, then v_1 is of degree 4. Just as above, let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 or α is colored with c_1, then Lemma 4 applies to v, so assume without loss of generality that v_2 has c_2, α has c_3, and v_4 has c_1 and is fully-flawed. If α' or v_1 is not colored with c_1, or is colored with c_1 but not fully-flawed, recolor α or

v_2 respectively with c_1 and use the unused color on v. Hence we proceed with the assumption that α' and v_1 are colored with c_1 and are fully-flawed. In order for the configuration to be allowed, v_6 is of degree 3.

We now consider the the location of the flaws of v_1 and v_4.

If v_5 has c_1, i.e. is the flaw for v_4, then v_6 must have a color that is not c_1. If γ has c_1, recolor v_6 to match β if needed, and recolor v_5 with whatever color is not used on β. Finally, use c_1 on v. If v_1' has c_1, then recolor v_1 with the color used on v_6, recolor v_6 with c_1, recolor v_5 with whatever color is not used on β, recolor v_2 with c_1, and use c_2 on v.

If instead, v_4' has c_1, i.e. is the flaw for v_4, then interchange the colors of v_4 and v_5, recoloring v_6 with the color not used on γ if necessary. Finally use c_1 on v. The needed coloring has been provided.

(c) If B is a $(3, 3, 5)$-triangle in which v_4 has degree 5, then v_6 has degree 3 and $d(v_1) = 4$. Let v_2 be v; remove v, color the remainder of the graph, and replace v. If v_3 or α is colored with c_1, then Lemma 4 applies to v, so assume without loss of generality that v_3 has c_3, α has c_2, and v_1 has c_1 and is fully-flawed. If α' or v_4 is not colored with c_1, or is colored with c_1 but not fully-flawed, recolor α or v_3 respectively with c_1 and use the unused color on v. Hence we proceed with the assumption that α' and v_4 are colored with c_1 and are fully-flawed.

If v_1' is colored with c_1, i.e. is the flaw for v_1, then switch the colors on v_1 and v_6; if needed recolor v_5 with whichever color is not used on β. Use c_1 on v. If v_6 is colored with c_1 i.e. is the flaw for v_1, then consider the location of the flaw on v_4.

If the flaw on v_4 is β, then recolor v_6 with the color not used on γ, recolor v_5 as necessary so that v_5 has the same color as γ, and use c_1 on v. If instead, the flaw on v_4 is on some neighbor not on the 6-face, recolor v_4 with the color not used on its neighbor off the 6-face not colored with c_1 (v_4'), recolor v_3 with c_1, recolor v_6

with the color not used on γ, recolor v_5 with c_1. Finally use c_3 on v. This produces a coloring with the required properties and with fewer flaws than the original and the result follows.

Secondly we consider the cases in which A is a $(3, 3, 4)$-triangle requiring a contribution of $\frac{3}{3}$ and in which v_2 and v_3 are of degree 3. If either v_1 or v_4 is of degree 3, then Lemma 5 applies to v_2 or v_3 respectively and the result follows. Assume now that both v_1 and v_4 are of degree at least 4. Note also that the weight contribution of the other two triangles must sum to at least $\frac{4}{3}$. We will consider cases based on triangle B.

(a) If B a $(3, 3, 4)$-triangle in which v_4 is of degree 4, then Lemma 5 applies to v_5 unless v_6 is of degree at least 4; so assume that v_6 is of degree at least 4. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 or the color not used on v_1 and use c_1 on v. Suppose now that v_4 is colored with c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. If v_5 has c_1, i.e. is the flaw for v_4, then recolor v_4 with the color not used on v_4', recolor β with c_1 or the color not used on β' if necessary, and use c_1 on v. Similarly, if β has c_1, recolor v_4 with the color not used on v_4', recolor v_5 with c_1 or the color not used on v_6 if necessary, and use c_1 on v. If v_4' has c_1, then switch the colors used on v_4 and v_5 and color v with c_1. The coloring has been extended in the needed way and the result follows.

(b) If B is a $(3, 4, 4)$-triangle in which $d(v_5) = 3$, then note that triangle C must require a contribution of at least $\frac{2}{3}$ and hence at least one of v_6 or γ is of degree 3.

(b.1) Suppose first that C is a $(3,4,4)$-triangle in which $d(\gamma) = 4$. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 or the color not used on v_1, and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. We consider the location of the flaws of v_4; note that the flaw will never be on β since β is of degree 4. If v_5 has c_1, i.e. is the flaw for v_4, then the flaw for v_1 is on v_1' off the 6-face. Switch the colors of v_1 and v_6, recolor v_2 with c_1, v_5 with the color not used on β and use c_2 on v_4. If the flaw for v_4 is v_4', then switch the colors on v_4 and v_5; if necessary, recolor v_6 with the color not used on γ. Use c_1 on v and the result follows.

(b.2) Secondly, suppose that C is a $(3,3,5)$-triangle in which v_1 is of degree 5. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. If v_5 is colored with c_1, i.e. is the flaw for v_4, and the flaw for v_1 is on γ, then recolor v_6 to match β, recolor v_5 with the color not used on β, and use c_1 to color v. If v_5 has c_1, i.e. is the flaw for v_4, and the flaw for v_1 is on v_1' off the 6-face, then recolor v_1 with whatever color is not used on its neighbor not colored with c_1 off the 6-face. If γ now has the same color as v_1, then switch the colors of v_6 and γ, or recolor γ with c_1. Recolor v_6 as needed with c_1; recolor v_5 with whatever color is not used on β; recolor v_2 with 1 and use c_2 on v. If v_4' is colored with c_1, i.e. is the flaw for v_4, then switch the colors of v_4 and v_5. If needed, recolor v_6 with the color not used on γ. Use c_1 on v and the result follows.

(b.3) Finally, suppose that C is a $(3,4,4)$-triangle in which $d(\gamma) = 3$. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. If v_5 is colored with c_1, i.e. is the flaw for v_4, then recolor v_5 with the color not used on β. If v_6 is does not now share a color with v_5, then color v with c_1. If instead v_6 and v_5 have the same color, then recolor v_6 with the color not used on v_6'. If v_6 now has c_1 and γ is colored with c_1, then recolor v_1 with whatever color is not used on v_1', recolor v_2 with c_1 and us c_1 on v. If v_6 now is colored with c_1, and γ is not colored with c_1, then v_1' is the flaw of v_1. Recolor v_1 with whatever color is not used on γ, recolor v_2 with c_1 and use c_2 on v. If instead v_6 and γ now share a color not c_1, then recolor γ with whatever color is not used on γ'. If γ receives any color but c_1, then end and use c_1 on v. If γ receives c_1, recolor v_1 with the color not used on v_6, recolor v_2 with c_1, and use c_2 on v. If v_4' is colored with c_1, i.e. is the flaw for v_4, then note that neither β or v_6 is colored with c_1. Switch the colors of v_4 and v_5, and use c_1 to color v. The result follows.

(c) If B is a $(3,3,5)$-triangle in which $d(v_4) = 5$, then note that triangle C must require a contribution of at least $\frac{2}{3}$ and hence at least one of v_6 or γ is of degree 3.

(c.1) Suppose first that C is a $(3,4,4)$-triangle in which γ is of degree 4. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 or the color not used on v_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. If v_5 is colored with c_1, i.e. is the flaw for v_4, then recolor v_5 with the color not used on β; if needed, recolor v_6 with c_1, v_1 with the color previously used on v_6, and v_2 with c_1. Use c_1 or c_2 on v and the result follows. If β is colored with c_1, i.e. is the flaw for v_4, then recolor β with the color not used on β', recolor v_5 with c_1 and follow the above procedure. If the flaw of v_4 is off the 6-face, then recolor v_4 with the color not used on its neighbor off the 6-face not colored c_1, recolor β with the color not used on β', if necessary, recolor v_5 with c_1, and if necessary recolor v_6 with the color not used on γ. Use c_1 on v and the result follows.

(c.2) Secondly, suppose that C is a $(3,4,4)$-triangle in which γ is of degree 3. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 or the color not used on v_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. If v_5 is colored with c_1, i.e. is the flaw for v_4, then recolor v_5 with the color not used on β. The recoloring will go the same as in (b.3). If v_6 is does not now share a color with v_5, then color v

with c_1. If instead v_6 and v_5 have the same color, then recolor v_6 with the color not used on v_6'. If v_6 now has c_1 and γ is colored with c_1, then recolor v_1 with whatever color is not used on v_1', recolor v_2 with c_1 and us c_1 on v. If v_6 now is colored with c_1, and γ is not colored with c_1, then v_1' is the flaw of v_1. Recolor v_1 with whatever color is not used on γ, recolor v_2 with c_1 and use c_2 on v. If instead v_6 and γ now share a color not c_1, then recolor γ with whatever color is not used on γ'. If γ receives any color but c_1, then end and use c_1 on v. If γ receives c_1, recolor v_1 with the color not used on v_6, recolor v_2 with c_1, and use c_2 on v. If β is colored with c_1, i.e. is the flaw for v_4, then recolor β with the color not used on β', recolor v_5 with c_1 and follow the above procedure. If the flaw of v_4 is off the 6-face, then recolor v_4 with the color not used on its neighbor off the 6-face not colored c_1, recolor v_5 or β with c_1 as needed. Use c_1 on v and the result follows.

(c.3) Finally, suppose that C is a $(3,3,5)$-triangle in which $d(v_1) = 5$. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If v_2 is returned with c_1, then use Lemma 4 on v. If α is returned with c_1 and fully-flawed, then recolor α with whatever color is not used on its neighbor off the triangle not colored with c_1. If necessary, recolor v_2 with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_1 has a color other than c_1, then recolor v_2 with c_1 and use c_2 on v; so assume that v_1 is colored with c_1 and fully-flawed. We consider the flaws on v_4. If v_5 is colored with c_1, i.e. is the flaw for v_4, then recolor v_5 with the color not used on β. The recoloring will go the same as in (b.3). If v_6 is does not now share a color with v_5, then color v with c_1. If instead v_6 and v_5 have the same color, then recolor v_6 with the color not used on v_6'. If v_6 now has c_1 and γ is colored with c_1, then recolor v_1 with whatever color is not used on v_1', recolor v_2 with c_1 and us c_1 on v. If v_6 now is colored with c_1, and γ is not colored with c_1, then v_1' is the flaw of v_1. Recolor v_1 with whatever color is not used on γ, recolor v_2 with c_1 and use c_2 on v. If instead v_6 and γ now share a color not c_1, then recolor γ with whatever color is not used on γ'. If γ receives any color but c_1, then end and use c_1 on v. If γ receives c_1, recolor v_1 with the color not used on v_6, recolor v_2 with c_1, and use c_2 on v. If β is colored with c_1, i.e. is the flaw for v_4, then recolor β with the color not used on β', recolor v_5 with c_1 and follow the above procedure. If the flaw of v_4 is off the 6-face, then recolor v_4 with the color not used on its neighbor off the 6-face not colored with c_1. Recolor β with the color not used on β', if needed. If v_5 and v_4 or β now have the same color, then recolor v_5 with c_1, recolor v_6 with the color not used on γ if necessary. Use c_1 on v and the result follows.

(d) The cases in which B is a $(3,4,4)$-triangle in which β is of degree 3 are mirror image the cases in which triangle C is a $(3,4,4)$-triangle with $d(\gamma) = 3$ above.

Finally we consider the cases in which A is a $(3,3,4)$-triangle in which v_3 and α are of degree 3. If v_4 is of degree 3, then Lemma 5 applies to v_3 and the result follows. Assume instead that v_4 is of degree at least 4. Note also that the weight contribution of the other two triangles must sum to at least $\frac{4}{3}$. We will consider cases based on triangle B.

(a) If B is a $(3,3,4)$-triangle in which v_4 is of degree 4, then v_6 is of degree at least 4 or Lemma 5 applies to v_5. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If α is returned with c_1, then use Lemma 4 on v. If v_2 is returned with c_1 and fully-flawed, then recolor v_2 with whatever color is not used on its neighbor not colored with c_1. If necessary, recolor α with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_5 is colored with c_1, i.e. is the flaw for v_4, then recolor v_4 with the color not used on v_4', recolor β with c_1 or the color not used on β' as necessary, and use c_1

on v. If β is colored with c_1, then recolor v_4 with the color not used on v_4', recolor v_5 with c_1 or the color not used on v_6 as necessary, and use c_1 on v. If v_4' is colored with c_1, then we consider cases based on triangle C.

Suppose first that C is a triangle in which v_6 is of degree 4 and v_1 is of degree 3 and γ is of degree either 3 or 4. If v_6 is not colored with c_1, then switch the colors used on v_4 and v_5. So assume v_6 is colored with c_1 and fully-flawed. If the flaw is v_1, then recolor v_1 with the color not used on γ, recolor v_2 as needed with c_1 or by switching the colors of v_2 and α. Switch the colors used on v_4 and v_5 and color v with either c_1 or c_2. If the flaw on v_6 is v_6', then switch the colors of v_6 and v_1, switch the colors of v_5 and v_4 and use c_1 on v. The result follows.

Suppose instead that C is a $(3,3,5)$-triangle in which v_6 is of degree 5. If v_6 is not colored with c_1, then switch the colors used on v_4 and v_5. So assume v_6 is colored with c_1 and fully-flawed. If the flaw is v_1, then recolor v_1 with the color not used on γ, recolor v_2 as needed with c_1 or by switching the colors of v_2 and α. Switch the colors used on v_4 and v_5 and color v with either c_1 or c_2. If the flaw on v_6 is γ, then recolor γ with the color not used on γ', recolor v_1 with c_1 and proceed as above. If the flaw on v_6 is off the 6-face, then recolor v_6 with whatever color is not used on its neighbor off the 6-face that is not colored with c_1. Recolor v_1 with c_1 and γ with the color not used on γ' as needed. Switch the colors of v_5 and v_4. Color v with c_1. The coloring has been extended in the needed way and the result follows.

(b) Secondly, consider the case in which B is a $(3,4,4)$-triangle in which v_5 is of degree 3. In this case, C must be a triangle that requires a contribution of at least $\frac{2}{3}$, hence at least one of v_1, v_6, or γ is of degree 3. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If α is returned with c_1, then use Lemma 4 on v. If v_2 is returned with c_1 and fully-flawed, then recolor v_2 with whatever color is not used on its neighbor not colored with c_1. If necessary, recolor α with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If v_4' has c_1, i.e. is the flaw for v_4 and v_6 is of degree 3, then recolor v_4 with the color used on v_5, recolor v_5 with c_1, recolor v_6 as necessary with the color not used on γ and v_1. Use c_1 on v. If v_4' has c_1, i.e. is the flaw for v_4 and v_6 is of degree 4, then v_1 is of degree 3. If v_1 is colored with c_1, recolor v_1 with the color not used on γ, if necessary recolor v_2 with c_1 or switch the colors of v_2 and α, switch the colors of v_5 and v_4 and use either c_1 or c_2 on v, whichever does not induce a flaw. If v_1 is colored with c_3, then v_2' must have c_1, otherwise we can recolor v_2. If γ has c_1, switch the colors of v_1 and γ, or recolor γ with the color not used on γ' and repeat the argument as for v_1 being colored with c_1. If v_6' has c_1, then switch the colors of v_1 and v_6, and those of v_5 and v_4 and use c_1 on v. The result follows.

Suppose instead that v_5 is colored with c_1, i.e. is the flaw for v_4. We consider the cases depending upon triangle C.

If C is a $(3,3,5)$-triangle in which γ is of degree 5, then recolor v_5 with the color not used on β. Recolor v_6 with c_1 or the color not used on γ, or switch the colors on v_6 and v_1 as necessary. Use c_1 on v.

If C is a $(3,3,5)$-triangle in which v_1 is of degree 5, then recolor v_5 with the color not used on β. Recolor v_6 with c_1, or switch the colors of v_6 and γ. If v_1 has c_1, and has a flaw off the 6-face, then recolor v_1 with the color not used on its neighbor off the 6-face not colored with c_1, recolor v_2 as needed with c_1 or by switching the colors of v_2 and α, recolor γ with c_1 if necessary, and finally use c_1 or c_2 on v, whichever does not produce a flaw.

If C is a $(3,4,4)$-triangle in which v_1 and γ are of degree 4, then recolor v_5 with the color not used on β. If v_1 has c_1 and is fully-flawed, switch the colors of v_6 and

v_1, recoloring v_2 as necessary. If v_1 has c_3, then v_2' must have c_1. Switch the colors on v_2 and α, and recolor v_1 with either c_1 or c_2, recolor v_6 with c_1, recolor γ with the color not used on its neighbor off the 6-face not colored with c_1 as necessary. Use c_1 on v and the result follows.

If C is a $(3,4,4)$-triangle in which v_6 and γ are of degree 4, then recolor v_5 with the color not used on β. Recolor v_6 with the color not used on v_6', off the 6-face, recolor v_1 and γ as needed, using the colors not used on their neighbors. Use c_1 on v and the result follows.

If C is a $(3,4,4)$-triangle in which v_1 and v_6 are of degree 4, then this configuration is forbidden.

(c) Finally consider the case that B is a $(3,3,5)$-triangle in which v_4 is of degree 5. In this case, triangle C must require a contribution of at least $\frac{2}{3}$, hence at least one of v_1, v_6, or γ is of degree 3. Due to the restrictions on faces allowed in $\mathcal{F}'_{\mathcal{N}}$, C may only be a $(3,3,5)$-triangle in which either γ or v_1 has degree 5.

(c.1) Assume first that γ has degree 5. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If α is returned with c_1, then use Lemma 4 on v. If v_2 is returned with c_1 and fully-flawed, then recolor v_2 with whatever color is not used on its neighbor not colored with c_1. If necessary, recolor α with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If α' has c_2, then recolor α with c_1 and use c_3 on v. Similarly, one of v_2's neighbors must also have c_1. If v_1 has c_1 and v_2' has c_3, then recolor v_1 with the color not used on its neighbor not colored with c_1, recolor v_2 with c_1, and use c_2 on v. Assume instead that v_2' has c_1. Now consider the location of the flaw on v_4.

Suppose that v_5 has c_1, i.e is the flaw for v_4. If v_1 also has c_1 and is not fully-flawed, then recolor v_6 with c_1, recolor v_5 with the color not used on β and use c_1

on v. If v_1 also has c_1 and is fully-flawed, then recolor v_6 with c_1 by switching the color of v_6 and v_1, recolor v_5 with the color not used on β and use c_1 on v, and switch the colors of v_2 and α as necessary. If v_1 instead has c_3, then v_6 must have c_2 and γ must have c_1, perform a c_2-c_3 switch from α to v_6, recolor v_5 with the color not used on β and use c_1 on v.

Suppose instead that β has c_1, i.e. is the flaw for v_4. Interchange the colors of v_5 and β, recolor β with the color not used on β' if necessary, and proceed as in the previous case.

Finally, suppose that the flaw for v_4 is off the 6-face. Recolor v_4 with the color not used on its neighbor off the 6-face not colored with c_1, recolor v_5 with c_1, following the recoloring procedures above to prevent inducing any flaws on v_5; recolor β as necessary, and use c_1 on v. The result follows.

(c.2) Secondly, assume that v_1 is of degree 5. Let v_3 be v; remove v, color the remainder of the graph, and replace v. If α is returned with c_1, then use Lemma 4 on v. If v_2 is returned with c_1 and fully-flawed, then recolor v_2 with whatever color is not used on its neighbor not colored with c_1. If necessary, recolor α with c_1 and use c_1 on v. Suppose now that v_4 has c_1 and is fully-flawed, and that v_2 has c_2 and α has c_3. If α' has c_2, then recolor α with c_1 and use c_3 on v. Similarly, one of v_2's neighbors must also have c_1.

If v_5 has c_1, i.e. is the flaw for v_4, then recolor v_5 with whatever color is not used on β, recolor v_6 with c_1 or by switching the colors of v_6 and γ. If v_6 and v_1 are both colored with c_1 at the end of this, recolor v_1 with whatever color is not used on the neighbor off the 6-face not colored with c_1. If needed, color γ with c_1 and/or switch the colors of v_2 and α. Use c_1 on v. Similarly, if β has c_1, then switch the colors of β and v_5, recolor β with the color not used on β' as necessary, and proceed exactly as before.

If the flaw for v_4 is instead on a vertex off the 6-face, then recolor v_4 with whatever color is not used on its neighbor off the 6-face not colored with c_1. If β now matches v_4, either recolor β with c_1 and use c_1 on v and end the algorithm, or switch the colors of v_5 and β. If v_5 is the same color as v_4, then recolor v_5 with c_1. If necessary, recolor v_6 with the color not used on its neighbors. Finally, use c_1 on v and the result follows.

Hence we have extended a coloring for each of the configurations. Since the graph G in $\mathcal{F}'_{\mathcal{N}}$ must contain at least one of these configurations, it is $(1,0,0)^{(3)}$-colorable and the result follows by induction.

□

Let $\mathcal{F}''_{\mathcal{N}}$ be the subfamily of $\mathcal{F}_{\mathcal{N}}$ in which the only faces of negative weight contained in a graph $G \in \mathcal{F}''_{\mathcal{N}}$ are those in $\mathcal{S}_{\mathcal{NC}}$, and any pair of these faces are at distance at least 3. We will $(1,0,0)^{(3)}$-color these graphs in the next theorem.

Theorem 5. *Let $G \in \mathcal{F}''_{\mathcal{N}}$. Then G is $(1,0,0)^{(3)}$-colorable.*

Proof. We proceed by induction on $|V(G)|$.

For the base case, observe that every graph on 3 vertices may be $(1,0,0)^{(3)}$-colored.

Let $G \in \mathcal{F}''_{\mathcal{N}}$ and assume that $G \setminus v$ has a $(1,0,0)^{(3)}$-coloring. We split the proof into two cases depending on the minimum degree of G.

Case 1: $\delta(G) \leq 2$

Let v be a vertex of degree 2 or less in G. Remove v. Color $G \setminus v$ using induction. Replace v. Since $d(v) \leq 2$, at most two colors can be used on the neighbors of v in G. Color v with the third color. This coloring does not introduce any flaws and the result follows.

Case 2: $\delta(G) = 3$

Select a degree 3 vertex from each of the faces of negative weight in G; call the set R. Remove all vertices in R and let $G' = G \setminus R$.

Claim: G' is properly 3-colorable.

Note that G' contains none of the faces from Appendix A, hence G' must contain a vertex of degree 2. Use induction to remove that vertex and 3-color the remainder. Since G' contains no faces of negative weight, each graph in the induction process also contains no faces of negative weight and hence must contain a vertex of degree 2. Hence the graph is properly 3-colorable at every stage of induction and hence G' is properly 3-colorable.

Properly 3-color G' and replace the vertices in R. Since the faces of negative weight are at distance at least 3, each vertex in R is at distance at least 3 from another vertex in R. Color each vertex in R with c_1, or the color not used on its neighbors. Any two vertices in R are distance at least 3, hence no vertex in G' can be adjacent to two vertices in R. Thus at most one flaw is introduced at any vertex colored with c_1 and since each member of R has degree 3, the flaw involves a vertex of degree 3 as needed for the coloring. The result follows. □

Taking Theorem 4 and Theorem 5 together allows us to prove Theorem 1.

Theorem 1. *A planar graph containing no 4-cycles, no 5-cycles, no chained triangles, and no pair of the faces listed in Appendix B at distance 2 or less can be* $(1,0,0)^{(3)}$*-colored.*

Proof. Let G be a planar graph containing no 4-cycles, no 5-cycles, no chained triangles, and no pair of the faces in $\mathcal{S}_{\mathcal{NC}}$ at distance 2 or less.

The proof proceeds by induction, which we split into cases depending upon the structure of G.

For the base case, note that any graph on 3 or fewer vertices may be $(1,0,0)^{(}3)$-colored.

Case 1: If G contains a vertex v of degree 2 or less, then remove v, color the remainder, replace v and use the color not used on v's neighbors to color v. If G does not contain a vertex of degree 2 or less, then G contains some member of the set of faces in Appendix A.

Case 2: If G contains only members of $\mathcal{G}_{\mathcal{NC}}$, then use Theorem 4.

Case 3: If G contains only members of $\mathcal{S}_{\mathcal{NC}}$ at distance at least 3 from each other, then use Theorem 5.

Case 4: If G contains both members of $\mathcal{G}_{\mathcal{NC}}$ and members of $\mathcal{S}_{\mathcal{NC}}$ at distance at least 3, then remove a specific vertex v from one of the members of $\mathcal{G}_{\mathcal{NC}}$. The choice of v will be determined from the member of $\mathcal{G}_{\mathcal{NC}}$ and by taking whichever vertex was used in the proof of the coloring of that configuration in Theorem 4.

Now, $(1,0,0)^{(3)}$-color the remainder of the graph using induction. Extend the coloring to v using the appropriate piece of the proof of Theorem 4.

The result follows by induction. $\square$

Based on Theorem 1, we propose the following conjecture, the solution of which would allow for the advancement of the state of Steinberg's Conjecture. We also observe that the family of graphs containing two or more faces from Appendix B at distance 2 or less provides some good structural characteristics for graphs that would provide counterexamples to Steinberg's Conjecture.

Conjecture 2. *Any planar graph G containing no 4-cycles, no 5-cycles, and no chained triangles contains no pair of the faces listed in Appendix B at distance* 2 *or less.*

CHAPTER 3

Coloring Planar Graphs Containing Chained Triangles

This third chapter centers around proving Theorem 2 using a method similar to that utilized for proving Theorem 1. Let $\mathcal{F}$ be the set of planar graphs containing no 4-cycles, no 5-cycles, and no pairs of faces listed in Appendix D at distance 2 or less. We begin by proving several useful lemmas that allow for the extension of a coloring while preserving a particular property. Next, we use a discharging procedure to prove the existence of certain substructures in the graphs in $\mathcal{F}$. Finally, we will show that every graph containing at least one of these substructures can be $(2,0,0)^{(4)}$-colored.

We use the colors c_1, c_2, and c_3. As in Chapter 2, we define a c_2-c_3 switch to be a recoloring scheme that simply swaps the colors c_2 and c_3 in the specified region of the graph. These switches will be performed only when the region specified is surrounded by vertices colored with c_1, so that they have no impact on the rest of the colors used in the graph.

Throughout this chapter we will refer to a set of k chained triangles as a k-*group*. Figure 11 illustrates a 3-group of triangles adjacent to a 6-face. We refer to a triangle that is not chained to any other triangle adjacent to the face considered as an *isolated triangle*.

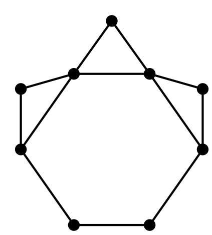

Figure 11. A 3-group of triangles adjacent to a 6-face.

3.1 Coloring Lemmas

In this section we prove several lemmas that allow the extension of a $(2,0,0)^{(4)}$ coloring on a given graph $G \in \mathcal{F}$. Each lemma begins with a $(2,0,0)^{(4)}$-coloring for $G \setminus v$ with a particular vertex v and then extends the coloring to vertex v. These lemmas will be used in the induction argument for the proof of Theorem 2.

Lemma 7. *Let G be a planar graph without 4-cycles or 5-cycles. Let G contain a vertex v with $d(v) \leq 4$ and such that every neighbor of v has degree no more than 4. A $(2,0,0)^{(4)}$-coloring of $G \setminus v$ can be extended to a $(2,0,0)^{(4)}$-coloring of G.*

Proof. Let G' be the graph obtained from G by deleting vertex v. Let u_i, $1 \leq i \leq 4$, be the neighbors of v. Note that each u_i is of degree no more than 3 in G'.

If $d(v) \leq 2$, then the $(2,0,0)^{(4)}$ coloring of G' can easily be extended by coloring v with whatever color was not used on u_1 and u_2.

Suppose instead that $d(v) = 3$ and the neighbors are labeled in such a way that each u_i received color c_i. If u_1 is not fully-flawed, then color v with c_1. If u_1 is fully-flawed, then recolor u_1 with the color not used on its non-c_1 neighbor. Now, v may be colored with c_1.

If $d(v) = 4$, then no more than one of the colors is repeated on the neighbors of v or else we would color v with whatever color is not used. If the repeated color is c_2 or c_3, follow the case above where u_1 was colored with c_1 and fully-flawed. If the repeated color is c_1, then assume that the neighbors are labeled in such a way that u_1 and u_4 have color c_1 and let u_2 and u_3 have colors c_2 and c_3 respectively. If u_1 or u_4 is fully-flawed, then recolor it as in the case with $d(v) = 3$, and use c_1 on v. If they are both not fully-flawed, then color v with c_1. Since no more than two of v's neighbors are colored c_1, v will not have more than 2 flaws. □

Lemma 8. *Let G be a planar graph without 4-cycles or 5-cycles. Suppose G contains a vertex v with $3 \leq d(v) \leq 4$ and neighbors u_1, u_2, u_3, u_4 such that*

$d(u_1) \leq 4$, $d(u_2) \leq 4$, $d(u_3) = 3$, $d(u_4) = 5$, and vu_3u_4 is a triangle. This configuration is illustrated in Figure 12. A $(2,0,0)^{(4)}$-coloring of $G \setminus v$ can be extended to a $(2,0,0)^{(4)}$-coloring of G.

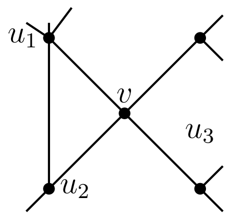

Figure 12. Configuration of v's neighbors in Lemma 5

Proof. Let $G' = G \setminus v$ have a $(2,0,0)^{(4)}$ coloring and replace v. At least three different colors must be used on the neighbors of v; otherwise we color v with whatever color was unused. Suppose first that u_3 is colored with c_1, u_4 is colored with c_2, u_2 has color c_3, and u_1 has any color. Note that u_3 cannot be fully-flawed. If u_1 is colored with c_1 and fully-flawed, recolor u_1 with whatever color is not used on its neighbor not colored with c_1. Color v with c_1 and the result follows.

Next, suppose that u_3 and u_4 are both colored with c_1. In this case u_1 and u_2 must have colors c_2 and c_3. Recolor u_3 with whatever color is not used on its neighbor and use c_1 on v.

Finally, if u_4 is colored with c_1 and is fully-flawed, but u_3 is not colored with c_1, then recolor u_4 with whatever color is not used on the neighbor off the triangle not colored with c_1. Recolor u_3, if necessary, with whatever color is not used on its neighbor. Recolor u_1 or u_2 if they are fully-flawed. This allows us to color v with c_1, giving the needed coloring of G. □

Lemma 9. *Let G be a planar graph without 4-cycles or 5-cycles and suppose G has a vertex v of degree 4 that is adjacent to two triangles, u_1u_2v and u_3u_4v with*

$d(u_1) = d(u_3) = 5$ *and* $d(u_2) = d(u_4) = 3$. *A* $(2,0,0)^{(4)}$*-coloring of* $G \setminus v$ *can be extended to a* $(2,0,0)^{(4)}$*-coloring of* G.

Proof. Let $G' = G \setminus v$ have a $(2,0,0)^{(4)}$ coloring and replace v. If one of u_1 or u_3 is colored with c_2 or c_3, say u_3, then follow the recoloring argument from Lemma 8 treating u_3 as one of the degree 4 neighbors of v in the proof of Lemma 8 since it does not need to be recolored.

Suppose instead that u_1 and u_3 are both colored with c_1. This forces u_2 and u_4 to have colors c_2 and c_3. If u_1 or u_3 is fully-flawed, then recolor them with whatever color is not used on the neighbor off the triangle not colored with c_1. If either u_1 or u_3 now has a color used on u_2 or u_4, then recolor u_2 or u_4 with c_1 or the color not used on their neighbors. Color v with c_1. This gives the needed coloring of G. $\square$

Lemma 10. *If* G *is a planar graph without* 4*-cycles or* 5*-cycles and contains an even face* C *in which all vertices have degree 3 and* v *is a vertex on the face, then a* $(2,0,0)^{(4)}$*-coloring of* $G \setminus v$ *can be extended to a* $(2,0,0)^{(4)}$*-coloring of* G.

Proof. Let v be fixed, let u_1 and u_2 be the neighbors of v on C, and u_3 be the neighbor of v off the face. If either of u_1 or u_2 is colored with c_1 and is fully-flawed, then we may recolor them in such a way that $c(u_1) = c(u_2)$ and color v with c_1.

Suppose instead that u_3 is colored with c_1 and is fully-flawed. Assume without loss of generality that u_1 is colored with c_3 and u_2 is colored with c_2. Since C is an even face and v is not yet colored, there must be some vertex $w \in C$ that is colored with c_1; if there is more than one, then let w be the vertex colored with c_1 that is closest to u_1. Let $u_1 x_1 x_2 \ldots x_l w$, $i = 1, 2, \ldots l$ be the path along C from u_1 to w and let $u_2 y_1 y_2 \ldots y_k w$, $i = 1, 2, \ldots k$ be the path along C from u_2 to w. Let x_i' be the neighbor of x_i off the face and similarly for y_i'. We will consider the case where w is fully-flawed and the case where it is not separately.

Case 1: If w is fully-flawed, then y_k is also colored with c_1. Change w to the color not used on x_l. Now y_k is not fully-flawed and move to Case 2 with y_k as w.

Case 2: Suppose w is not fully-flawed. If x'_l and w do not share a color, then recolor x_l with c_1 and move along the path $wx_l \ldots x_2x_1$ toward u_1, recoloring each x_i with the color not used on its neighbors previous in the path or on x'_i. If x'_l and w do share c_1, then move along the path $wx_l \ldots x_2x_1$ toward u_1, recoloring each x_i with the color not used on its neighbors previous in the path or on x'_i. The final step will change u_1 to either c_1 or c_2 and we use c_3 on v, giving us the needed coloring for G. □

Lemma 11. *Let G be a planar graph without 4-cycles or 5-cycles that contains an even face C in which one vertex z has degree 4 or 5 and all other vertices are of degree 3. Let v be a vertex of degree 3 on C such that both of its neighbors on C have degree 3. A $(2,0,0)^{(4)}$-coloring of $G \backslash v$ can be extended to a $(2,0,0)^{(4)}$-coloring of G.*

Proof. Let u_1 and u_2 be the neighbors of v on C, and let u_3 be the neighbor of v not on the face. If either of u_1 or u_2 is colored with c_1 and is fully-flawed, then recolor it with another color and use c_1 to color v.

Assume instead that u_3 is colored with c_1 and is fully-flawed, and that u_2 is colored with c_2 and u_1 is colored with c_3. Since C is an even face, there is some vertex $w \in C$ that is colored with c_1. If the only vertex on the face colored with c_1 is z, then let z be w. Otherwise, let w be the vertex of degree 3, colored with c_1 and closest to v via a path of vertices of degree 3. We examine each scenario separately.

Case 1: The vertex w is of degree 3. Let $x_1x_2x_3 \ldots x_k$ be the path between v and w along C consisting of vertices of degree 3 and let x'_i be the neighbor of x_i not on C. Here, x_1 is either u_1 or u_2.

If w is fully-flawed, then recolor w, if necessary, to match x'_k, recolor x_k with c_1 or the color not used on x'_k, and proceed along the path $wx_k \ldots x_1 v$ changing the color of each x_i to the color not used on x_{i+1} or x'_i. This will result in changing the color of u_1 to either c_2 or c_1, or the color of u_2 to either c_3 or c_1, as the case may be and allowing us to color v with c_3 or c_2.

If w is not fully-flawed, then recolor x_k with c_1 or the color not used on x'_k and proceed as above, changing the colors along the path $x_k x_{k-1} \ldots x_1$ and using the unused color on v.

Case 2: The vertex w is z and has degree 4. Let $x_1 x_2 x_3 \ldots x_k$ be the path between v and w along C consisting of vertices of degree 3 and let x'_i be the neighbor of x_i not on C. Here we specify $x_1 = u_1$. In this case, no other vertex on C is colored with c_1. If w is not fully-flawed, then we recolor x_k with c_1 or the color not used on x'_k and proceed by changing the colors along the path as in Case 1.

If w is fully-flawed then both neighbors of w off the face must be colored with c_1. Let the neighbor of w on the cycle that is not x_k be y. Note that y and x_k have different colors since it is an even face. Now recolor w with the color of x_k and proceed along the path $x_k \ldots x_1 v$ changing the color of each vertex to that not used previously in the path or on x'_i. This frees color c_3 to be used on v.

Case 3: The vertex w is z and has degree 5. Note that, in this case, w is the only vertex colored with c_1. Let y_l and x_k be the neighbors of w on C and let $Y = y_l y_{l-1} \ldots y_1 u_2$ be the path along C from w to u_2 and $X = x_k x_{k-1} \ldots x_1 u_1$ be the path along C from w to u_1.

If w is not fully-flawed, then recolor x_k with either c_1 or the color used on x_{k-1} and proceed along X changing the color of each x_i to a color not used on x_{i+1} or x'_i.

If w is fully-flawed, then both of its neighbors sharing color c_1 must be off of C. Hence, only one other color may be used on a neighbor of w not on C. Recolor w with the color not used on this neighbor. Move now to x_k or y_l, whichever has the same color as w and recolor it with the unused color. Proceed along the path X or Y, recoloring each vertex on the path. The final step of recoloring will change the color of u_1 or u_2, whichever the case may be, allowing us to use either c_2 or c_3 to color v. $\square$

Lemma 12. *Let G be a planar graph without 4-cycles or 5-cycles that contains a 2-group of triangles involving only one vertex of degree 6 as in Figure 13. A $(2,0,0)^{(4)}$-coloring of $G \setminus v$ can be extended to a $(2,0,0)^{(4)}$-coloring of G.*

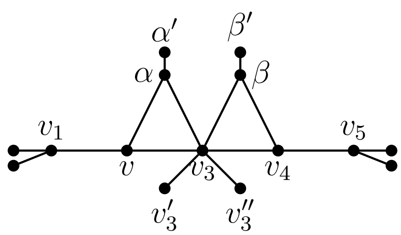

Figure 13. Configuration of the 2-group in Lemma 12.

Proof. Let v be the vertex labeled v in Figure 13; remove v and assume that $G \setminus v$ has a $(2,0,0)^{(4)}$-coloring. Replace v.

Note that each of v_1, v_3, and α must have different colors, otherwise we color v with the color not used on its neighbors and the result follows. Furthermore, the neighbor colored with c_1 must be fully-flawed, otherwise we color v with c_1 and the result follows.

If v_1 is colored with c_1 and fully-flawed, then recolor v_1 with c_2 or c_3 and use c_1 on v. If α is colored with c_1, then it is not fully-flawed and we may use c_1 on v. So assume without loss of generality that v_1 is colored with c_2, α is colored with

c_3, and v_3 is colored with c_1 and is fully-flawed. We consider the location of the flaws of v_3.

If v_4 and β are colored with c_1, i.e. are the flaws for v_3, then recolor v_4 with whatever color is not used on v_5 and use c_1 on v. If v_4 and v_3' are colored with c_1, i.e are the flaws for v_3, then recolor β with either c_1 or the color of v_3'', recolor v_3 with the color not used on v_3'', color v with c_1, and if necessary recolor α with c_1 or c_2. Similarly if β and v_3' are colored with c_1, i.e. are the flaws for v_3, then recolor v_4 with either c_1 or the color used on v_3'', recolor v_3 with the color not used on v_3'', use c_1 on v, and if necessary recolor α with c_1 or c_2.

If v_3' and v_3'' are colored with c_1, i.e. are the flaws for v_3, then recolor v_3 with the color not used on α, recolor v_4 with c_1, recoloring v_5 if needed with a color other than c_1, this is possible since v_5 is of degree 3. Recolor β with the color not used on v_3 if necessary, and finally color v with c_1.

Since only vertices of degree 3 received c_1 during recoloring, no flaws were introduced that involve a vertex of degree 5 or more and the result follows. $\square$

Lemma 13. *Let G be a planar graph without 4-cycles or 5-cycles that contains a 3-group of triangles involving only one vertex of degree 6 as in Figure 14 and with $d(\gamma) \leq 4$. A $(2,0,0)^{(4)}$-coloring of $G \setminus v$ can be extended to a $(2,0,0)^{(4)}$-coloring of G.*

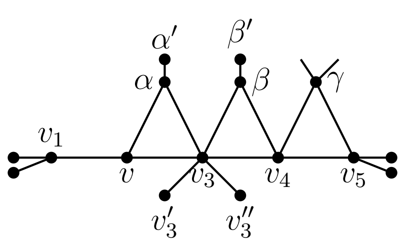

Figure 14. Configuration of the 3-group in Lemma 13.

Proof. Let v be the vertex labeled v in Figure 13; remove v and assume that $G \setminus v$ has a $(2,0,0)^{(4)}$-coloring. Replace v.

Note that each of v_1, v_3, and α must have different colors, otherwise we color v with the color not used on its neighbors and the result follows. Furthermore, the neighbor colored with c_1 must be fully-flawed, otherwise we color v with c_1 and the result follows.

If v_1 is colored with c_1 and fully-flawed, then recolor v_1 with c_2 or c_3 and use c_1 on v. If α is colored with c_1, then it is not fully-flawed and we may use c_1 on v. So assume without loss of generality that v_1 is colored with c_2, α is colored with c_3, and v_3 is colored with c_1 and is fully-flawed. We consider the location of the flaws of v_3.

If v_4 and β are colored with c_1, i.e. are the flaws for v_3, then recolor β with whatever color is not used on β' and use c_1 on v. If v_4 and v_3' are colored with c_1, i.e are the flaws for v_3, then recolor β with either c_1 or the color of v_3'', recolor v_3 with the color not used on v_3'', and color v with c_1. If α and v_3 now have the same color, then recolor α with the color not used on α'.

If β and v_3' are colored with c_1, then recolor v_4 with either the color of v_3'' or c_1, whichever does not cause a flaw. If necessary, recolor v_5 or γ with the color not used on its neighbor not colored with c_1. Finally recolor v_3 with whatever color was not used on v_3'', recolor α with c_1 as needed, and color v with c_1.

If v_3' and v_3'' are both colored with c_1, then recolor v_3 with the color used on v_4, recolor v_4 with c_1. Note that this is always possible since if either of v_5 or γ are colored with c_1 and fully-flawed, they may be recolored with whatever color is not used on the neighbor not colored with c_1. Finally, color v with c_1 and we have obtained the necessary coloring. $\square$

Each of the above lemmas will allow us to extend colorings using an induction argument on some graph G in $\mathcal{F}$.

3.2 Discharging Procedure and Forced Substructures in $G \in \mathcal{F}$

This section centers around proving several structural properties of $G \in \mathcal{F}$ and that any $G \in \mathcal{F}$ with minimum degree 3 contains one of the configuration of faces listed in Appendix C. The proof of this fact relies on a traditional discharging procedure.

Theorem 6. *Let G be a member of $\mathcal{F}$ with minimum degree 3. Then G contains at least one of the faces listed in Appendix C.*

Proof. Assign weights to the vertices and faces of G as follows: weight each vertex v_i with $d(v_i) - 4$ and each face f_i with $|f_i| - 4$, where $|f_i|$ is the number of vertices on the boundary of f_i. The total weight on the graph after assignment is:

$$\begin{aligned}
\sum_{v_i \in V(G)} w(v_i) + \sum_{f_i \in F(G)} w(f_i) &= \sum_{v_i \in V(G)} (d(v_i) - 4) + \sum_{f_i \in F(G)} |f_i| - 4 \\
&= 2e - 4n + 2e - 4f \\
&= -4(n + f - e) \\
&= -8.
\end{aligned}$$

Note that every triangle has weight -1 and every other face has weight at least 2 since there are no 4-faces or 5-faces. Furthermore, only vertices of degree 3 have negative weight. We will proceed to redistribute the weights in such a way as to make every triangle and every vertex have non-negative weight. This will leave only faces of size 6 or more with potentially negative weight.

In the discharging rules, we classify triangles by the degree of each of the vertices. For example, a $(3, 3, 4)$-triangle is one in which two of the vertices are of degree 3 and one is of degree 4. We will also refer to a face based on its position relative to the vertices of an attached triangle. For example, we may refer to "the face across from the degree 4 vertex of a $(3, 3, 4)$-triangle" to mean the face which shares the $3, 3$ edge with the triangle. We also say a *vertex is of degree* $k+$ to indicate that it is of degree k or more.

The weight discharged from a face to a triangle will be referred to as a *contribution.*

With this terminology in place, redistribute the weights according to the following discharging rules:

- Each triangle gives 1 to any vertex of degree 3 on its boundary.
- Each vertex of degree 5 that is part of one or more triangles gives $\frac{6}{12}$ to each triangle, each vertex of degree 6 gives $\frac{8}{12}$ to each triangle, and each vertex of degree $7+$ gives 1 to each triangle.
- Each face of size 6 or more gives $\frac{4}{12}$ to any degree three vertex on its boundary that is not part of a triangle.
- Each face adjacent to a $(3, 3, 3)$-triangle gives $\frac{16}{12}$ to the triangle.
- Each face adjacent to a $(3, 3, 4)$-triangle gives $\frac{12}{12}$ to the triangle.
- Each face adjacent to a $(3, 3, 5)$-triangle gives $\frac{10}{12}$ to the triangle.
- The face across from the vertex of degree 6 in a $(3, 3, 6)$-triangle gives $\frac{10}{12}$ to the triangle while the faces across from the vertices of degree 3 in a $(3, 3, 6)$-triangle give $\frac{9}{12}$ to the triangle.
- Each face adjacent to a $(3, 3, 7+)$-triangle gives $\frac{8}{12}$ to the triangle.
- Each face across from a vertex of degree 4 in a $(3, 4, 4)$-triangle gives $\frac{10}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3, 4, 4)$-triangle gives $\frac{4}{12}$ to the triangle.

- The face across from the vertex of degree 5 in a $(3,4,5)$-triangle gives $\frac{8}{12}$ to the triangle. The face across from the vertex of degree 4 in a $(3,4,5)$-triangle gives $\frac{8}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,4,5)$-triangle gives $\frac{2}{12}$ to the triangle.

- The face across from the vertex of degree 6 in a $(3,4,6)$-triangle gives $\frac{8}{12}$ to the triangle. The face across from the vertex of degree 4 in a $(3,4,6)$-triangle gives $\frac{6}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,4,6)$-triangle gives $\frac{2}{12}$ to the triangle.

- The face across from the vertex of degree 7+ in a $(3,4,7+)$-triangle gives $\frac{6}{12}$ to the triangle. The face across from the vertex of degree 4 in a $(3,4,7+)$-triangle gives $\frac{4}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,4,7+)$-triangle gives $\frac{2}{12}$ to the triangle.

- Each face across from a vertex of degree 5 in a $(3,5,5)$-triangle gives $\frac{6}{12}$ to the triangle. The face sharing the vertices of degree 5 gives nothing to the triangle.

- The face across from the vertex of degree 6 in a $(3,5,6)$-triangle gives $\frac{6}{12}$ to the triangle. The face across from the vertex of degree 5 in a $(3,5,6)$-triangle gives $\frac{4}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,5,6)$-triangle gives nothing to the triangle.

- The face across from the vertex of degree 7+ in a $(3,5,7+)$-triangle gives $\frac{4}{12}$ to the triangle. The face across from the vertex of degree 5 in a $(3,5,7+)$-triangle gives $\frac{2}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,5,7+)$-triangle gives nothing to the triangle.

- Each face across from a vertex of degree 6 in a $(3,6,6)$-triangle gives $\frac{4}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,6,6)$-triangle gives nothing to the triangle.

- The face across from the vertex of degree 7+ in a $(3,6,7+)$-triangle gives $\frac{2}{12}$ to the triangle. The face across from the vertex of degree 6 in a $(3,6,7+)$-triangle gives $\frac{2}{12}$ to the triangle. The face across from the vertex of degree 3 in a $(3,6,7+)$-triangle gives nothing to the triangle.

- Each face adjacent to a $(4,4,4)$-triangle gives $\frac{4}{12}$ to the triangle.

- Each face adjacent to a $(4,4,5)$-triangle gives $\frac{2}{12}$ to the triangle.

- The face across from the vertex of degree 6 in a $(4,4,6)$-triangle gives $\frac{2}{12}$ to the triangle. Each face across from a vertex of degree 4 in a $(4,4,6)$-triangle gives $\frac{1}{12}$ to the triangle.

- Each vertex of degree 5 that is part of only one triangle gives $\frac{3}{12}$ to each adjacent face that is not adjacent to the triangle. Each vertex of degree 5 that is part of no triangles gives $\frac{2}{12}$ to each adjacent face. Each vertex of degree 6 that is part of two or fewer triangles gives $\frac{4}{12}$ to each adjacent face that is not adjacent to the triangles. Each vertex of degree 7 that is part of 3 or fewer triangles gives $\frac{5}{12}$ to each adjacent face. Each vertex of degree k ($k \geq 8$) that is part of $\lfloor \frac{k}{2} \rfloor - 1$ or fewer triangles gives $\frac{6}{12}$ to each adjacent face that is not adjacent to the triangles.

Notice that for each type of triangle, the positive contribution to the triangle from the adjacent faces cancels out the negative weight on the triangle. For example, a $(3,3,3)$-triangle has weight -4 after the first step in the redistribution of weights, and each adjacent face contributes $\frac{16}{12}$, totaling 0.

After redistributing the weights, the weight on each triangle is non-negative and the weight on each vertex of degree 3 not part of a triangle is 0. Hence some of the faces of size 6 or more must have negative weight.

Observe that all faces of size 12 or more will have non-negative weight. Indeed, the weight on any given k-face is $|k| - 4$. Note that to discharge the maximum weight from a k-face, it must be adjacent to the maximum number of $(3,3,3)$-triangles, or the maximum number of 2-groups requiring a contribution of $\frac{24}{12}$ as each of these circumstances can be thought of requiring $\frac{8}{12}$ from each vertex. Hence the maximum weight discharged from a k-face is $\frac{k}{2}(\frac{16}{12}) = \frac{2k}{3}$ if k is even and $\frac{k-3}{2}(\frac{16}{12}) + \frac{24}{12} = \frac{2k}{3}$ if k is odd. Observe that $k - 4 \geq \frac{2k}{3}$ whenever $k \geq 12$; hence any face of size 12 or more has weight at least 0 after discharging.

Thus all faces of negative weight must be of size 11 or less. Appendix C lists all of the faces of size 11 or less that have negative weight. Since the initial weight on G was negative, G must contain at least one of these faces and the result follows. □

Theorem 6 gives us a list of possible structural characteristics of a given graph $G \in \mathcal{F}$. These guaranteed faces allow us to apply an induction argument to coloring G.

3.3 Coloring members of $\mathcal{F}$

With a number of coloring lemmas and substantial knowledge of the structures that must be in any graph $G \in \mathcal{F}$ we now proceed to show that members of this family are $(2,0,0)^{(4)}$-colorable. The proof will proceed by induction on the vertex set of G. During the inductive step, the lemmas from Section 3.1 will be frequently invoked to do the bulk of the coloring.

Define the set $\mathcal{S}_\mathcal{C}$ to be the set of faces listed in Appendix B and the set $\mathcal{G}_\mathcal{C}$ to be the set of faces listed in Appendix A minus $\mathcal{S}_\mathcal{C}$.

We begin by coloring a subfamily, $\mathcal{F}'$ consisting of all planar graphs containing no C_4, no C_5, and in which the only faces of negative weight are those in $\mathcal{G}_{\mathcal{C}}$.

Theorem 7. *Let $G \in \mathcal{F}'$. Then there is a $(2,0,0)^{(4)}$-coloring of G.*

Proof. We proceed by induction on $|V(G)|$.

For the base case, observe that every graph on 3 vertices may be $(2,0,0)^{(4)}$-colored.

Let $G \in \mathcal{F}'$ and assume that $G \setminus v$ has a $(2,0,0)^{(4)}$-coloring for any vertex v in G. We split the proof into two cases depending on the minimum degree of G.

Case 1: $\delta(G) \leq 2$

Let v be a vertex of degree 2 or less in G. Remove v. Color $G \setminus v$ using induction. Replace v. Since $d(v) \leq 2$, at most two colors can be used on the neighbors of v in G. Color v with the third color. This coloring does not induce any flaws and the result follows.

Case 2: $\delta(G) = 3$

In this case, G must contain one of the substructures listed in Appendix C but not those listed in Appendix D. These faces will provide us with a choice of v to remove, replace, and color. We further split the proof into subcases reflecting each of the possible faces that G could contain.

Subcase: 11-faces

For the weight on an 11-face to be negative, the contribution required by adjacent triangles and vertices of degree 3 must be at least $\frac{85}{12}$. This may occur in one of two configurations. We deal with each separately.

(i) Adjacent to six triangles: one 2-group and four isolated isolated triangles Since any three of the isolated triangles are $(3,3,3)$-triangles, two of them are next to each other and Lemma 7 applies to a vertex on one of these triangles.

(ii) Adjacent to seven triangles: one isolated triangle and three 2-groups Since the isolated triangle is a $(3,3,3)$-triangle requiring a contribution of $\frac{16}{12}$ contribution, we may use Lemma 7.

Hence we are able to use the coloring lemmas to color any 11-faces with negative weights.

Subcase: 10-faces

For the weight on a 10-face to be negative, the contribution required by adjacent triangles and vertices of degree 3 must be at least $\frac{73}{12}$. This may occur in one of five configurations. We deal with each separately.

(i) Adjacent to seven triangles: one 3-group and two 2-groups Since five or more of the triangles require a contribution of $\frac{12}{12}$, at least one of the 2-groups is composed entirely of $(3,3,4)$-triangles and we may use Lemma 7 with the center of this 2-group.

(ii) Adjacent to six triangles: three 2-groups

This argument goes exactly the same as the one above.

(iii) Adjacent to six triangles: two 2-groups and two isolated triangles Since at least one of the isolated triangles requires a contribution of $\frac{16}{12}$ and a neighbor of that triangle is of degree 4 or less, Lemma 7 applies and the result follows.

(iv) Adjacent to five triangles: one 2-group and three isolated triangles

(a) If all three of the isolated triangles require a contribution of $\frac{16}{12}$ then Lemma 7 can be applied using two isolated triangles that are next to each other.

(b) If the 2-group requires a contribution of $\frac{22}{12}$ or more, then Lemma 7 or Lemma 8 applies to the center of the 2-group and the result follows.

(v) Adjacent to five triangles: all isolated Since three or more of the triangles require a contribution of $\frac{16}{12}$, then we may use Lemma 7 with whichever pair is next to each other and the result follows.

We have shown that the coloring lemmas can be used in all 10-face configurations that have negative weight.

Subcase: 9-faces

A 9-face with negative weight can appear in one of eight configurations, each of which is dealt with separately below.

(i) Adjacent to four triangles: one 2-group and two isolated triangles In order for the weight on the 9-face to be negative, the 2-group requires a contribution totaling $\frac{22}{12}$ or $\frac{24}{12}$. Hence, Lemma 7 applies to the center of the 2-group and the result follows.

(ii) Adjacent to four triangles: each isolated Since three or more of the triangles require a contribution of $\frac{16}{12}$, and we may apply Lemma 7 to a vertex from whichever two are next to each other.

(iii) Adjacent to five triangles: one 3-group and two isolated triangles: Note that one of the isolated triangles is next to the 3-group. Since both of the isolated triangles are $(3, 3, 3)$-triangles and the corner of the 3-group is of degree no more than 4, Lemma 7 applies and the result follows.

(iv) Adjacent to five triangles: two 2-groups and one isolated triangle Since at least one of the 2-groups must require a contribution of $\frac{22}{12}$ or more, Lemma 7 may be applied to the center of that 2-group and the result follows.

(v) Adjacent to five triangles: one 2-group and three isolated triangles This subcase is illustrated in Figure 15. In order for the weight on the 9-face to be negative, at least one of the isolated triangles must be a $(3, 3, 3)$-triangle.

If triangle A is a $(3, 3, 3)$-triangle, then one of $d(v_1) \leq 4$ or $d(v_7) \leq 4$ and Lemma 7 applies. The case in which triangle D is a $(3, 3, 3)$-triangle is symmetric.

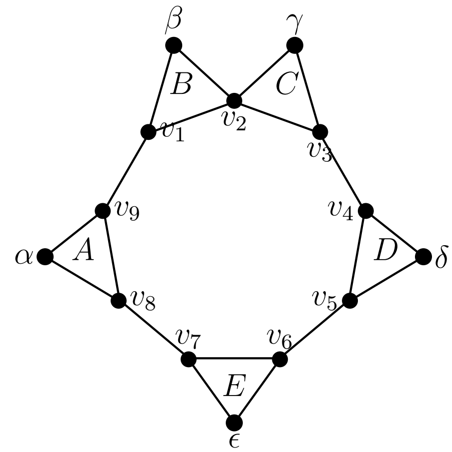

Figure 15. A 9-face adjacent to five triangles, one 2-group and three isolated triangles.

If triangle E is a $(3,3,3)$-triangle, then either $d(v_8) \leq 4$ or $d(v_5) \leq 4$ and Lemma 7 may be used and the result follows.

(vi) Adjacent to six triangles: one 4-group and two isolated triangles Since both isolated triangles are $(3,3,3)$-triangles and are next to each other, Lemma 7 applies and the result follows.

(vii) Adjacent to six triangles: one 3-group, one 2-group, and one isolated triangle If the isolated triangle is a $(3,3,3)$-triangle, then the neighbor of the isolated triangle that is a member of the 2-group has degree no more than 4 and Lemma 7 applies.

Otherwise the 2-group requires a contribution totaling $\frac{22}{12}$ or more and Lemma 7 can be used with the center of the 2-group.

(viii) Adjacent to six triangles: three 2-groups Note that at least one of the 2-groups requires a contribution totaling $\frac{22}{12}$ and Lemma 7 applies to the center of that 2-group.

Hence a coloring lemma may be used on any 9-face with negative weight that appears in the graph.

Subcase: 8-faces

Thirteen configurations of 8-faces and their adjacent triangles result in negative weight on the 8-face. We deal with each configuration separately below.

(i) Adjacent to three triangles: one 2-group and one isolated triangle Since the 2-group requires a contribution of $\frac{22}{12}$ or more, Lemma 7 applies to the center of the 2-group and the result follows.

(ii) Adjacent to three triangles: each isolated Note that since at least two of the triangles are $(3,3,3)$-triangles, either they are next to each other around the 8-face, or one of them has a degree 3 vertex as a neighbor. In both of these situations, Lemma 7 will apply to one of the corners of the triangle.

(iii) Adjacent to four triangles: one 3-group and one isolated Since the isolated triangle is a $(3,3,3)$-triangle and both of the vertices not on triangles are of degree 3, Lemma 7 will apply to one of the corners of the isolated triangle and the result follows.

(iv) Adjacent to four triangles: two 2-groups Since one 2-group requires a contribution of at least $\frac{22}{12}$, Lemma 7 applies to the center of the 2-group and the result follows.

(v) Adjacent to four triangles: one 2-group and two isolated triangles

(a) If the 2-group requires a contribution at least $\frac{22}{12}$, then Lemma 7 applies to the center of the 2-group.

(b) If both of the isolated triangles require a contribution of $\frac{16}{12}$, then Lemma 7 applies to a corner of one of them and the result follows.

(c) If one isolated triangle is a $(3,3,3)$-triangle, then it is neighbors with either the vertex not on a triangle or one of the corners of the 2-group. Lemma 7 applies to a corner of this triangle and the result follows.

(vi) Adjacent to four triangles: all isolated triangles Observe that in each scenario [8.11]-[8.16] in Appendix C, one of the triangles is a $(3, 3, 3)$-triangle which has at least one neighbor of degree no more than 4 and Lemma 7 applies.

(vii) Adjacent to five triangles: a 4-group and one isolated triangle Since the isolated triangle is a $(3, 3, 3)$-triangle and one of its neighbors is of degree 3, Lemma 7 applies to one corner and the result follows.

(viii) Adjacent to five triangles: one 3-group and one 2-group

(a) If the 2-group requires a contribution of $\frac{22}{12}$ or more, then Lemma 7 applies to the center of the 2-group. This corresponds to configuration [8.18].

(b) If instead, one of the ends of the 3-group requires $\frac{12}{12}$, then Lemma 7 will apply to the outside corner of this triangle. This corresponds to configuration [8.19].

(ix) Adjacent to five triangles: one 3-group and two isolated triangles

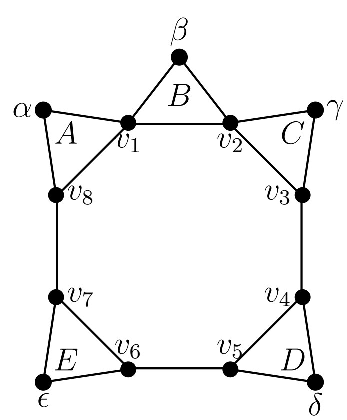

Figure 16. An 8-face adjacent to five triangles.

(a) If one of the isolated triangles requires a contribution of $\frac{16}{12}$, then Lemma 7 will apply to one of its corners. Configuration [8.20]

(b) If triangle D is a $(3, 3, 4)$-triangle, triangle E is a $(3, 3, 4)$-triangle, and either triangle A or C is a $(3, 3, 4)$-triangle, then Lemma 7 applies to either v_8 or

v_3, whichever is degree 3. Configuration [8.21]

(c) If triangle D is a $(3,3,4)$-triangle, triangle E requires a contribution of $\frac{9}{12}$ or $\frac{10}{12}$, and both triangle A and C are a $(3,3,4)$-triangles, then Lemma 7 applies to either v_8 or v_3. Configuration [8.22]

(x) Adjacent to five triangles: two 2-groups and one isolated triangle

(a) If any of the 2-groups require a contribution of $\frac{22}{12}$ or more, then Lemma 7 will apply to the center of that 2-group.

So we may assume that the total contribution from each 2-group is no more than $\frac{20}{12}$.

(b) If the isolated triangle is a $(3,3,3)$-triangle, then Lemma 7 will apply v_1 or v_2.

(c) Similarly, if the isolated triangle requires a contribution of $\frac{12}{12}$, the corner that has degree 3 can be used with Lemma 7, or Lemma 7 may be used with v_5 or v_6, depending on the degrees of v_3 and v_8.

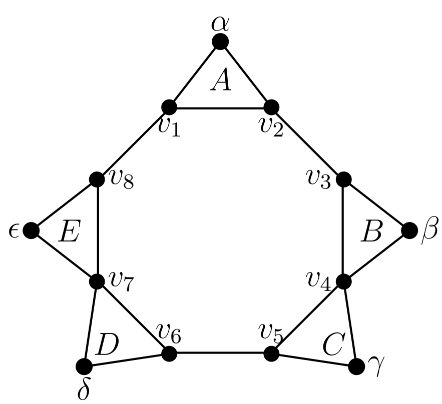

Figure 17. An 8-face adjacent to five triangles.

(d) If every triangle requires a contribution of exactly $\frac{10}{12}$, then these triangles A and B are $(3,3,5)$- or $(3,4,4)$-triangles. If v_4 is of degree 4, Lemma 7 applies to it. If v_4 is of degree 5, Lemma 8 applies to v_5.

(xi) Adjacent to six triangles: a 5-group and an isolated triangle If the isolated triangle requires a contribution of $\frac{16}{12}$, then Lemma 7 applies to one of its corners.

(xii) Adjacent to six triangles: a 4-group and a 2-group

(a) If the 2-group requires a contribution totaling $\frac{22}{12}$ or more, Lemma 7 applies to the center of the 2-group.

(b)If one of the ends of the 4-group is a $(3, 3, 4)$-traingle, then Lemma 7 applies to a corner of this triangle.

(xiii) Adjacent to six triangles: two 3-groups Since one of the triangles at the end of the 3-groups is a $(3, 3, 4)$-triangle and the neighbor of this triangle on the 8-face is of degree no more than 4, Lemma 7 applies and the result follows.

Hence we have a coloring for every 8-face that resulted in negative weight.

Subcase: 7-faces

For a 7-face to have negative weight after the redistribution it must be adjacent to triangles requiring contributions of at least $\frac{37}{12}$. There are several ways this could occur, we deal with each situation below.

(i) Adjacent to two triangles: one 2-group Since the 2-group requires a contribution of $\frac{22}{12}$ or $\frac{24}{12}$, Lemma 7 applies to the vertex at the center of the 2-group.

(ii) Adjacent to two triangles: next to each other In each configuration that results in negative weight on the 7-face, triangle A in Figure 18 is a $(3, 3, 3)$-triangle and one of v_1 or v_4 is of degree no more than 4. Hence Lemma 7 applies and the result follows.

(iii) Adjacent to two triangles: on opposite sides Just as in the case above, one of the triangles is a $(3, 3, 3)$-triangle with a neighbor on the 7-face of degree 4 or less and then Lemma 7 applies.

(iv) Adjacent to three triangles: one 3-group Since one of the triangles

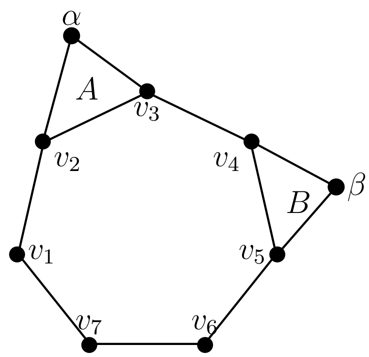

Figure 18. A 7-face adjacent to two triangles, next to each other.

is a $(3,3,4)$-triangle and it has a neighbor on the 7-face that is of degree 3, Lemma 7 applies to the corner of this triangle.

(v) Adjacent to three triangles: one 2-group and one isolated triangle We consider each of the possible configurations [7.9]-[7.12] below.

(a) If the 2-group requires a contribution of $\frac{22}{12}$ or more, then Lemma 7 applies to the center of the 2-group and the result follows.

(b) If the isolated triangle is a $(3,3,3)$-triangle and at least one of the vertices not part of any triangles on the 7-face is of degree 3, then Lemma 7 applies, since the $(3,3,3)$-triangle will have a neighbor of degree no more than 4.

(c) If the isolated triangle is $(3,3,5)$- or $(3,4,4)$-triangle, then Lemma 7 or Lemma 8 applies to one corner of this triangle since both neighbors are of degree 3 on the 7-face.

If the isolated triangle is a $(3,3,6)$-triangle, then Lemma 11 applies since the face has no more than one vertex of degree 4 or 5.

(d) If the isolated triangle is a $(3,3,6)$-triangle with the vertex of degree 6 on the 7-face and the 2-group is either a pair of $(3,4,4)$-triangle, a pair of $(3,3,5)$-triangles, or one $(3,4,5)$-triangle and one $(3,3,4)$-triangle then Lemma 7 or Lemma 8 applies to either the center of the 2-group or one of the vertices on the corners in the 2-group.

(e) If the isolated triangle is a $(3, 3, 6)$-triangle with the vertex of degree 6 on the 7-face and the 2-group is composed of a $(3, 3, 4)$-triangle and a $(3, 4, 6)$-triangle, then consider where the isolated triangle is with respect to the 2-group.

If the isolated triangle is not next to the 2-group, then Lemma 7 applies to the $(3, 3, 4)$-triangle in the 2-group and the result follows.

If instead, the isolated triangle is next to the 2-group and the vertex of degree 6 forms the neighbor of the $(3, 3, 4)$-triangle, then let v be v_5 in Figure 19 below. This causes v_4 and α to be of degree 6 to fit the configuration.

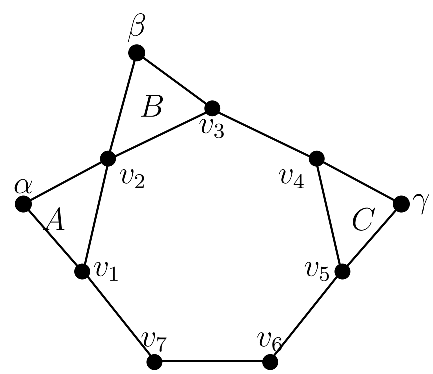

Figure 19. A 7-face adjacent to three triangles as a 2-group and one isolated triangle.

Remove v, $(2, 0, 0)^{(4)}$-color the remainder, and replace v. If either v_6 or γ is colored with c_1 and fully-flawed, then recolor it and use c_1 on v. So assume instead that v_6 has color c_3, γ has color c_2, and v_4 has c_1 and is fully-flawed.

If v_7 has color 1 and is fully-flawed, recolor v_6 with c_1 or c_2, whichever is not used on v_6', recolor v_7 with c_3 as necessary, and use c_3 on v. Hence v_6' must be colored with c_1 and fully-flawed, and v_7 must have color c_2. Applying a similar argument to v_7 forces v_1 to be colored with c_3 and v_7' to be colored with c_1 and fully-flawed.

Now, if v_2 is colored with c_1 and fully-flawed and the flaws are α and another vertex, then recolor v_7 with c_3, v_6 with c_2, and use c_3 on v. If v_2 is colored with c_1

and fully-flawed and the flaws are v_3 and β then recolor v_1 with c_1 or c_2, v_2 with the color not used on α. Recolor v_7 with c_3, v_6 with c_2, and use c_3 on v. Hence we may assume that α is colored with c_1 and is fully-flawed and that v_2 is colored with c_2.

If both β and v_3 are colored with c_1, then perform a c_2-c_3 color switch along v_2, v_1, v_7, v_6 and use c_3 to color v. If v_3 is colored with c_1 and β is colored with c_3, then recolor β with c_1 if β' is colored c_2 and perform a c_2-c_3 color switch along v_2, v_1, v_7, v_6 and use c_3 to color v; if β' has color c_1, perform a c_2-c_3 color switch along $\beta, v_2, v_1, v_7, v_6$ and use c_3 to color v. If β is colored with c_1 and v_3 is colored with c_3 then perform a c_2-c_3 color switch along v_3, v_2, v_1, v_7, v_6 and use c_3 to color v. The result follows.

(f) If the 2-group requires a contribution of $\frac{18}{12}$ or less and the isolated triangle is a $(3, 3, 3)$-triangle, then Lemma 7 applies to one of the corners of the isolated triangle. If the 2-group requires a contribution of $\frac{18}{12}$ or less and the isolated triangle is a $(3, 3, 4)$-triangle, then both of the vertices not on a triangle are of degree 3 and Lemma 7 applies to one corner of the isolated triangle.

(vi) Adjacent to three triangles: each isolated

(a) If all three triangles require a contribution of $\frac{12}{12}$, Lemma 7 will apply to either v_2 or v_7, whichever is of degree 3.

(b) In configurations [7.15]-[7.21], Lemma 7 or Lemma 8 applies to one of the triangles and the result follows. In configuration [7.22], Lemma 8 applies if one of A or C is a $(3, 3, 5)$-triangle and all other configurations are forbidden.

(vii) Adjacent to four triangles: one 4-group Since at least one triangle on the end of the 4-group must require a contribution of $\frac{12}{12}$ and both of the vertices on the 7-face that are not part of triangles are of degree 3, Lemma 7 and the result follows.

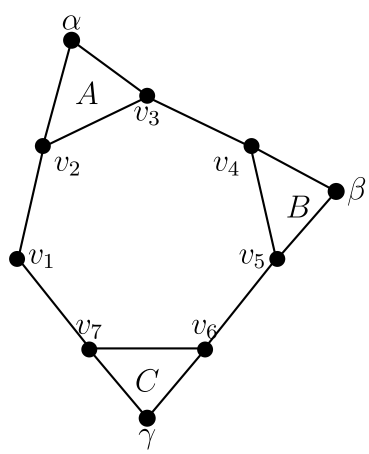

Figure 20. A 7-face adjacent to three isolated triangles.

(viii) Adjacent to four triangles: one 3-group and one isolated This configuration is illustrated in Figure 21 below.

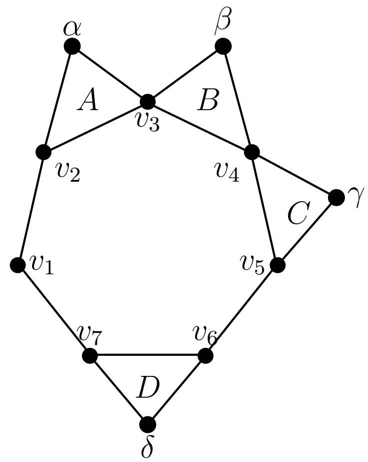

Figure 21. A 7-face adjacent to four triangles as a 3-group and one isolated triangle.

(a) If triangle D is a $(3,3,3)$-, $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle in which $d(v_6) = 3$, then Lemma 7 or Lemma 8 applies to v_6 and the result follows. Similarly, if triangle C is a $(3,3,4)$-, $(3,4,4)$-, or $(3,3,5)$-triangle in which $d(v_5) = 3$ and $d(v_6) \leq 4$, then Lemma 7 or Lemma 8 applies to v_5. This covers configurations [7.24], [7.29], [7.31], and [7.32].

(b) If triangle A is a $(3,3,4)$-, $(3,4,4)$-, or $(3,3,5)$-triangle and $d(v_1) \leq 4$, then Lemma 7 or Lemma 8 applies to v_2 and the result follows. This covers configurations [7.28] and [7.30].

(c) If triangles A and B require a total contribution of $\frac{16}{12}$, then Lemma 7 applies to v_3. Similarly if triangles C and B require a total contribution of $\frac{16}{12}$, then Lemma 7 applies to v_4. This covers configurations [7.25] and [7.26].

Since one of the above configurations must occur in order for the weight on the 7-face to be negative, the result follows.

(ix) Adjacent to four triangles: two 2-groups This configuration is illustrated in Figure 22 below.

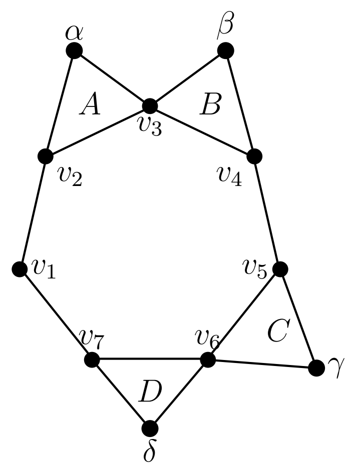

Figure 22. A 7-face adjacent to four triangles as two 2-groups.

(a) If either 2-group requires a contribution of $\frac{22}{12}$ or more, then Lemma 7 applies to the center of the 2-group.

(b) If triangles A and B are both $(3,4,4)$-triangles, or one $(3,4,4)$-triangle and one $(3,4,5)$-triangle, then Lemma 7, Lemma 8, or Lemma 9 applies to the center of the 2-group and the result follows.

(c) If triangle A (or B) is a $(3,3,5)$-triangle, and $d(v_1) \leq 4$ (or $d(v_5) \leq 4$), then Lemma 8 applies and the result follows.

(d) If triangles A and B are $(3,3,6)$-triangles, and $d(v_1) = d(v_5) = 3$, then Lemma 12 and the result follows.

(e) If triangle A (or B) is a $(4,4,4)$-triangle, triangle B (or A) is a $(3,3,4)$-$(3,4,4)$-, or $(3,4,5)$-triangle, then Lemma 7 or Lemma 8 applies to v_2 and the result follows.

(f) If triangle A (or B) is a $(3,5,7+)$-triangle, triangle B (or A) is a $(3,3,5)$-triangle, $d(v_1) = 3$, and $d(v_5) = 3$, then Lemma 8 applies to v_4 (or v_2) and the result follows.

(g) If triangle A is a $(3,4,4)$-triangle $d(v_1) \geq 5$, triangle B is a $(3,4,6)$-triangle, and triangle C is a $(3,3,4)$-, $(3,4,4)$- or $(3,3,5)$-triangle, then Lemma 7 or Lemma 8 applies to v_5.

(h) If triangle A requires a contribution $\frac{8}{12}$ or less, then one of triangles B or C to require a contribution of at least $\frac{10}{12}$ and Lemma 8 or Lemma 7 applies and the result follows.

(x) Adjacent to four triangles, one 2-group and two isolated triangles: This configuration is illustrated in Figure 23.

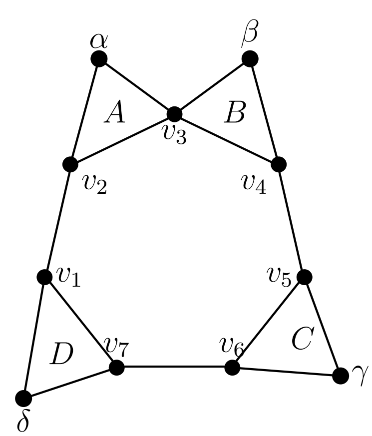

Figure 23. A 7-face adjacent to four triangles, one 2-group, two isolated.

(a) If the two group requires a contribution of more than $\frac{20}{12}$, then Lemma 7 applies to v_3.

(b) If v_1 and v_5 are of degree 3, and the contribution required by the 2-group totals $\frac{18}{12}$ or more, then Lemma 7 applies to one of v_2 or v_3 or v_4, Lemma 8 applies to one of v_2 or v_4, or Lemma 12 applies to v_2. If the contribution required by the 2-group totals $\frac{16}{12}$ or less, then the contribution required by one of triangles C or D is at least $\frac{12}{12}$.

If triangle C is a $(3,3,3)$-triangle and $d(v_7) \leq 4$ or $d(v_4) \leq 4$, then Lemma 7 applies to v_6 or v_5. If triangle C is a $(3,3,3)$-triangle and $d(v_7) \geq 5$, and $d(v_4) \geq 5$, then triangle A requires a contribution of at least $\frac{10}{12}$ and Lemma 7 or Lemma 8 applies to v_2.

If triangle C is the $(3,3,4)$-triangle, and $d(v_4) \leq 5$, then Lemma 7 applies to v_5. If triangle C is a $(3,3,4)$-triangle and $d(v_4) \geq 5$, then triangle A requires a contribution of at least $\frac{10}{12}$ and Lemma 7 or Lemma 8 applies to v_2.

(c) Assume that v_1 is of degree 3 and v_4 is of degree 4. If either of triangle A or B requires a contribution of $\frac{10}{12}$, then Lemma 7 or Lemma 8 applies to either v_2 or v_4 respectively. So the total contribution required by the 2-group is no more than $\frac{18}{12}$.

If triangle D is a $(3,3,3)$-triangle, then Lemma 7 applies unless both $d(v_2) \geq 5$ and $d(v_6) \geq 5$, in which case the weight on the face is non-negative.

If triangle D is a $(3,3,4)$-triangle, then Lemma 7 applies unless $d(v_2) \geq 5$. This forces triangle B to require a contribution of at least $\frac{10}{12}$ and Lemma 7 or Lemma 8 applies to v_4.

(d) Assume now that v_1 and v_5 have degree 4. If either of triangle A or B requires a contribution of $\frac{10}{12}$, then Lemma 7 or Lemma 8 applies to either v_2 or v_4 respectively. If the total contribution required by the 2-group is no more than $\frac{18}{12}$, then for the weight on the 7-face to be negative triangle C (respectively D) requires a contribution of either $\frac{12}{12}$ or $\frac{10}{12}$ and v_7 (respectively v_6) is of degree 4 or less. Lemma 7 or Lemma 8 now applies to v_6 (respectively v_7) and the result follows.

(e) Suppose that v_1 is of degree at least 5. If triangle C is a $(3,3,3)$-triangle, then Lemma 7 applies to v_6 unless, $d(v_7) \geq 5$ in which case the weight on the 7-face is non-negative. Now, for the weight on the 7-face to be negative, v_7 must be of degree 3.

If triangle C is a $(3,3,4)$-triangle and $d(v_6) = 3$, then Lemma 7 applies to v_6. If triangle C is a $(3,3,4)$-triangle and $d(v_5) = 3$, then Lemma 7 applies to v_5 unless $d(v_4) \geq 5$ in which case the weight on the 7-face is non-negative.

If triangle C is a $(3,4,4)$ or $(3,3,5)$-triangle, then Lemma 7 or Lemma 8 will apply to v_6 or v_5 just as above.

Suppose now that triangle C is a $(3,3,6)$-triangle in which v_5 and v_6 are of degree 3. If triangle D requires a contribution of $\frac{10}{12}$, then Lemma 8 will apply to v_7. If triangle B requires a contribution of $\frac{10}{12}$, then Lemma 8 will apply to v_4. If triangle B requires a contribution of $\frac{9}{12}$, then this configuration is forbidden.

If triangles C and D both require contributions of $\frac{9}{12}$ or less, then the total contribution required by the 2-group must be at least $\frac{19}{12}$. This may happen in two ways, either both triangles require a contribution of $\frac{10}{12}$ or, one triangle requires a contribution of $\frac{12}{12}$ and the other a contribution of $\frac{8}{12}$. If both triangles A and B are $(3,4,4)$-triangles, then Lemma 7 applies to v_3 and the result follows. If either A or B is a $(3,4,4)$-triangle and the other is a $(3,3,5)$-triangle then Lemma 8 applies to v_3 and the result follows.

(xi) Adjacent to five triangles: as a 5-group For the weight on the 7-face to be negative, the vertex not part of any triangle on the 7-face must be of degree 3 and one of the triangles at the ends of the 5-group must require a contribution of at least $\frac{12}{12}$. Lemma 7 applies to outer corner of this triangle and the result follows.

(xii) Adjacent to five triangles: one 4-group and one isolated triangle This configuration is illustrated in Figure 24 below. For the weight on the 7-face to be negative, at least one of v_1 or v_7 must be of degree 3 and both of v_2 and v_6 must be of degree 3.

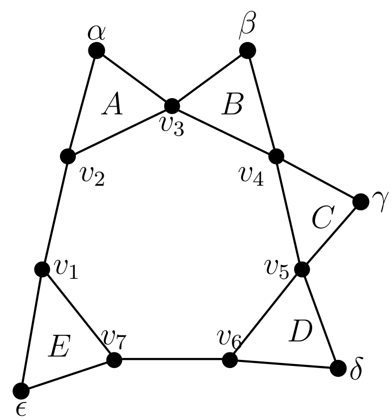

Figure 24. A 7-face adjacent to five triangles as a 4-group and one isolated triangle.

(a) If triangle E is a $(3,3,3)$-triangle or a $(3,3,4)$-triangle, then Lemma 7 applies to either v_1 or v_7, whichever is of degree 3. Assume now that triangle E requires a contribution of no more than $\frac{10}{12}$.

(b) If both triangles A and D require a contribution of $\frac{10}{12}$ or more, then Lemma 7 or Lemma 8 will apply to v_2 or v_6, depending upon which of v_1 or v_7 is of degree 3.

(c) If triangles B and C require a total contribution of $\frac{8}{12}$, then Lemma 7 applies to v_4. Now, in order for the weight on the 7-face to be negative, at least one of triangle B or C must require a contribution of $\frac{4}{12}$. For a negative weight on the 7-face, triangle A (respectively D) must require a contribution of $\frac{12}{12}$ and triangle D (respectively A) must require a contribution of $\frac{9}{12}$. This forces triangle C (respectively B) to require a contribution of no more than $\frac{2}{12}$. Hence Lemma 7 applies to v_3 and the result follows.

(xiii) Adjacent to five triangles: one 3-group and one 2-group This configuration is illustrated in Figure 25. For the weight on the 7-face to be negative, at least one of v_1, or v_6, and one of v_2, or v_5, must be of degree 3 and all four of these vertices must have degree no more than 4.

If either of triangle D or E requires a contribution of $\frac{10}{12}$ or more, then Lemma 7 or Lemma 8 applies to either v_1 or v_6. For the rest of this section, we assume that the 2-group requires a total contribution of no more than $\frac{18}{12}$.

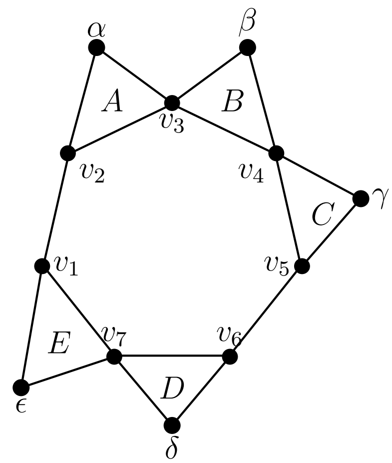

Figure 25. A 7-face adjacent to five triangles as a 3-group and a 2-group.

If triangle A requires a contribution of $\frac{10}{12}$ or more, then Lemma 7 or Lemma 8 applies to v_2 since v_1 has degree no more than 4. Similarly if triangle C requires a contribution of $\frac{10}{12}$ or more, Lemma 7 or Lemma 8 applies to v_5. Hence the maximum contribution of triangles A and C totals no more than $\frac{18}{12}$. Now, in order for the weight on the 7-face to be negative now, both v_2 and v_5 must have degree 3.

If the total contribution required by the 2-group is $\frac{18}{12}$, then each of triangle D and E are $(3, 3, 6)$-triangles with v_7 being of degree 6 and Lemma 12 applies to v_1. Hence the maximum contribution required by the two group now is $\frac{16}{12}$ and the weight on the 7-face is non-negative in this scenario. The result follows.

(xiiii) Adjacent to six triangles: as a 6-group For the weight on the 7-face to be negative, at least one triangle at the end of the 6-group must require a contribution of $\frac{12}{12}$ and the other a contribution of at least $\frac{9}{12}$. This allows us to apply Lemma 7 to the vertex of degree 3 on the 7-face and triangle requiring a $\frac{12}{12}$ contribution and the result follows.

Subcase: 6-faces

For the weight on a 6-face to be negative, the contribution required by the triangles and degree 3 vertices adjacent to it must be at least $\frac{25}{12}$. This may occur in several configurations; we deal with each separately below.

(i) Adjacent to one triangle:

(a) If the triangle requires a contribution of $\frac{16}{12}$, then three or more of the vertices on the 6-face but not on the triangle are of degree 3. Lemma 7 applies to one of the vertices on the triangle and 6-face and the result follows.

(b) If instead, the triangle requires a contribution of $\frac{12}{12}$, then all four of the other vertices on the 6-face are of degree 3 and Lemma 10 or Lemma 11 applies to the 6-face and the result follows.

(ii) Adjacent to two triangles: one 2-group This configuration is illustrated in Figure 26 below.

(a) If the 2-group requires a contribution totaling $\frac{22}{12}$ or more, then Lemma 7 applies to v_3 and the result follows.

(b) If triangle A requires a contribution of $\frac{10}{12}$ or more and $d(v_1) \leq 4$, then Lemma 7 or Lemma 8 applies to v_2 and the result follows. This covers configurations [6.5] and [6.7] in Appendix C.

(c) If triangle A is a $(3,3,4)$-triangle, $d(v_1) = 5$, $d(v_5) = d(v_6) = 3$, and triangle B is a $(3,4,5)$-triangle, then Lemma 8 applies to v_3 and the result follows.

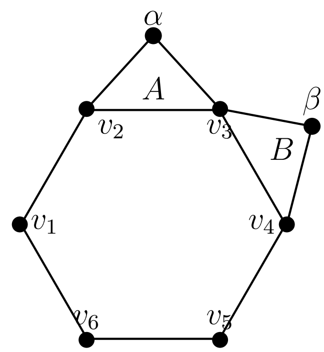

Figure 26. A 6-face adjacent to a 2-group of triangles.

(c.1) If triangle A is a $(3,3,4)$-triangle, $d(v_1) = 5$, $d(v_5) = d(v_6) = 3$, and triangle B is a $(3,4,6)$-triangle, then let v_2 be v, remove v, $(2,0,0)^{(4)}$-color the remainder and replace v. Note that, if α or v_3 is colored with c_1 and fully-flawed, then we may recolor it and use c_1 on v. Assume instead that v_1 is colored with c_1 and fully-flawed, v_3 is colored with c_3 and α is colored with c_2.

If α' is colored with c_3 or with c_1 and not fully-flawed, then recolor α with c_1 and use c_2 on v. If v_4 is colored with c_1 and flawed, then recolor it with either c_3 or the color not used on v_5 or β, recolor v_3 with c_1 and use c_3 on v. If v_4 is colored with c_1 and not flawed, then recolor v_3 with c_1 and use c_3 on v. So assume now that α' is colored with c_1 and fully-flawed, β is colored with c_1 and fully-flawed, and v_4 is colored with c_2. We now consider the locations of the flaws of v_1.

If the flaws of v_1 are v_1' and v_1'', off the 6-face, then recolor v_1 with the color not used on v_1''', its neighbor off the 6-face not colored with c_1. If necessary, recolor v_6 with c_1 or the color not used on v_6', recolor v_5 with c_1 or the color not used on v_5', and perform a c_2-c_3 color switch on v_4, v_3, and α. Use c_1 on v.

If the flaws of v_1 are v_1' and v_6, then recolor v_6 with whatever color is not used on v_6'. If necessary, recolor v_6 with c_1 or the color not used on v_6', recolor v_5 with c_1 or the color not used on v_5', and perform a c_2-c_3 color switch on v_4, v_3, and α. Use c_1 on v. The result follows. This covers configuration [6.6] in Appendix C.

(d) If triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) \geq 5$, triangle B requires a contribution of $\frac{10}{12}$ and $d(v_5) = 3$, then Lemma 7 or Lemma 8 applies to v_4 and the result follows. This covers configuration [6.8] in Appendix C.

(e) If triangles A and B require contributions of $\frac{9}{12}$ and $d(v_1) = d(v_5) = 3$, then Lemma 12 applies to v_2. This covers configuration [6.9] in Appendix C.

(f) In this configuration v_1, v_2, v_4, v_6, α, and β are of degree 3, v_3 is of degree 6, and v_5 is of degree 4.

In this case, let v_2 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph, and replace v. If either of v_1 or α is colored with c_1 and fully-flawed, recolor it with another color and use c_1 to color v. Assume instead that v_1 is colored with c_2, α is colored with c_3, and v_3 is colored with c_1 and fully-flawed. Note that, each neighbor of v_1 and α must have different colors otherwise they may be recolored and the unused color used on v. Furthermore, if v_6 is colored with c_1 and fully-flawed, then recolor v_6 with c_2, v_1 with c_3, and use c_2 on v. If v_6 is colored with c_1 and not fully-flawed, then recolor v_6 with c_1, v_1 with c_3, and use c_2 on v. Hence v_1' must be colored with c_1 and fully-flawed, and v_6 must be colored with c_3. Similarly, if v_5 is colored with c_1 and fully-flawed, recolor v_5 with the color not used on its neighbor not colored with c_1, recolor v_6 with c_1, v_1 with c_3 and use c_2 on v. So assume that v_5 is colored with c_2. We now consider the locations of the flaws of v_3 separately.

If v_4 and β are colored with c_1, recolor v_4 with whatever color is not used on v_5 and use c_1 to color v. If v_4 and one of the neighbors of v_3 off the 6-face (say v_3') are colored with c_1, recolor v_3 with the color not used on v_3'', recolor β and α with c_1 or the color not used on their neighbors as necessary and use c_1 on v. Similarly, if β and v_6' are colored with c_1, recolor v_3 with the color not used on v_3'', recolor α and v_4 with c_1 or the color not used on their neighbors as necessary and use c_1 on v. Finally, if v_3' and v_3'' are colored with c_1, then recolor v_3 with the color used on v_4, recolor v_4 with c_1, recolor α as necessary with c_1 and use c_1 to color v. The needed coloring has been produced. This covers configuration [6.10] in Appendix C.

(g) Suppose that all three of v_1, v_5, and v_6 are of degree 3 and the 2-group requires a contribution of between $\frac{13}{12}$ and $\frac{16}{12}$, each of triangles A and B requiring a contribution of at most $\frac{9}{12}$. If v_3 has degree 4 or 5, then Lemma 11 applies to the 6-face and the result follows.

Assume instead that v_3 is of degree 6, for the weight on the 6-face to be negative, at least one of α or β is of degree 3, assume it is α without loss of generality.

Now let v_2 be v, remove v, $(2, 0, 0)^{(4)}$-color the remainder of the graph, and replace v. If either of v_1 or α is colored with c_1 and fully-flawed, recolor it with another color and use c_1 to color v. Assume instead that v_1 is colored with c_2, α is colored with c_3, and v_3 is colored with c_1 and fully-flawed. Note that, each neighbor of v_1 and α must have different colors otherwise they may be recolored and the unused color used on v. Furthermore, if v_6 is colored with c_1 and fully-flawed, then recolor v_6 with c_2, v_1 with c_1, and use c_2 on v, hence v_1' must be colored with c_1 and fully-flawed, and v_6 must be colored with c_3. Similarly, if v_5 is colored with c_1 and fully-flawed, recolor v_5 with the color not used on its neighbor not colored with c_1, recolor v_6 with c_1, v_1 with c_3 and use c_2 on v. So assume that v_5 is colored with c_2. Repeating this argument with v_4, leads to the conclusion that v_4 must be colored with c_3.

If the flaws of v_3 are β and some v_3', then perform a c_2-c_3 switch along the 6-face, and color v with c_2. If the flaws of v_3 are v_3' and v_3'', then recolor v_3 with the color used on v_4, recolor v_4 with c_1, recolor α with c_1 if needed and use c_1 on v. This covers the allowed members of the configuration family [6.11*] in Appendix

C

(iii) Adjacent to two triangles: next to each other This situation is illustrated in Figure 27.

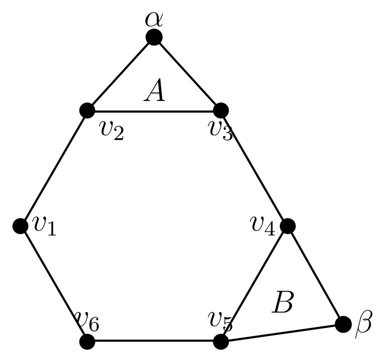

Figure 27. A 6-face adjacent to two triangles, next to each other.

(a) If triangle A requires a contribution of $\frac{16}{12}$ and one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$, then Lemma 7 applies to v_2 or v_3.

(b) If triangle A is a $(3, 3, 3)$-triangle, and $d(v_1) \geq 5$, $d(v_4) \geq 5$ and if triangle B requires a contribution of $\frac{10}{12}$, then Lemma 7 or Lemma 8 applies to v_5.

Assume now that v_1 is of degree 5.

(b.1) If triangle B requires a contribution of $\frac{9}{12}$, then v_4 is of degree 6. Let v_5 be v, remove v, $(2, 0, 0)^{(4)}$-color the remainder of the graph and replace v. If either of v_6 or β are colored with c_1 and fully-flawed, then recolor them and use c_1 to color v. Assume instead that v_6 is colored with c_2, β is colored with c_3, and v_4 is colored with c_1 and fully-flawed. If β' is colored with c_1, then recolor β with c_2 and use c_3 on v, assume instead that β' is colored with c_2. If v_3 is colored with c_1 and fully-flawed, then recolor v_3 with whatever color is not used on its neighbors and color v with c_1. If v_3 is colored with c_1 and not fully-flawed, then swap the colors of v_3 and v_2; if necessary, recolor v_1 with whatever color is not used on its neighbor off the 6-face not colored with c_1 and recolor v_6 with c_1. Finally color v with c_1 or c_2, whichever is available. If instead the flaws of v_4 occur on neighbors of v_4 off the 6-face, then recolor v_4 with whatever color is not used on its neighbor not part of the 6-face or triangle and not colored with c_1. Recolor v_3 and β as necessary with c_1 or another color, and use c_1 to color v.

(b.2) If triangle B requires a contribution of $\frac{8}{12}$ and is a $(3,4,5)$-triangle, then v_4 is of degree 5 and β is of degree 4. Let v_5 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph and replace v. If either of v_6 or β are colored with c_1 and fully-flawed, then recolor them and use c_1 to color v. Assume instead that v_6 is colored with c_2, β is colored with c_3, and v_4 is colored with c_1 and fully-flawed. If v_3 is colored with c_1 and fully-flawed, then recolor v_3 with whatever color is not used on its neighbors and color v with c_1. If v_3 is colored with c_1 and not fully-flawed, then swap the colors of v_3 and v_2; if necessary, recolor v_1 with whatever color is not used on its neighbor off the 6-face not colored with c_1 and recolor v_6 with c_1. Finally color v with c_1 or c_2, whichever is available. If instead the flaws of v_4 occur on neighbors of v_4 off the 6-face, then recolor v_4 with whatever color is not used on β, recolor v_3 as necessary with either c_1 or the color used on β, and use c_1 to color v.

This handles the non-forbidden configurations of the type [6.13*] in Appendix C.

(c) If triangle A is a $(3,3,4)$-triangle with $d(v_2) = 3$, and $d(v_1) \leq 4$, then Lemma 7 applies to v_2. If triangle A is a $(3,3,4)$-triangle with $d(v_3) = 3$, and $d(v_4) \leq 4$, then Lemma 7 applies to v_3. This handles configurations [6.14] and [6.16] in Appendix C

(c.1) If triangle A is a $(3,3,4)$-triangle with $d(v_2) = 3$, $d(v_1) \geq 5$, triangle B is a $(3,3,4)$-triangle, and $d(v_6) = 3$, then Lemma 7 applies to v_4 or v_5 and the result follows. This handles configuration [6.15] in Appendix C

(c.2) We now consider the cases in which triangle A is a $(3,3,4)$-triangle with

$d(v_3) = 3$, and $d(v_4) \geq 5$.

If triangle B requires a contribution of $\frac{10}{12}$, then Lemma 7 or Lemma 8 applies to v_5 and the result follows.

If both v_6 and v_1 are of degree 3 and triangle B requires a contribution of $\frac{9}{12}$, then let v_5 be v, remove v, $(2,0,0)^{(4)}$-color the remainder, and replace v. If either of v_6 or β are colored with c_1 and fully-flawed, then recolor them and use c_1 to color v. Assume instead that v_6 is colored with c_2, β is colored with c_3, and v_4 is colored with c_1 and fully-flawed. If β' is colored with c_1, then recolor β with c_2 and use c_3 on v, assume instead that β is colored with c_2. Furthermore, if v_1 is colored with c_1 and fully-flawed, then recolor it, recolor v_6 with c_1, and use c_2 to color v. Hence v_6' must be colored with c_1 and fully-flawed, and v_1 must be colored with c_3. This argument can be repeated with v_1 instead of v_6 to determine that v_1' must be colored with c_1 and v_2 must be colored with c_2. We now consider the locations of the flaws of v_4.

If v_3 is colored with c_1 and fully-flawed, then recolor it with the color not used on its neighbor not colored with c_1, and use c_1 to color v. If v_3 is colored with c_1 and not fully-flawed, then either recolor α with c_1, v_3 with the color formerly used on α, and use c_1 on v; or, if α' is colored with c_1, perform a c_2-c_3 switch from α to v_6, and use c_2 to color v.

If both flaws of v_4 are v_4' and v_4'', not part of the 6-face or triangle, then recolor v_4 with whatever color is not used on v_4'''. Recolor β and v_3 as necessary with c_1 and use c_1 to color v.

If both v_6 and v_1 are of degree 3 and triangle B is a $(3,4,5)$-triangle, then let v_5 be v, remove v, $(2,0,0)^{(4)}$-color the remainder, and replace v. If either of v_6 or β are colored with c_1 and fully-flawed, then recolor them and use c_1 to color v. Assume instead that v_6 is colored with c_2, β is colored with c_3, and v_4 is colored with c_1 and fully-flawed. Furthermore, if v_1 is colored with c_1 and fully-flawed, then recolor it, recolor v_6 with c_1, and use c_2 to color v. Hence v_6' must be colored with c_1 and fully-flawed, and v_1 must be colored with c_3. This argument can be repeated with v_1 instead of v_6 to determine that v_1' must be colored with c_1 and v_2 must be colored with c_2. We now consider the locations of the flaws of v_4.

If v_3 is colored with c_1 and fully-flawed, then recolor it with the color not used on its neighbor not colored with c_1, and use c_1 to color v. If v_3 is colored with c_1 and not fully-flawed, then either recolor α with c_1, v_3 with the color formerly used on α, and use c_1 on v; or, if α' is colored with c_1, perform a c_2-c_3 switch from α to v_6, and use c_2 to color v.

If both flaws of v_4 are v_4' and v_4'', not part of the 6-face or triangle, then recolor v_4 with whatever color is not used on β. Recolor v_3 as necessary with c_1 and use c_1 to color v. The result follows at this point.

This handles the non-forbidden configurations from [6.17*] in Appendix C.

(d) If triangle A is a $(3, 3, 5)$-triangle in which v_2 is of degree 3, then Lemma 8 applies.

If triangle A is a $(3, 4, 4)$-triangle in which v_2 is of degree 3, then Lemma 7 applies to v_2.

If triangle A is a $(3, 3, 6)$-triangle in which v_2 and v_3 are of degree 3 and no more than one of v_4 or v_5 is of degree 4 or 5 (the other is of degree 3), then Lemma 11 applies to the face.

If triangle A is requires a contribution of $\frac{10}{12}$, $d(v_2) \geq 4$, and $d(v_4) \leq 4$, then Lemma 7 or Lemma 8 applies to v_3.

This handles the non-forbidden configurations from [6.18] and [6.19*] in Appendix C.

(iv) Adjacent to two triangle: on opposite sides This configuration is illustrated in Figure 28. As with the previous 6-face, we will examine the situations that result from triangle A requiring various contributions, and assume that triangle A always requires the maximum contribution of the triangles.

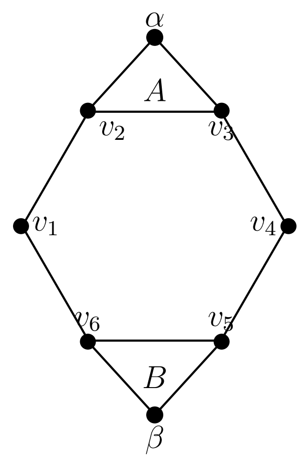

Figure 28. A 6-face adjacent to two triangles, on opposite sides.

(a.1) If triangle A is a$(3,3,3)$-triangle and $d(v_1) \leq 4$ or $d(v_4) \leq 4$, then Lemma 7 applies to v_2 or v_3

(a.2) If triangle A is a$(3,3,3)$-triangle and v_1 and v_4 are both of degree 5, then let v_2 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of G, and replace v. If either α or v_3 is colored with c_1 and fully-flawed, then recolor it and use c_1 on v. So assume instead that v_3 and α have colors c_3 and c_2 respectively. If α' (respectively v_4) is colored with anything other than c_1, then recolor α (respectively v_4) with c_1 and use c_2 (respectively c_3) on v. Instead both α' and v_4 must be colored with c_1 and fully-flawed. We consider the locations of the flaws of v_1.

If v_6 is colored with c_1, and fully-flawed, recolor v_6 with whatever color is not used on its neighbors, and use c_1 to color v. If v_6 is colored with c_1 and not fully-flawed, then note that both flaws of v_4 must be vertices off the 6-face. Recolor v_4 with whatever color is not used on its neighbors off the 6-face, recolor v_3 and v_5 with c_1 as needed and use c_3 on v. If the flaws of v_1 are both vertices off the 6-face, then recolor v_1 with the color not used on its neighbors off the 6-face. Recolor v_6 with c_1 as needed, and use c_1 to color v. The result follows.

This handles the non-forbidden configurations from [6.21] and [6.22*] in Appendix C.

(b.1) If triangle A is a $(3,3,4)$-triangle in which $d(v_2) = 3$, $d(v_1) \leq 4$, then Lemma 7 applies to v_2.

(b.2) If triangle A is a $(3,3,4)$-triangle in which $d(v_2) = 3$, $d(v_1) \geq 5$, $d(v_4) = 3$, and forces triangle B to be a $(3,3,4)$-triangle in which $d(v_5) = 3$, then Lemma 7 applies to v_5.

This handles the non-forbidden configurations from [6.23] and [6.24*] in Appendix C.

(c.1) If triangle A is a $(3,3,5)$-triangle or a $(3,4,4)$-triangle, then Lemma 7 or Lemma 8 applies to one of v_2, or v_3.

(c.2) If triangle A is a $(3,3,6)$-triangle and $d(v_5) \leq 5$, then Lemma 11 or Lemma 10 applies to the 6-face.

This handles the non-forbidden configurations from [6.25*] in Appendix C

(v) Adjacent to three triangles: one a 3-group This configuration is illustrated in Figure 29. We will, as usual, consider the different cases resulting from different contributions required by triangle A, and assume that triangle A requires the largest contribution of the triangles in the 3-group.

(a.1) If triangle A is a $(3,3,4)$-triangle and $d(v_6) \leq 4$, then Lemma 7 applies to v_1.

(a.2) If triangle A is a $(3,3,4)$-triangle and triangle B is a $(3,4,4)$- or $(4,4,4)$-triangle, then Lemma 7 applies to v_2.

(a.3) If triangle A is a $(3,3,4)$-triangle, triangle B requires a contribution of $\frac{2}{12}$ or less, $d(v_6) \geq 5$, $d(v_5) = 3$, and triangle C requires a contribution of $\frac{10}{12}$ or

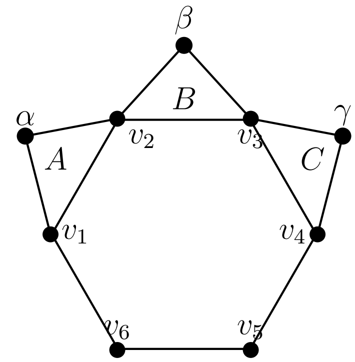

Figure 29. A 6-face adjacent to a 3-group of triangles.

more, then Lemma 7 or Lemma 8 applies to v_4. If instead, triangle C requires a contribution of $\frac{9}{12}$, then triangle B must be a $(3,4,6)$-triangle and Lemma 13 applies to v_4 and the result follows.

This covers configurations [6.27], [6.28], and [6.29] in Appendix C.

(b.1) If triangle A requires a contribution of $\frac{10}{12}$ and $d(v_6) \leq 4$, then Lemma 7 or Lemma 8 applies to v_1.

(b.2) If triangle A requires a contribution of $\frac{10}{12}$ and triangle B is a $(3,4,4)$- or $(4,4,4)$-triangle, then Lemma 7 applies to v_2.

This covers configurations [6.30] and [6.31] in Appendix C.

(c) If triangle A requires a contribution of $\frac{9}{12}$, then triangle B requires a contribution of no more than $\frac{2}{12}$ and both v_5 and v_6 are of degree 3. We consider the options for triangle C.

(c.1) If triangle C is a $(3,3,6)$-triangle in which v_3 is of degree 6, a $(3,4,5)$-triangle in which v_3 is of degree 5, or a $(3,3,7+)$-triangle in which v_3 is of degree 7+, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph, and replace v. If either of α or v_6 is colored with c_1 and fully-flawed, then recolor them with the color not used on their neighbors, and use c_1 to color v. So assume instead that v_2 is colored with c_1 and is fully-flawed, that v_6 is colored with c_3, and that

α is colored with c_2. If α' is also colored with c_1, then recolor α with c_3 and use c_2 to color v. So assume that α' is colored with c_2. If v_5 is colored with c_1 and fully-flawed, then recolor v_5 with c_3, recolor v_6 with c_1 and use c_3 on v. Hence we may assume that v_6' is colored with c_1 and fully-flawed, and v_5 is colored with c_2. Applying a similar argument to v_5 and v_4 allows us to assume that v_4 is colored with c_3, v_5' is colored with c_1 and fully-flawed, γ is colored with c_2 and v_3 is colored with c_1 and fully-flawed. Note that this forces one of the flaws of v_2 to be v_3, which is not possible since every flaw must involve a vertex of degree no more than 4, so one one of the cases above must hold and the result follows.

(c.2) If triangle C requires a contribution of $\frac{8}{12}$ and is a $(3,4,5)$-triangle in which γ has degree 5, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph, and replace v. If either of α or v_6 is colored with c_1 and fully-flawed, then recolor them with the color not used on their neighbors, and use c_1 to color v. So assume instead that v_2 is colored with c_1 and is fully-flawed, that v_6 is colored with c_3, and that α is colored with c_2. If α' is also colored with c_1, then recolor α with c_3 and use c_2 to color v. So assume that α' is colored with c_2. If v_5 is colored with c_1 and fully-flawed, then recolor v_5 with c_3, recolor v_6 with c_2 and use c_3 on v. Hence we may assume that v_6' is colored with c_1 and fully-flawed, and v_5 is colored with c_2. Applying a similar argument to v_5 and v_4 allows us to assume that v_4 is colored with c_3, v_5' is colored with c_1 and fully-flawed, γ is colored with c_1 and fully-flawed, and v_3 is colored with c_2. If β is colored with c_1, then perform a c_2-c_3 color switch along the 6-face from v_3 to v_6, and use c_3 to color v. If instead, β is colored with c_3, then both the flaws of v_2 are vertices off the 6-face. Recolor v_2 with the color not used on β, recolor α with c_1 if necessary. Recolor v_3 with c_1, and v_4 and γ with c_1 or the colors not used on their neighbors as needed. Finally use c_1 to color v and the result follows.

If triangle C requires a contribution of $\frac{6}{12}$, then it is a $(3, 4, 7+)$-triangle and triangle B must require a contribution of $\frac{2}{12}$, forcing β to be of degree 3. Let v_1 be v, remove v, $(2, 0, 0)^{(4)}$-color the remainder of the graph, and replace v. If either of α or v_6 is colored with c_1 and fully-flawed, then recolor them with the color not used on their neighbors, and use c_1 to color v. So assume instead that v_2 is colored with c_1 and is fully-flawed, that v_6 is colored with c_3, and that α is colored with c_2. If α' is also colored with c_1, then recolor α with c_3 and use c_2 to color v. So assume that α' is colored with c_2. If v_5 is colored with c_1 and fully-flawed, then recolor v_5 with c_3, recolor v_4 with c_2 and use c_3 on v. Hence we may assume that v_6' is colored with c_1 and fully-flawed, and v_5 is colored with c_2. Applying a similar argument to v_5 and v_4 allows us to assume that v_4 is colored with c_3, v_5' is colored with c_1 and fully-flawed, γ is colored with c_1 and fully-flawed, and v_3 is colored with c_2. If β is colored with c_1, then perform a c_2-c_3 color switch along the 6-face from v_3 to v_6 and use c_3 to color v. Suppose instead β is colored with c_3. If β' is colored with c_1, then perform a similar c_2-c_3 color switch along the 6-face from β to v_6 and use c_3 on v. Assume now that β' is colored with c_2. Then recolor β with c_1, recolor v_2 with c_3 since both v_2' and v_2'' must be colored with c_1, and use c_1 to color v. The result follows.

This covers all of the non-forbidden configurations from [6.32*] in Appendix C.

(vi) Adjacent to three triangles: one 2-group and one isolated This configuration is illustrated in Figure 30. As noted in Appendix D, many of these faces are forbidden from graphs in $\mathcal{F}'$. For this configuration, we consider the cases generated by both the contribution required by triangle C and the contribution required by the 2-group.

(a) if the contribution required by the 2-group totals $\frac{22}{12}$ or more, then Lemma

7 applies to v_2 and the result follows.

(b) If the contribution required by triangle C is $\frac{16}{12}$ and one of $d(v_6) \leq 4$ or $d(v_3) \leq 4$, then Lemma 7 applies to either v_4 or v_5.

We will assume that the 6-face has at least two vertices of degree 4 or more, otherwise Lemma 11 applies to the 6-face.

(c) If triangle C is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle and either $d(v_4) = 3$ and $d(v_3) \leq 4$, or $d(v_5) = 3$ and $d(v_6) \leq 4$, then Lemma 7 or Lemma 8 applies to v_4 or v_5.

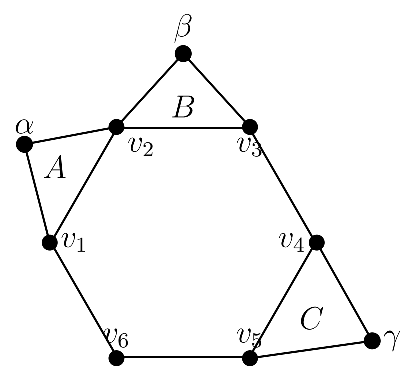

Figure 30. A 6-face adjacent to a 3-group of triangles.

(d) Suppose first that the contribution required by the 2-group is $\frac{20}{12}$. This could occur in several ways; we deal with each separately below.

(d.1) If triangles A and B are both $(3,4,4)$-triangles, then Lemma 7 applies to v_2 and the result follows.

(d.2) If triangle A is a $(3,3,4)$-triangle and triangle B is a $(3,4,5)$-triangle, then Lemma 8 applies to v_2 and the result follows.

(d.3) If triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,4,6)$-triangle and $d(v_6) \leq 4$, then Lemma 7 applies to v_1.

(d.4) If triangle B is a $(3,3,4)$-triangle and triangle A is a $(3,4,5)$-triangle, then Lemma 8 applies to v_2.

(d.5) If triangle B is a $(3,3,4)$-triangle, triangle A is a $(3,4,6)$-triangle, and $d(v_4) \leq 4$, then Lemma 7 applies to v_3.

(d.6) If both triangles A and B are $(3,3,5)$-triangles and one of $d(v_6) \leq 4$ or $d(v_4) \leq 4$, then Lemma 8 applies to either v_1 or v_3.

This covers the non-forbidden configurations from [6.37*] in Appendix C.

(e) Secondly, suppose the contribution required by the 2-group is $\frac{18}{12}$; this may occur in numerous ways.

(e.1) If both triangle A and B are $(3,3,6)$-triangles, $d(v_6) = 3$ and $d(v_4) = 3$, then Lemma 12 applies.

(e.2) If triangle A (respectively B) is a $(3,4,4)$-triangle and triangle B (respectively A) is a $(3,4,5)$-triangle, then Lemma 8 applies to v_2.

(e.3) If triangle A is a $(3,3,4)$-triangle and triangle B is a $(3,4,7+)$-triangle, or if triangle A is a $(3,4,4)$-triangle and triangle B is a $(3,4,6)$-triangle, or if triangle A is a $(3,3,5)$-triangle and triangle B is a $(3,4,5)$-triangle and $d(v_6) \leq 4$, then Lemma 7 or Lemma 8 applies to v_1.

(e.4) If triangle A is a $(3,4,7+)$-triangle and triangle B is a $(3,3,4)$-triangle or if triangle B is a $(3,4,4)$-triangle and triangle A is a $(3,4,6)$-triangle, or if triangle B is a $(3,3,5)$-triangle and triangle A is a $(3,4,5)$-triangle, and $d(v_4) \leq 4$, then Lemma 7 or Lemma 8 applies to v_3.

(f) We now consider the configurations in which the 2-group requires a contribution of $\frac{16}{12}$.

(f.1) If both triangles A and B are $(3,4,5)$-triangles in which v_2 is of degree 4, then Lemma 8 applies to v_2.

(f.2) If triangle A is a a $(3,3,5)$-triangle, triangle B is a $(3,5,5+)$-triangle, and $d(v_6) \leq 4$, then Lemma 8 applies to v_1.

(f.3) If triangle A is a a $(3,3,5)$-triangle, triangle B is a $(3,5,5+)$-triangle,

$d(v_6) \geq 5$, triangle C is a $(3,3,4)$-triangle, with $d(v_4) = 3$, then Lemma 7 applies to v_4.

(g.1) If the 2-group requires a contribution of $\frac{14}{12}$, triangle A is a $(3,3,5)$-triangle, triangle B is a $(3,5,7+)$-triangle, and $d(v_6) \leq 4$, then Lemma 8 applies to v_1.

(g.2) If the 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,3,5)$-triangle, triangle B is a $(3,5,7+)$-triangle, $d(v_6) \geq 5$ and triangle C is a $(3,3,3)$-triangle, then Lemma 7 applies to v_4 and the result follows.

(g.3) If the 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,5,7+)$-triangle, triangle B is a $(3,3,5)$-triangle, $d(v_4) \leq 4$, then Lemma 8 applies to v_3.

(h) If the triangle A is a $(3,4,5)$-triangle and triangle B is a $(4,4,4)$-triangle, then Lemma 8 applies to v_2 and the result follows.

(vii) Adjacent to three triangles: all isolated As with case **(vi)** above, many of the possible configurations are forbidden in $\mathcal{F}'$. This configuration is illustrated in Figure 31. As usual, we split into subcases depending upon the contribution required by triangle A, and assume that triangle A requires the maximum contribution of any of the triangles in the configuration, followed by triangle B.

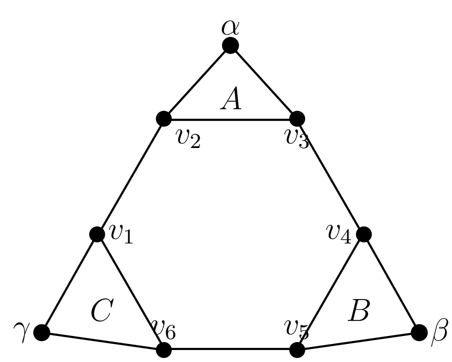

Figure 31. A 6-face adjacent to three triangles.

(a.1) If triangle A is a $(3,3,3)$-triangle and one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$, then Lemma 7 applies to one of v_2 or v_3.

We now consider the case when both $d(v_1) \geq 5$ and $d(v_4) \geq 5$, and break into possibilities for triangle B.

(a.2) If triangle B (symmetrically C) is a $(3,3,5)$-triangle, then Lemma 8 applies to v_5 unless v_6 is also of degree at least 5. In this case, let v_3 be v, remove v, $(2,0,0)^{(4)}$-color the remainder, and replace v. If either v_2 or α is colored with c_1 and fully-flawed, then recolor it and use c_1 on v. Assume instead, that v_2 has color c_2, α has color c_3, and v_4 is colored with c_1 and fully-flawed. Note that, if α' is colored with c_2, then we may recolor α with c_1 and use c_3 to color v. Similarly, if v_1 is colored with c_3, then we may recolor v_2 with c_1 and use c_2 on v. Hence we assume that v_1 and α' are colored with c_1 and fully-flawed. We consider the locations of the flaws on v_4. If v_5 and β are both colored with c_1, i.e. are flaws for v_4, then recolor v_5 with the color not used on v_6, and use c_1 to color v. If v_5 and v_4' are colored with c_1, then recolor v_4 with the color not used on v_4'', recolor β with c_1 or with the color not used on β' as necessary, and use c_1 to color v. If β and v_4' are colored with c_1, proceed exactly as above, replacing β with v_5 and β' with v_6. If v_4' and v_4'' are colored with c_1, i.e. both flaws are off the 6-face, then recolor v_4 with whatever color is used on v_5. Recolor v_5 with c_1. Note that v_6 cannot be colored with c_1 since it is of degree at least 5, and v_1 is colored with c_1 and is of degree 5. Finally color v with c_1 and the result follows.

(b) Suppose now that triangle A requires a contribution of $\frac{12}{12}$. There are two versions of the triangle that require a contribution of $\frac{12}{12}$.

(b.1) If triangle A is a $(3,3,4)$-triangle in which α has degree 4 and one of v_1 and v_4 is degree no more than 4, then Lemma 7 applies to v_1 or v_3.

(b.2) If triangle A is a $(3,3,4)$-triangle in which α has degree 4 and both of v_1 and v_4 are degree at least 5, both $d(v_5) \leq 4$ and $d(v_6) \leq 4$, and one of triangle B or triangle C is a $(3,3,5)$-triangle, then Lemma 8 applies to one of v_5 or v_6.

(b.3) If triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4 and $d(v_4) \leq 4$, then Lemma 7 applies to v_3.

For the case in which triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4, $d(v_4) \geq 5$, we consider now the different contributions that may be required by triangles B and C.

If triangle C is a $(3,3,4)$-, $(3,4,4)$- or $(3,3,5)$-triangle in which either $d(v_6) = 3$ or $d(v_1) = 3$, then Lemma 7 or Lemma 8 applies and the result follows.

If triangle B is a $(3,3,5)$-triangle and $d(v_6) \leq 4$, then Lemma 8 applies to v_5.

If triangle B is a $(3,4,5)$-triangle in which $d(\beta) = 4$, and triangle C is a $(3,3,6+)$-triangle in which $d(v_6) = 3$ and $d(v_1) = 3$, then let v_3 be v, remove v, $(2,0,0)^{(4)}$-color the remainder, and replace v. If α is colored with c_1 and fully-flawed, then color v with the color not used on v_4. If v_2 is colored with c_1 and fully-flawed, then recolor v_2 with the color not used on α and use c_1 on v. So assume that v_4 is colored with c_1 and fully-flawed, and that v_2 has c_2 and α has c_3. Note that α' must be colored with c_1 and fully-flawed, otherwise we may recolor α with c_1 and use c_3 on v. Similarly, at least one of v_1 or v_2' must be colored with c_1. We now consider the locations of the flaws on v_4.

If the flaws are v_5 and β, then recolor v_5 with the color not used on v_6 and use c_1 on v. If the flaws are v_5 and v_4', then recolor v_5 with the color not used on β. If necessary, recolor v_6 with c_1 or (if this is not possible) perform a c_2-c_3 color switch between v_6 and α. Use c_1 on v. If the flaws are β and v_4' and v_6 is colored c_1, then recolor v_5 to match v_4'', recolor v_4 with the color not used on v_4'', and use c_1 on v. If the flaws are β and v_4' and v_6 is not colored with c_1, then recolor v_5 with c_1, v_4 with the color not used on v_4'' and use c_1 on v. If the flaws are v_4' and v_4'', then recolor v_5 with c_1, and v_6 with the color not used on γ if needed. Recolor v_4 with the color formerly used on v_5, and use c_1 on v.

(c) Suppose now that triangle A requires a contribution of $\frac{10}{12}$. Just as before, we consider the different forms that triangle A could take.

(c.1) If triangle A is a $(3,3,5)$-triangle in which $d(\alpha) = 5$, and one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$, then Lemma 8 applies to one of v_2 or v_3.

If triangle A is a $(3,3,5)$-triangle with $d(\alpha) = 5$, both of $d(v_1) \geq 5$ and $d(v_4) \geq 5$, triangle B (or C) forms a $(3,3,5)$-triangle, and $d(v_4) \leq 4$ and $d(v_6) \leq 4$, then Lemma 8 applies and the result follows.

(c.2) If triangle A is a $(3,3,5)$-triangle in which $d(v_2) = 5$ and $d(v_4) \leq 4$, then Lemma 8 applies to v_3.

If instead triangle B forms a $(3,3,5)$-triangle with $d(v_4) = 5$, then Lemma 8 applies since both of $d(v_5) \leq 4$ and $d(v_5) \leq 4$. Similarly, if triangle C forms a $(3,3,5)$-, or $(3,4,4)$-triangle in which v_6 is of degree 3, then Lemma 8 or Lemma 7 applies to v_6 and the result follows.

(c.3) If triangle A is a $(3,4,4)$-triangle in which $d(v_3) = 3$ and $d(v_4) \leq 4$, then Lemma 7 applies to v_3.

If triangle A is a $(3,4,4)$-triangle in which $d(v_3) = 3$, $d(v_4) \geq 5$, and triangle B forms a $(3,3,5)$-triangle, then Lemma 8 applies to v_5 and the result follows. Similarly, if triangle C forms a $(3,3,5)$-, or $(3,4,4)$-triangle in which either v_6 or v_1 is of degree 3, then Lemma 8 or Lemma 7 applies to v_6 or v_1 and the result follows.

(viii) Adjacent to four triangles: as a 4-group The configuration is illustrated in Figure 32.

(a) If triangles B and C require a contribution totaling $\frac{8}{12}$, then Lemma 7 applies to v_3.

(b) We consider the cases in which triangle A is a $(3,3,4)$-triangle.

(b.1) If triangle A requires a contribution of $\frac{12}{12}$ and v_6 is of degree 4 or less,

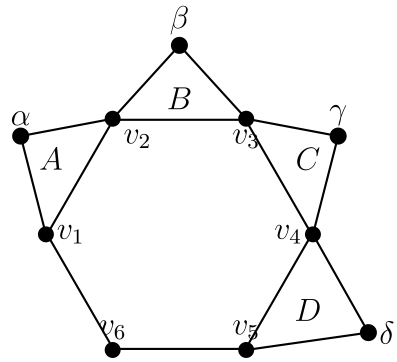

Figure 32. A 6-face adjacent to a 4-group of triangles.

then Lemma 7 applies to v_1.

(b.2) If triangle A requires a contribution of $\frac{12}{12}$, triangle B requires a contribution of $\frac{4}{12}$, then Lemma 7 applies to v_2.

(b.3) If v_6 is of degree 5 and triangle B requires a contribution of $\frac{2}{12}$ or less, then triangle D requires a contribution of at least $\frac{10}{12}$. If triangle D requires a contribution of $\frac{10}{12}$, then triangle C must require a contribution of $\frac{4}{12}$ and Lemma 7 applies to v_4.

If instead, triangle D requires a contribution of $\frac{12}{12}$, then triangle C must require a contribution of $\frac{2}{12}$ or more, and if the contribution required by triangle C is larger than $\frac{2}{12}$, then Lemma 7 applies.

This leaves us with one of the following configurations for which triangle C requires a contribution of $\frac{2}{12}$: either β and γ are of degree 5+ or v_3 is of degree 5+. In both these situations, v_6 is of degree 5, otherwise the weight on the 6-face would be non-negative.

If β and γ are of degree 5+ and one of them is of degree exactly 5, say γ, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph, and replace v. If either of α or v_2 are colored with c_1 and fully-flawed, then recolor them with the color not used on their neighbors, and use c_1 to color v. Assume instead that

v_6 is colored with c_1 and is fully-flawed, and that α has c_3 and v_2 has c_2. Observe that α' must be colored with c_1 and fully-flawed, otherwise we may recolor α and use c_3 on v. Similarly, β must be colored with c_1 and fully-flawed, otherwise we may recolor v_2 and use c_2 on v. Furthermore, v_3 must be colored with c_3, since if it is colored with c_1 and not fully-flawed, we may switch the colors of v_3 and v_4, recoloring v_4, v_5, and β with c_1 or the colors not used on their neighbors as needed. Following the same argument, we can show that γ must be colored with c_1 and fully-flawed, and v_4 must be colored with c_2. We now consider the location of the flaws of v_6. If v_5 is colored with c_1 and δ is colored with c_3, then recolor γ with the color not used on its neighbor off the triangle not colored with c_1, recolor v_3, v_2, and α as necessary by performing a c_2-c_3 color switch, recolor v_4 with c_1, recolor v_5 with c_2 and use c_1 on v. If instead, v_5 is colored with c_1 and δ is also colored with c_1, then recolor v_5 with c_3 and use c_1 to color v. If the flaws of v_6 are both off the 6-face, then recolor v_6 with the color not used on its neighbor off the 6-face not colored with c_1, recolor v_5 with c_1, and use c_1 to color v.

If v_3 is of degree 5, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph, and replace v. If either of α or v_2 are colored with c_1 and fully-flawed, then recolor them with the color not used on their neighbors, and use c_1 to color v. Assume instead that v_6 is colored with c_1 and is fully-flawed, and that α has c_3 and v_2 has c_2. Observe that α' must be colored with c_1 and fully-flawed, otherwise we may recolor α and use c_3 on v. If β is colored with c_1 and fully-flawed, then recolor β with whatever color is not used on its neighbor not colored with c_1, recolor v_2 with c_1 and use c_2 to color v; so assume instead that v_3 is colored with c_1 and fully-flawed. If the flaws of v_3 are γ and v_3', recolor v_3 with the color not used on β, recolor v_4 with c_1 as necessary, recolor v_2 with c_1, and use c_2 on v. If the flaws of v_3 are γ and v_4, then we may either recolor v_4 with the color not used on one of its neighbors, or both v_5 and δ have colors c_2 or c_3. In this second case, then the flaws of v_6 are vertices off the 6-face, and we may recolor v_6 with whatever color is not used on its neighbor off the 6-face not colored with c_1, recolor v_5 and v_4 as needed by switching colors, and use c_1 to color v. Finally, if the flaws of v_3 are v_4 and v_3', and γ is colored with c_2, then note that v_5 must have color c_1, otherwise we recolor v_1 as above. Now recolor v_4 with the color not used on γ, recolor δ as necessary, recolor v_2 with c_1, and use c_2 to color v. The result follows.

If v_3 is of degree 6 or 7, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder of the graph, and replace v. If either of α or v_2 are colored with c_1 and fully-flawed, then recolor them with the color not used on their neighbors, and use c_1 to color v. Assume instead that v_6 is colored with c_1 and is fully-flawed, and that α has c_3 and v_2 has c_2. Observe that α' must be colored with c_1 and fully-flawed, otherwise we may recolor α and use c_3 on v. If β is colored with c_1 and fully-flawed, then recolor β with either c_2 or c_3, recolor v_2 with c_1 and use c_2 to color v; so assume instead that v_3 is colored with c_1 and fully-flawed. We consider the locations of the flaws of v_6. If both flaws of v_6 are vertices off the 6-face, then recolor v_6 with the color not used on its neighbor off the 6-face not colored with c_1, recolor v_5 with c_1, and v_4 or δ as necessary, and use c_1 to color v. If instead, v_5 is colored with c_1, then the flaws of v_3 must both be off the 6-face and triangles, otherwise, we may either recolor v_5 with the color not used on δ, or simply switch the colors of v_5, v_4, and if needed γ. Now recolor v_3 with whatever color is not used on its neighbor off the 6-face not colored with c_1 (if of degree 7, then $d(\beta) = d(\alpha) = 3$), recoloring β or v_2 as necessary, then switch the colors of v_4 and v_5, and use c_1 or c_2 to color v, whichever is unused on its neighbors. The result follows.

(c) If triangle A requires a contribution of $\frac{10}{12}$, then v_6 must be of degree no more than 4 and Lemma 7 or Lemma 8 applies to v_1.

(d) If triangle A requires a contribution of $\frac{9}{12}$, and triangle C requires a contribution of $\frac{4}{4}$, then Lemma 13 applies and the result follows.

(e) If triangle A requires a contribution of $\frac{8}{12}$, then $d(v_6) = 3$ and $d(v_5) = 3$. We will consider the various options for triangle A.

(e.1) If triangle A is a $(3, 4, 5)$-triangle, with $d(v_2) = 4$, then let v_1 be v, remove v, $(2, 0, 0)^{(4)}$-color the remainder, and replace v. If v_6 is colored with c_1 and fully-flawed, then recolor v_6 with any other color and use c_1 on v. Similarly, if v_2 is colored with c_1 and fully-flawed, recolor v_2 with whatever color is not used on α, and use c_1 on v. Assume instead that v_6 is colored with c_3, v_2 is colored with c_2, and α is colored with c_1 and fully-flawed.

Note that, if v_5 is colored with c_1 and fully-flawed, then we may recolor v_5 with the color of v_6', recolor v_6 with c_1 and use c_3 on v. Hence we may assume that v_6' is colored with c_1 and is fully-flawed, and that v_5 has c_2. Similarly, if v_4 is colored with c_1 and fully-flawed, then recolor v_5 with c_1, v_4 with the color not used on its neighbor not colored with c_1, v_6 with c_2 and use c_2 on v. Hence assume that v_4 is colored with c_3 and δ with c_1 and fully-flawed. This forces v_3 to have c_1. If v_3 is flawed with β, recolor v_2 with c_3 and use c_2 on v. If v_3 is flawed with γ, then recolor v_4 with c_2, v_5 with c_3, v_6 with c_2 and use c_3 on v. So assume instead that v_3 has no flaws. In this situation, recolor α with the color not used on its neighbor off the triangle that is not colored c_1, recolor v_2 with c_1 and use either c_1 or c_2 on v. The result follows.

(e.2) If triangle A is a $(3, 4, 6)$-triangle and triangle B is a $(3, 4, 5)$-triangle, follow (e.1) in mirror image.

(e.3) If triangle A is a $(3, 4, 5)$-triangle in which v_2 is of degree 5, then triangle B requires a contribution of $\frac{2}{12}$ or less and triangle C is either a $(3, 4, 4)$- or $(4, 4, 4)$-

triangle. If triangle B is a $(3,4,5)$-triangle, then Lemma 8 applies to v_3. If triangle D is a $(3,4,5)$-triangle, then Lemma 8 applies to v_4. We now consider the degrees of β and γ.

If $d(\beta) = 3$, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder, and replace v. As in (d.1), if v_6 is colored with c_1 and fully-flawed, then recolor v_6 with any other color and use c_1 on v. Similarly, if α is colored with c_1 and fully-flawed, recolor α with whatever color is not used on v_2, and use c_1 on v. Assume instead that v_6 is colored with c_3, α is colored with c_2, and v_2 is colored with c_1 and fully-flawed. Note that, if v_5 is colored with c_1 and fully-flawed, then we may recolor v_5 with the color of v_6', recolor v_6 with c_1 and use c_3 on v. Hence we may assume that v_6' is colored with c_1 and is fully-flawed, and that v_5 has c_2. Similarly, if v_4 is colored with c_1 and fully-flawed, then recolor v_5 with c_1, v_4 with the color not used on its neighbor not colored with c_1, v_6 with c_2 and use c_2 on v. Hence assume that v_4 is colored with c_3 and δ with c_1 and fully-flawed. We consider now the locations of the flaws for v_2.

If the flaws of v_2 are v_3 and β, then recolor β with the color not used on β' and use c_1 on v. If the flaws of v_2 are v_3 and v_2', then recolor v_2 with c_3, recolor β with c_1 or c_2 as needed and use c_1 on v. If the flaws of v_2 are v_2' and β, then recolor v_2 with c_3 and use c_1 on v.

Secondly, if $d(\beta) = 4$ and $d(\gamma) = 3$, then let v_1 be v, remove v, $(2,0,0)^{(4)}$-color the remainder, and replace v. As in (d.1), if v_6 is colored with c_1 and fully-flawed, then recolor v_6 with any other color and use c_1 on v. Similarly, if α is colored with c_1 and fully-flawed, recolor α with whatever color is not used on v_2, and use c_1 on v. Assume instead that v_6 is colored with c_3, α is colored with c_2, and v_2 is colored with c_1 and fully-flawed. Note that, if v_5 is colored with c_1 and fully-flawed, then we may recolor v_5 with the color of v_6', recolor v_6 with c_1 and use c_3 on v. Hence we may assume that v_6' is colored with c_1 and is fully-flawed, and that v_5 has c_2. Similarly, if v_4 is colored with c_1 and fully-flawed, then recolor v_5 with c_1, v_4 with the color not used on its neighbor not colored with c_1, v_6 with c_2 and use c_2 on v. Hence assume that v_4 is colored with c_3 and δ with c_1 and fully-flawed. We consider the locations for the flaws of v_2.

If the flaws of v_2 are v_3 and β, then recolor v_3 with the color used on γ, recolor γ with the color not used on γ' if necessary, and use c_1 on v. If the flaws of v_2 are v_3 and v_2', then assume that β is colored with c_3, otherwise recolor v_2 with c_3 and use c_1 on v. Recolor v_3 with c_2, γ with the color not used on γ', and perform a c_2-c_3 switch on v_4, v_5, and v_6 if necessary. Use c_1 on v. Finally, if the flaws of v_2 are v_2' and β, then v_3 must be colored with c_2 and γ must be colored with c_1. Recolor v_2 with c_3 and use c_1 on v. The result follows.

(e.4) If triangle A is a $(3, 3, 7+)$-triangle and triangle D is a $(3, 4, 5)$-triangle, then Lemma 8 applies to v_4 and the result follows.

(ix) Adjacent to four triangles: one 3-group and one isolated The configuration is illustrated in Figure 33. We will consider options based upon the contribution required by triangle D.

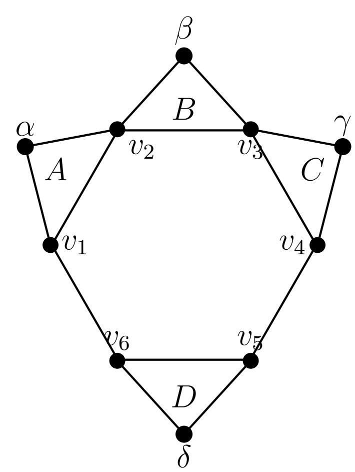

Figure 33. A 6-face adjacent to a 3-group of triangles and one isolated triangle.

(a) If triangle D is a $(3,3,3)$-triangle and one of v_1 or v_4 is of degree 4 or less, then Lemma 7 applies and the result follows.

(b.1) If triangle D is a $(3,3,4)$-triangle in which δ is of degree 4, then Lemma 7 applies and the result follows.

(b.2) Suppose now that triangle D is a $(3,3,4)$-triangle in which δ is of degree 3 and $d(v_6) = 4$. If $d(v_4) \leq 4$, then Lemma 7 applies to v_5.

(b.3) Suppose now that triangle D is a $(3,3,4)$-triangle in which δ is of degree 3 and $d(v_4) \geq 5$. We consider the options based on triangle A.

If triangle A is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle then Lemma 7 or Lemma 8 applies to v_1.

If triangle A is a $(3,4,5)$-triangle, then Lemma 8 applies to v_2 and the result follows.

(c) If triangle D requires a contribution of between $\frac{6}{12}$ and $\frac{10}{12}$, then we consider the weight required by triangle A.

(c.1) If triangle A is a $(3,3,4)$-triangle and $d(v_6) \leq 4$, then Lemma 7 applies to v_1. If triangle A is a $(3,3,4)$-triangle $d(v_6) \leq 4$, D is a $(3,3,5)$-triangle, $d(v_4) \geq 5$, and triangle B requires a contribution of $\frac{4}{12}$, then Lemma 7 applies to v_2.

(c.2) If triangle A is a $(3,4,4)$- or $(3,3,5)$-triangle and $d(v_6) \leq 4$, then Lemma 7 or Lemma 8 applies to v_1. If triangle A is a $(3,4,4)$- or $(3,3,5)$-triangle, $d(v_6) \geq 5$, and $d(v_4) \leq 4$, then Lemma 8 applies to v_5.

(c.3) If triangle A is a $(3,3,6)$-triangle and triangle D is a $(3,3,5)$- or $(3,4,4)$-triangle in which $d(v_6) = 3$, then Lemma 7 or Lemma 8 applies.

If triangle A is a $(3,3,6)$-triangle and triangle D is a $(3,3,5)$- or $(3,4,4)$-triangle in which $d(v_6) \geq 4$ and $d(v_5) = 3$, then Lemma 7 or Lemma 8 applies to v_5 and the result follows.

(c.4) If triangle A requires a contribution of $\frac{8}{12}$ and triangle D is a $(3,3,5)$- or

$(3, 4, 4)$-triangle in which one of $d(v_6) = 3$ or $d(v_5) = 3$, then Lemma 7 or Lemma 8.

(c.5) If triangle A requires a contribution of $\frac{6}{12}$ and triangle D is a $(3, 3, 5)$- or $(3, 4, 4)$-triangle, then Lemma 7 or Lemma 8.

(d) There are multiple configurations in which triangle D requires a contribution of $\frac{4}{12}$; we consider each of them below.

(d.1) If D is a $(3, 4, 4)$- or $(4, 4+, 4+)$-triangle with $d(v_6) = 4$ and triangle A requires a contribution of at least $\frac{10}{12}$, then Lemma 7 or Lemma 8 applies.

(d.2) If D is a $(3, 5, 7+)$-, $(3, 6, 5)$-, $(3, 6, 6)$-, or $(3, 4, 7+)$-triangle in which $d(v_6) = 3$ and triangle A is a $(3, 3, 4)$-, $(3, 4, 4)$-, or $(3, 3, 5)$-triangle, then Lemma 7 or Lemma 8 applies to v_1 and the result follows.

If triangle A is a $(3, 3, 6)$-triangle, $d(\beta) = 3$, and triangle C is a $(3, 3, 4)$-, $(3, 4, 4)$-, or $(4, 4, 4)$-triangle, then Lemma 13 applies and the result follows.

(x) Adjacent to four triangles: two 2-groups This configuration is illustrated in Figure 34.

(a) If either two group requires a contribution of $\frac{22}{12}$ or more, then one of Lemma 7 or Lemma 8 applies.

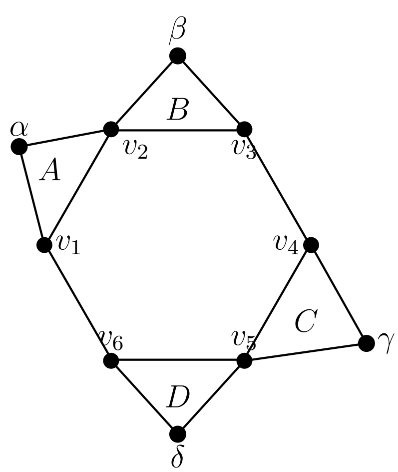

Figure 34. A 6-face adjacent to two 2-groups.

We consider a number of different cases depending on the contribution required by triangle A.

(b.1) If A is a $(3,3,4)$-triangle and $d(v_6) \leq 4$, then Lemma 7 applies to v_1.

(b.2) If A is a $(3,3,4)$-triangle and B is a $(3,4,5)$-, $(3,4,4)$-, or $(4,4,4)$-triangle, then Lemma 7 or Lemma 8 applies to v_2.

(b.3) If triangle A is a $(3,3,4)$-triangle, $d(v_6) \geq 5$, $d(v_3) \leq 4$, and triangle C is a $(3,3,4)$-, $(3,4,4)$-, or $(3,3,5)$-triangle in which $d(v_4) = 3$, Lemma 7 or Lemma 8 applies to v_4 and the result follows.

(c.1) If triangle A is a $(3,4,4)$-triangle in which $d(v_1) = 3$ and $d(v_6) \leq 4$, then Lemma 7 applies to v_1.

(c.2) If triangle A is a $(3,4,4)$-triangle in which $d(v_1) = 3$ and triangle B is a $(3,4,5)$-, $(3,4,4)$-, or $(4,4,4)$-triangle, then Lemma 8 or Lemma 7 applies to v_2.

(c.3) If triangle A is a $(3,4,4)$-triangle in which $d(v_1) = 3$, $d(v_6) \geq 5$, and triangle C is a $(3,3,4)$-, $(3,4,4)$-, or $(3,3,5)$-triangle in which $d(v_4) = 3$, then Lemma 7 or Lemma 8 applies to v_4 and the result follows.

(d.1) If A is a $(3,3,5)$-triangle, and $d(v_6) \leq 4$, then Lemma 8 applies to v_1.

(d.2) If triangles A and B are $(3,3,5)$-triangles, $d(v_6) \geq 5$,, then Lemma 8 applies to v_3, since $d(v)4 \leq 4$.

(d.3) If A is a $(3,3,5)$-triangle, $d(v_6) \geq 4$, $d(v_3) \leq 4$ and triangle C is a $(3,3,4)$-, $(3,4,4)$-, or $(3,3,5)$-triangle in which $d(v_4) = 3$, Lemma 7 or Lemma 8 applies to v_4 and the result follows.

(e) If A and B are $(3,3,6)$-triangles and both of $d(v_4) = 3$ and $d(v_6) = 3$, then Lemma 12 applies.

(f) If A and B are $(3,4,5)$-triangles, then Lemma 9 applies and the result follows.

(xi) Adjacent to five triangles: as a 5-group This configuration is illustrated in Figure 35.

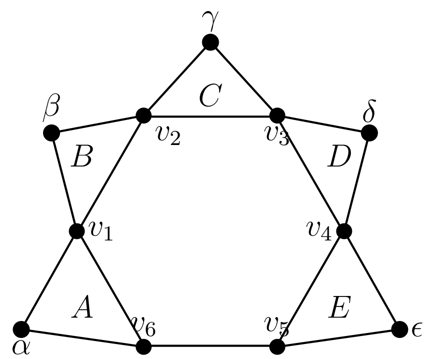

Figure 35. A 6-face adjacent to five triangles as a 5-group.

We will consider the different contributions required by triangle A.

(a) If triangle A requires a contribution of $\frac{12}{12}$ and triangle B requires a contribution of $\frac{4}{12}$, then Lemma 7 applies to either v_1. If triangle A requires a contribution of $\frac{12}{12}$ and v_5 is of degree 3, then Lemma 7 applies v_6 respectively.

(b) If triangle A requires a contribution of $\frac{10}{12}$ and v_5 is of degree 3, then Lemma 7 or Lemma 8 applies to v_6.

(c) If triangles B and C (or C and D) require a contribution totaling $\frac{8}{12}$, then Lemma 7 applies to v_2 (or v_3).

(d.1) If triangle A requires a contribution of $\frac{9}{12}$, triangle B requires a contribution of $\frac{2}{12}$, and triangle C is requires a contribution of $\frac{4}{12}$, then Lemma 13 applies to v_6 and the result follows.

(d.2) If A requires a contribution of $\frac{9}{12}$, and triangle E is a $(3,4,5)$-triangle, and triangle D requires a contribution of $\frac{4}{12}$, then Lemma 8 applies to v_5 and the result follows.

(e) If triangle A requires a contribution of $\frac{8}{12}$ and is a $(3,4,5)$-triangle, and triangle B requires a contribution of $\frac{4}{12}$, then Lemma 8 applies to v_1.

Thus we have extended a $(2,0,0)^{(4)}$-coloring for each of the configurations allowed in a graph $G \in \mathcal{F}'$ and the result follows by induction.

$\square$

Let $\mathcal{F}''$ be the subfamily of $\mathcal{F}$ in which the only faces of negative weight contained in a graph $G \in \mathcal{F}''$ are those in $\mathcal{S}_{\mathcal{C}}$, and any pair of these faces are at distance at least 3. We will $(2,0,0)^{(4)}$-color these graphs in the next theorem.

Theorem 8. *Let $G \in \mathcal{F}''$. Then G is $(2,0,0)^{(4)}$-colorable.*

Proof. We proceed by induction on $|V(G)|$.

For the base case, observe that every graph on 3 vertices may be $(2,0,0)^{(4)}$-colored.

Let $G \in \mathcal{F}''$ and assume that $G \setminus v$ has a $(2,0,0)^{(4)}$-coloring. We split the proof into two cases depending on the minimum degree of G.

Case 1: $\delta(G) \leq 2$

Let v be a vertex of degree 2 or less in G. Remove v. Color $G \setminus v$. Replace v. Since $d(v) \leq 2$, at most two colors can be used on the neighbors of v in G. Color v with the third color. This coloring does not introduce any flaws and the result follows.

Case 2: $\delta(G) = 3$

Select a degree 3 vertex from each of the faces of negative weight in G; call the set R. Remove all vertices in R and let $G' = G \setminus R$.

Claim: G' is properly 3-colorable.

Note that G' contains none of the faces from Appendix C, hence G' must contain a vertex of degree 2. Use induction to remove that vertex and 3-color the remainder. Since G' contains no faces of negative weight, each graph in the induction process also contains no faces of negative weight and hence must contain a vertex of degree 2. Hence the graph is properly 3-colorable at every stage of induction and hence G' is properly 3-colorable.

Properly 3-color G' and replace the vertices in R. Since the faces of negative weight are at distance at least 3, each vertex in R is at distance at least 3 from another vertex in R. Color each vertex in R with c_1, or the color not used on its neighbors. Any two vertices in R are distance at least 3, hence no vertex in G' can be adjacent to two vertices in R. Thus at most one flaw is introduced at any vertex colored with c_1 and since each member of R has degree 3, the flaw involves a vertex of degree 3 as needed for the coloring. The result follows. □

Taking Theorem 7 and Theorem 8 together allows us to prove Theorem 2.

Theorem 2. *A planar graph containing no 4-cycles, no 5-cycles and no pair of the faces listed in Appendix D at distance 2 or less can be* $(2,0,0)^{(4)}$*-colored.*

Proof. Let G be a planar graph containing no 4-cycles, no 5-cycles, and no pair of the faces in $\mathcal{S}_{\mathcal{C}}$ at distance 2 or less.

The proof proceeds by induction, which we split into cases depending upon the structure of G.

For the base case, note that any graph on 3 or fewer vertices may be $(2,0,0)^(4)$-colored.

Case 1: If G contains a vertex v of degree 2 or less, then remove v, color the remainder, replace v and use the color not used on v's neighbors to color v.

If G does not contain a vertex of degree 2 or less, then G contains some member of the set of faces in Appendix A.

Case 2: If G contains only members of $\mathcal{G}_{\mathcal{C}}$, then use Theorem 7.

Case 3: If G contains only members of $\mathcal{S}_{\mathcal{C}}$ at distance at least 3 from each other, then use Theorem 8.

Case 4: If G contains both members of $\mathcal{G}_{\mathcal{C}}$ and members of $\mathcal{S}_{\mathcal{C}}$ at distance at least 3, then remove a specific vertex v from one of the members of $\mathcal{G}_{\mathcal{C}}$. The choice of v will be determined from the member of $\mathcal{G}_{\mathcal{C}}$ and by taking whichever vertex was used in the proof of the coloring of that configuration in Theorem 7.

Now, $(2,0,0)^{(4)}$-color the remainder of the graph using induction. Extend the coloring to v using the appropriate piece of the proof of Theorem 7.

The result follows by induction. □

Based on Theorem 2, we propose the following conjecture, the solution of which would allow for the advancement of the state of Steinberg's Conjecture. We also observe that the family of graphs containing two or more faces from Appendix D at distance 2 or less provides some good structural characteristics for graphs that may provide counterexamples to Steinberg's Conjecture.

Conjecture 3. *Any planar graph G containing no 4-cycles, no 5-cycles, and no chained triangles contains no triple of the faces listed in Appendix D at distance* 2 *or less.*

CHAPTER 4

Earth-Moon Graphs and Colorings

We now step away from strictly planar graphs to consider the colorings of another class of graphs. An *earth-moon graph* consists of two planar graphs on the same set of vertices; here, we will typically draw an earth-moon graph as two graphs, with labeled vertices, as in Figure 36. These earth-moon graphs spring from the idea of having a map of countries on the Earth each with a corresponding colony on the Moon. One natural question is to try to color this map in such a way that each country's territory receives the same color but no adjacent territories receive the same color. Since the colonies may not necessarily be arranged in the same order as their corresponding Earth countries, these types of maps will require more colors than a simple map on only one planet.

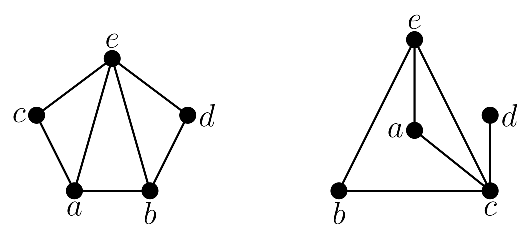

Figure 36. An Earth-Moon Graph on 5 Vertices

The idea of extending a coloring to earth-moon graphs were first introduced by Ringel in 1950 [1]. Hutchinson collated and improved upon many of the early results in [2] and posed a number of new conjectures in the 1990s. This idea was generalized to more than two planar graphs by Beineke and White in [3] where they define a graph of *thickness* k to be a graph consisting of k planar graphs on the same set of vertices. In this notation, an earth-moon graph is a thickness 2 graph. Further problems concerning earth-moon type graphs were posed by Jackson and

Ringel in 2000 [4], but have remained largely unstudied.

In 1890, Heawood [5] proved that any earth-moon graph can be colored using no more than 12 colors without defining earth-moon graphs specifically. Many years later, Gardner provided an example of an earth-moon graph that requires at least 9 colors in his 1980 paper [6]. Other than these results, very little is known about colorings of earth-moon graphs.

In the next section, we initiate the study of Steinberg-type earth-moon graphs. A *Steinberg-type earth-moon graph* is an earth-moon graph in which both the earth graph and the moon graph contain no 4-cycles and no 5-cycles, we will denote this family of graphs by $\mathcal{EM}_{\mathcal{S}}$.

4.1 Coloring Steinberg-type Earth-Moon Graphs

In this section, we prove a few results that allow us to establish some bounds for the number of colors needed to color an earth-moon graph without 4-cycles or 5-cycles.

Theorem 9. *There exists a graph $G \in \mathcal{EM}_{\mathcal{S}}$ which requires at least 5 colors to be properly colored.*

The proof of this goes by constructing a graph that requires at least 5 colors.

Proof. Let G be the graph given in Figure 37 below. When G is considered as one graph, each vertex is connected to every other vertex forming a K_5. This forces $\chi(G) = 5$. Hence such a graph exists and the result follows. □

In [7], Füredi gives the maximum number of edges in any graph on 6 vertices that does not contain a C_4 to be 7.

Note that this shows that there does not exist a Steinberg-type earth-moon graph containing a K_6, since the total number of edges in a K_6 is 15, and either the

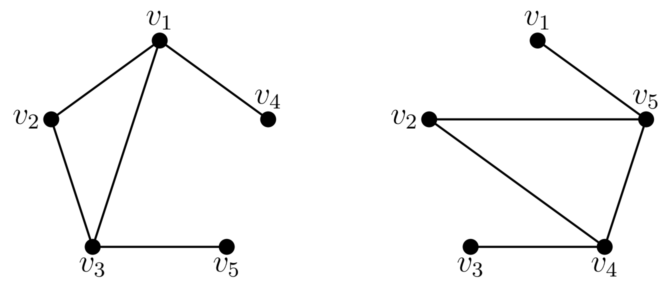

Figure 37. Graph G; a Steinberg-type earth-moon graph that requires 5 colors

earth graph or moon graph must have 8 edges on 6 vertices and therefore contain a C_4.

We now provide an upper bound for the chromatic number of any Steinberg-type earth-moon graph.

Theorem 10. *Let $G \in \mathcal{EM}_{\mathcal{S}}$, then $\chi(G) \leq 8$.*

Proof. Let $G \in \mathcal{EM}_{\mathcal{S}}$ with vertex set V and edge set E and let G_{Earth} represent the planar earth graph and G_{Moon} represent the planar moon graph that compose G. Since G_{Earth} and G_{Moon} are planar, and contain no C_4 or C_5, they contain no adjacent triangles and Lemma 1 applies to each of them. Hence the number of edges in G_{Earth} is at most:

$$|E(G_E)| \leq 2|V(G)| - 4,$$

and the number of edges in G_{Moon} is also no more than $2|V(G)| - 4$.

Since $|E(G)|$ is at most the total of the sizes of the edge sets of G_{Earth} and G_{Moon}, we observe that $|E(G)| \leq 4|V(G)| - 8$. This forces the average degree of G to be less than 8. Thus in any $G \in \mathcal{EM}_{\mathcal{S}}$ there exists some vertex v, with $d(v) \leq 7$.

Now proceed by induction on $|V(G)|$.

For the base case, note that any earth-moon graph on less than 8 vertices is 8-colorable.

Let $G \in \mathcal{EM}_{\mathcal{S}}$, and let v be a vertex of degree no more than 7 in G. Remove v and color the remainder of G using induction. Replace v and use the color not used on v's neighbors to color v. The result follows by induction. $\square$

Taking Theorem 9 and Theorem 10 together gives the following Corollary.

Corollary 2. *We have* $5 \leq \chi(\mathcal{EM}_{\mathcal{S}}) \leq 8$.

It is interesting to notice that Theorem 10 gives strong evidence that Steinberg-type graphs can be colored using strictly fewer colors than graphs in the same families without the Steinberg restrictions. This is seen immediately by observing that the upper bound for the chromatic number of Steinberg-type earth-moon graphs is already lower than the constructed lower bound for earth-moon graphs given by Gardner [6].

4.2 Future Work

The indication that Steinberg-type graphs can be colored using strictly fewer colors than graphs in the same families without the Steinberg restrictions opens up several interesting questions. We pose a few of them here, in hopes that the resolution of these conjectures will provide further insight into Steinberg's Conjecture, and into coloring non-planar graphs.

Because the proof of the upper bound depends only on the average degree of the the graph, we conjecture that a stronger upper bound could be proven using the structure provided by the Steinberg-type conditions.

Conjecture 4. *Every graph* $G \in \mathcal{EM}_{\mathcal{S}}$ *is 7-colorable.*

Conjecture 5. *There exists a graph* $G \in \mathcal{EM}_{\mathcal{S}}$ *with* $\chi(G) = 7$.

Another interesting branch of coloring may be opened by asking the question: what results may be obtained when the earth-moon graph under consideration has no 4-cycles or 5-cycles?

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APPENDIX A

Faces of Negative Weight required to be in any G in $\mathcal{F}_{\mathcal{N}}$ with minimum degree 3

In each section we present the configurations of faces of size k that are required to be in any graph of minimum degree 3 in $\mathcal{F}_{\mathcal{N}}$. The general configuration is given in a figure and each of the vertices are labeled; specific configurations are then listed by giving the degree of each labeled vertex. Families of configurations that may be members of the set required to be at distance at least 3 from each other will be denoted by *, and then detailed in Appendix B.

Note furthermore, that every configuration which has negative weight must have at least one vertex of degree 3. Certainly, if all vertices are of degree 4 or more, then the maximum weight discharged from a k-face is $\frac{k}{3} \leq k-4$ for all $k \geq 6$.

10-Faces

A 10-face has weight $\frac{18}{3}$ before the discharging. In order for the weight to become negative after the redistribution of weights, the 10-face must be adjacent to five triangles.

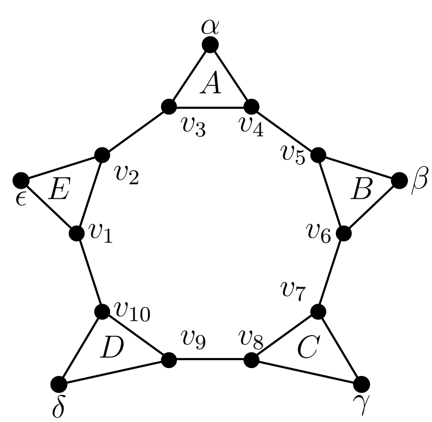

Figure A.1. A 10-Face adjacent to 5 triangles

10.1 In order for the weight on the face to be negative, at least four of the five triangles must be $(3,3,3)$-triangles requiring a contribution of $\frac{4}{3}$ from the 10-face. Suppose they are A, B, C, and D. Furthermore, triangle E must require a contribution of at least $\frac{3}{3}$.

9-Faces

A 9-face has weight $\frac{15}{3}$ before the discharging. For the weight on the face to become negative after the redistribution of weights, the 9-face must be adjacent to four triangles.

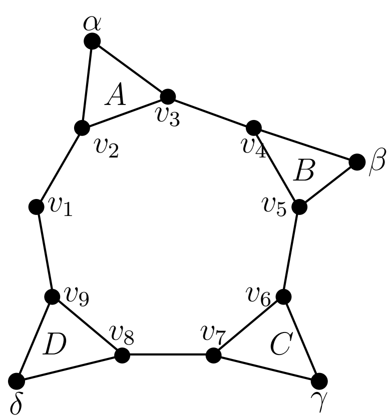

Figure A.2. A 9-Face adjacent to 5 triangles

At least three of these triangles must be $(3,3,3)$-triangles requiring a contribution of $\frac{4}{3}$. These may be either triangles A, B, and C, or triangles A, B, and D. The weight required by v_1 and the third triangle may vary as below.

9.1 Triangles A, B, C, and D are all $(3,3,3)$-triangles and $d(v_1) \geq 3$

9.2 Triangles A, B, and C are $(3,3,3)$-triangles, triangle D is a $(3,3,4)$-triangle, and $d(v_1) = 3$

9.3 Triangles A, B, and D are $(3,3,3)$-triangles, triangle C is a $(3,3,4)$-triangle, and $d(v_1) = 3$

8-faces

An 8-face has weight $\frac{12}{3}$ before the discharging procedure is applied. In order for the 8-face to have negative weight after the redistribution it must be adjacent to three or four triangles. We detail each of these options below.

Adjacent to Three Triangles

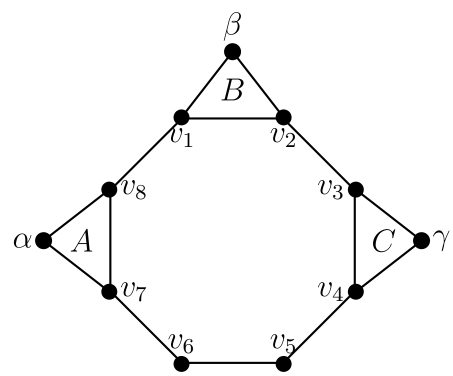

Figure A.3. An 8-face adjacent to 3 triangles.

In order for the 8-face to have negative weight, at least two of the triangles must require a contribution of at least $\frac{4}{3}$ and at least one of v_5 or v_6 must be degree 3.

8.1 Triangles A, B, and C are $(3,3,3)$-triangles and $d(v_5) = 3$, $d(v_6) \geq 3$

8.2 Triangles A and B are $(3,3,3)$-triangles, triangle C is a $(3,3,4)$-triangle and $d(v_5) = d(v_6) = 3$

8.3 Triangles A and C are $(3,3,3)$-triangles, triangle B is a $(3,3,4)$-triangle and $d(v_5) = d(v_6) = 3$

Adjacent to Four Triangles

In order for the weight on the 8-face to be negative after discharging, at least one of the triangles must require a contribution of $\frac{4}{3}$.

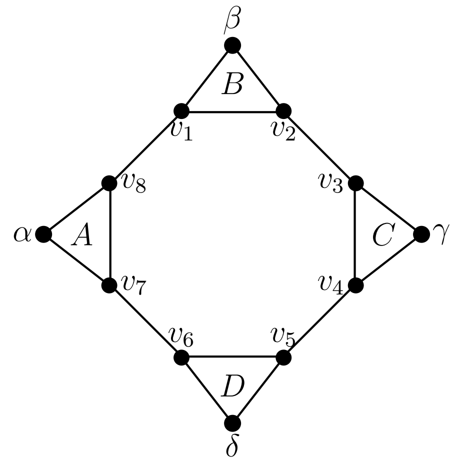

Figure A.4. An 8-face adjacent to four triangles.

8.4 Triangles A, B, C, and D are all $(3,3,3)$-triangles.

8.5 Triangles A, B, and C are $(3,3,3)$-triangles, and triangle D is either a $(3,3,4)$, $(3,3,5)$, $(3,4,4)$, $(3,3,6+)$, $(3,4,5)$, or $(4,4,4)$-triangle.

8.6 Triangles A, B, and D are $(3,3,3)$-triangles, and triangle C is either a $(3,3,4)$, $(3,3,5)$, $(3,4,4)$, $(3,3,6+)$, $(3,4,5)$, or $(4,4,4)$-triangle.

8.7 Triangles A and B are $(3,3,3)$-triangles, and triangles C and D are $(3,3,4)$-triangles.

8.8 Triangles A and B are $(3,3,3)$-triangles, triangle C is a $(3,3,4)$-triangle, and triangle D is a $(3,3,5)$- or $(3,4,4)$-triangle.

8.9 Triangles A and C are $(3,3,3)$-triangles, and triangles B and D are $(3,3,4)$-triangles.

8.10 Triangles A and C are $(3,3,3)$-triangles, triangle B is a $(3,3,4)$-triangle, and triangle D is a $(3,3,5)$- or $(3,4,4)$-triangle.

8.11 Triangle A is a $(3,3,3)$-triangle, and triangles B, C, and D are $(3,3,4)$-triangles.

7-Faces

A 7-face has weight $\frac{9}{3}$ before discharging. In order for it to return with negative weight after the redistribution of weights, the 7-face must be adjacent to either two or three triangles. Each type is listed below.

Adjacent to Two Triangles, Next to Each Other

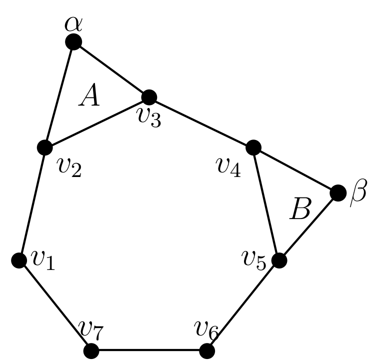

Figure A.5. A 7-face adjacent to two triangles.

In order for the weight on the 7-face to be negative at least one of the triangles must require a contribution of $\frac{4}{3}$

7.1 Triangles A and B are $(3,3,3)$-triangles, $d(v_1) = 3$, $d(v_6) = 3$, and $d(v_7) \geq 3$.

7.2 Triangles A and B are $(3,3,3)$-triangles, $d(v_1) = 3$, $d(v_6) \geq 3$, and $d(v_7) = 3$.

7.3 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-triangle, and $d(v_1) = d(v_6) = d(v_7) = 3$.

Adjacent to Two Triangles on Opposite Sides

In order for the weight on the 7-face to be negative at least one of the triangles must require a contribution of $\frac{4}{3}$.

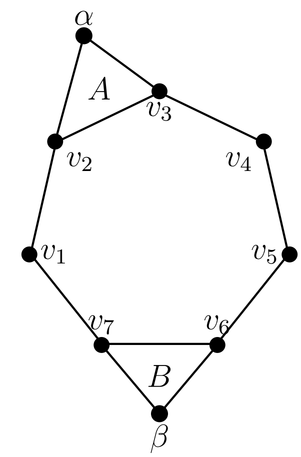

Figure A.6. A 7-face adjacent to two triangles.

7.4 Triangles A and B are $(3, 3, 3)$-triangles, $d(v_1) = 3$, $d(v_4) = 3$, and $d(v_7) \geq 3$.

7.5 Triangles A and B are $(3, 3, 3)$-triangles, $d(v_1) = 3$, $d(v_4) \geq 3$, and $d(v_7) = 3$.

7.6 Triangle A is a $(3, 3, 3)$-triangle, triangle B is a $(3, 3, 4)$-triangle, and $d(v_1) = d(v_4) = d(v_7) = 3$.

Adjacent to Three Triangles

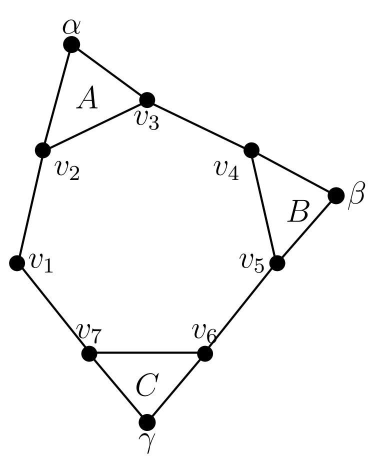

Figure A.7. A 7-face adjacent to three triangles.

In order for the weight on the 7-face to be negative, either all three triangles must require a contribution of $\frac{3}{3}$, or at least one requires a contribution of $\frac{4}{3}$ and at least one requires a contribution of $\frac{4}{3}$.

7.7 Triangles A, B, and C are $(3,3,3)$-triangles, and $d(v_1) \geq 3$.

7.8 Triangles A and B are $(3,3,3)$-triangles, triangle C is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle, and $d(v_1) = 3$.

7.9 Triangles A and C are $(3,3,3)$-triangles, triangle B is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle, and $d(v_1) = 3$.

7.10* Triangle A is a $(3,3,3)$-triangle, triangles B and C are $(3,3,4)$-triangles, and $d(v_1) \geq 3$.

7.11 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-triangle, triangle C is a $(3,3,5)$- or $(3,4,4)$-triangle, and $d(v_1) = 3$.

7.12* Triangle A is a $(3,3,3)$-triangle, triangle C is a $(3,3,4)$-triangle, triangle B is a $(3,3,5)$- or $(3,4,4)$-triangle, and $d(v_1) = 3$.

7.13* Triangle B is a $(3,3,3)$-triangle, triangles B and C are $(3,3,4)$-triangles, and $d(v_1) \geq 3$.

6-Faces

A 6-face has weight $\frac{6}{3}$ prior to the discharging. To have negative weight after discharging, it must be adjacent to one, two, or three triangles.

Adjacent to One Triangle

In order for the weight on the 6-face to be negative, the triangle must require a contribution of at least $\frac{3}{3}$.

6.1 Triangle A is a $(3,3,3)$-triangle, $d(v_1) = d(v_4) = d(v_5) = 3$, and $d(v_6) \geq 3$.

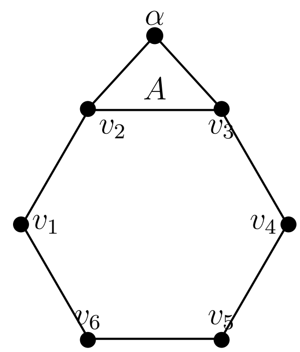

Figure A.8. A 6-face adjacent to one triangle.

6.2 Triangle A is a $(3,3,3)$-triangle, $d(v_1) = d(v_5) = d(v_6) = 3$, and $d(v_4) \geq 3$.

6.3 Triangle A is a $(3,3,4)$-triangle, and $d(v_1) = d(v_4) = d(v_5) = d(v_6) = 3$.

6-Faces Adjacent to Two Triangles, next to each other:

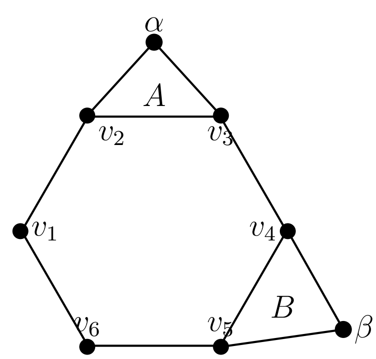

Figure A.9. A 6-face adjacent to two triangles, next to each other.

In order for the weight on the 6-face to be negative after the discharging procedure, one of the triangles must require a contribution of at least $\frac{3}{3}$.

6.4 Triangles A and B are $(3,3,3)$-triangles, and $d(v_1) \geq 3$, $d(v_6) \geq 3$.

6.5 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-triangle, $d(v_1) \geq 3$, and $d(v_6) \geq 3$.

6.6 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-traingle, $d(v_1) = 3$, and $d(v_6) \geq 3$.

6.7* Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,4,4)$-triangle, $d(v_1) \geq 3$, and $d(v_6) = 3$.

6.8 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,4,4)$-triangle, $d(v_1) = 3$, and $d(v_6) \geq 3$.

6.9 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,5)$-triangle, $d(v_1) = 3$, and $d(v_6) \geq 3$.

6.10* Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,5)$-triangle, $d(v_1) \geq 3$, and $d(v_6) = 3$.

6.11 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,6)$-, $(3,4,5)$-, $(4,4,4)$- or $(4,4,5)$-triangle, $d(v_1) = 3$, and $d(v_6) = 3$.

6.12* Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,3,4)$-triangle, $d(v_1) = 3$, and $d(v_6) \geq 3$.

6.13* Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,3,4)$-triangle, $d(v_1) \geq 3$, and $d(v_6) = 3$.

6.14 Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,4,4)$-, or $(3,3,4)$-triangle, and $d(v_1) = d(v_6) = 3$.

6-Faces Adjacent to Two Triangles, on opposite sides:

6.15* Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,3)$ or $(3,3,4)$-triangle, $d(v_1) \geq 3$, and $d(v_4) \geq 3$.

6.16 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,4,4)$-, $(4,4,4)$-, or $(3,3,5)$-triangle, $d(v_1) = 3$, and $d(v_4) \geq 3$.

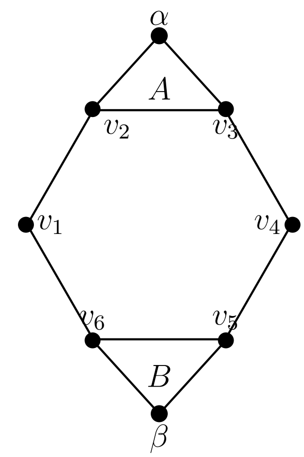

Figure A.10. A 6-face adjacent to two triangles, on opposite sides.

6.17 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,6)$-, $(3,4,5)$-, or $(4,4,5)$-triangle, and $d(v_1) = d(v_4) = 3$.

6.18* Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,3,4)$-triangle, $d(v_1) \geq 3$, and $d(v_4) = 3$.

6.19 Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,3,5)$-, or $(3,4,4)$-triangle, and $d(v_1) = d(v_4) = 3$.

6-Faces Adjacent to Three Triangles:

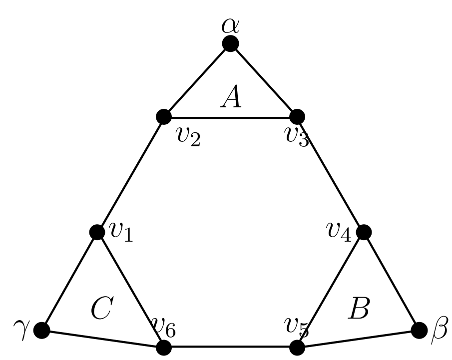

Figure A.11. A 6-face adjacent to three triangles.

6.20 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,3)$-triangle, and triangle C is any type of triangle.

6.21 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-triangle, and triangle C is any type of triangle.

6.22* Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,4,4)$-triangle, and triangle C is any type of triangle requiring a contribution of at least $\frac{1}{3}$.

6.23* Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,5)$-triangle, and triangle C is any type of triangle requiring a contribution of at least $\frac{1}{3}$.

6.24* Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,3,4)$-triangle, and triangle C is any type of triangle requiring a contribution of at least $\frac{1}{3}$.

6.25* Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,4,4)$-triangle, and triangle C is any type of triangle requiring a contribution of at least $\frac{2}{3}$.

6.26* Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,3,5)$-triangle, and triangle C is any type of triangle requiring a contribution of at least $\frac{2}{3}$.

6.27* Triangle A is a $(3,3,4)$-triangle, triangle C is a $(3,3,4)$-triangle, and triangle B is any type of triangle requiring a contribution of at least $\frac{1}{3}$.

APPENDIX B

Faces at Distance at least 3 from each other in any graph G in $\mathcal{F}_{\mathcal{N}}$

In each section we present the configurations of faces of size k that are required to be distance at least 3 from each other in the family $\mathcal{F}_{\mathcal{N}}$. The general configuration is given in a figure and each of the vertices are labeled; specific configurations are then listed by giving the degree of each labeled vertex. For each configuration, we also list which vertex v is to be selected for the set R of vertices removed in the proof of Theorem 5.

7-Faces

We first detail the three 7-faces, each of which is adjacent to three triangles.

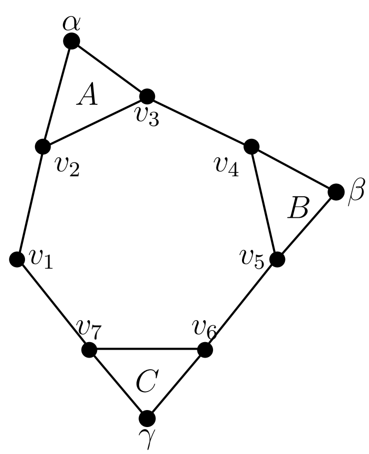

Figure B.1. A 7-face adjacent to three triangles.

7.10* Forbidden configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 4$, $d(v_7) = 3$, $d(\gamma) = 3$. Vertex v_3 will be the member of R.

7.12* Forbidden configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) =$

4, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$, $d(v_7) = 3$, $d(\gamma) = 3$. Vertex v_6 will be the member of R.

7.13* Forbidden configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 4$, $d(v_7) = 3$, $d(\gamma) = 3$. Vertex v_5 will be the member of R.

6-Faces

Multiple 6-faces are required to be distance at least 3 from each other in any graph in $\mathcal{F}_{\mathcal{N}}$. We deal with each type of configuration separately.

6-Faces Adjacent to Two Triangles, next to each other:

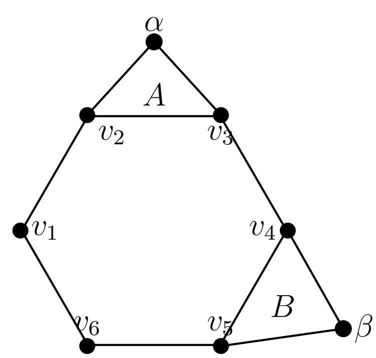

Figure B.2. A 6-face adjacent to two triangles, next to each other.

6.7*a Forbidden configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 3$, $d(\beta) = 4$, $d(v_6) = 3$. Vertex v_3 will be the member of R.

6.7*b Forbidden configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$. Vertex v_3 will be the member of R.

6.10* In this configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 5$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 3$. Vertex v_3 will be the member of R.

6.12* In this configuration: $d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) \geq 5$. Vertex v_3 will be the member of R.

6.13* In this configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$. Vertex v_4 will be the member of R.

6-Faces Adjacent to Two Triangles, on opposite sides:

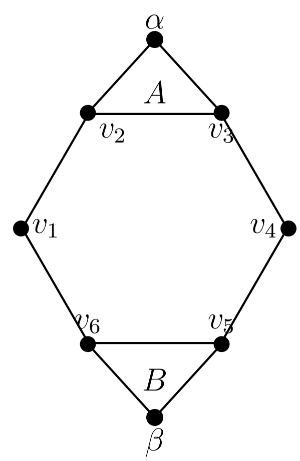

Figure B.3. A 6-face adjacent to two triangles, on opposite sides.

6.15* In this configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) \geq 4$, $d(v_5) = 3, 4$, $d(\beta) \geq 3$, $d(v_6) \geq 3$, and if $d(v_6) \geq 4$, then $d(v_5) = 3$. Vertex v_3 will be the member of R.

6.18* In this configuration: $d(v_1) \geq 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$. Vertex v_2 will be the member of R.

6-Faces Adjacent to Three Triangles:

6.22*a Forbidden configuration: $d(v_1) = 5, 6$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 3$, $d(\beta) = 4$, $d(v_6) = 3$, $d(\gamma) = 3, 4$. Vertex v_3 will be the member of R.

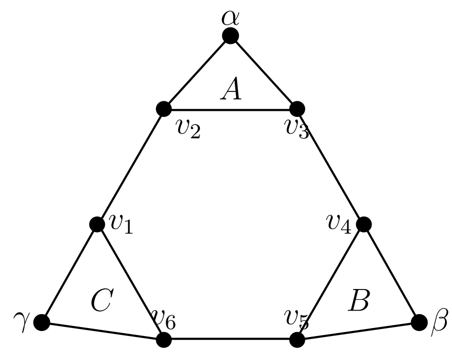

Figure B.4. A 6-face adjacent to three triangles.

6.22*b Forbidden configuration: $d(v_1) = 4, 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 4, 5$, $d(\gamma) = 3, 4$. Vertex v_3 will be the member of R.

6.22*c Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 4$, $d(v_4) = 4$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 4$, $d(\gamma) = 3$. Vertex v_3 will be the member of R.

6.23*a Forbidden configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 5$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 4$. Vertex v_3 will be the member of R.

6.23*b Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 5$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 4$, $d(\gamma) = 4$. Vertex v_3 will be the member of R.

6.23*c Forbidden configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(v_3) = 3$, $d(\alpha) = 3$, $d(v_4) = 5$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 4$, $d(\gamma) = 3$. Vertex v_3 will be the member of R.

6.24*a Forbidden configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 4$. Vertex v_4 will be the member of

R.

6.24*b Forbidden configuration: $d(v_1) = 6$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_4 will be the member of R.

6.24*c Forbidden configuration: $d(v_1) = 4, 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 4, 5$, $d(\gamma) = 3, 4$. Vertex v_4 will be the member of R.

6.24*d Forbidden configuration: $d(v_1) = 4, 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 4, 5$, $d(\gamma) = 3, 4$, and not both v_1 and v_6 are of degree 5. Vertex v_2 will be the member of R.

6.25*a Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 4$. Vertex v_2 will be the member of R.

6.25*b Forbidden configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 4$, $d(\beta) = 4$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_4 will be the member of R.

6.25*c Forbidden configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 4$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_6 will be the member of R.

6.25*d Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 4$, $d(v_5) = 3$, $d(\beta) = 4$, $d(v_6) = 4$, $d(\gamma) = 3$. Vertex v_2 will be the member of R.

6.26*a Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) =$

3, $d(v_5) = 5$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 4$. Vertex v_2 will be the member of R.

6.26*b Forbidden configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 5$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_4 will be the member of R.

6.26*c Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 5$, $d(\beta) = 3$, $d(v_6) = 4$, $d(\gamma) \geq 3$. Vertex v_2 will be the member of R.

6.26*d Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 5$, $d(v_5) = 3$, $d(\beta) = 3$, $d(v_6) = 4$, $d(\gamma) = 3$. Vertex v_2 will be the member of R.

6.27*a Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 5$, $d(\beta) = 4$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_2 will be the member of R.

6.27*b Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 3$, $d(v_5) = 6$, $d(\beta) = 3$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_2 will be the member of R.

6.27*c Forbidden configuration: $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 4$, $d(\alpha) = 3$, $d(v_4) = 4, 5$, $d(v_5) = 4, 5$, $d(\beta) = 3, 4$, $d(v_6) = 3$, $d(\gamma) = 3$. Vertex v_2 will be the member of R.

APPENDIX C

Faces of Negative Weight required to be in any G in $\mathcal{F}$ with minimum degree 3

In each section we present the configurations of faces of size k that are required to be in any graph of minimum degree 3 in $\mathcal{F}$. The general configuration is given in a figure and each of the vertices are labeled; specific configurations are then listed by giving the degree of each labeled vertex. Families of configurations that may be members of the set required to be at distance at least 3 from each other will be denoted by *, and then detailed in Appendix D.

For the discharging procedure, we will often compute the maximum contribution required by various groups of triangles. Observe that the maximum contribution required by any k-group is $2\left(\frac{12}{12}\right) + (k-2)\left(\frac{4}{12}\right)$.

Note furthermore, that every configuration which has negative weight must have at least one vertex of degree 3. Certainly, if all vertices are of degree 4 or more, then the maximum weight discharged from a k-face is $\frac{4}{12}(k) = \frac{k}{3} \leq k-4$ for all $k \geq 6$.

11-faces

For the weight on an 11-face to be negative, the contribution required by adjacent triangles and vertices of degree 3 must be at least $\frac{85}{12}$. This may occur in one of two configurations, each detailed below.

Adjacent to six triangles as one 2-group, and four isolated:

11.1 Three of the isolated triangles must be $(3,3,3)$-triangles.

Otherwise, if two or fewer of the isolated triangles are $(3,3,3)$-triangles requiring a contribution of $\frac{16}{12}$ then the total weight contribution required of the 11-face is no more than $4(\frac{12}{12}) + 2(\frac{16}{12}) = \frac{80}{12}$ and the weight on the face is non-negative.

Adjacent to seven triangles, three 2-groups, and one isolated:

11.2 The isolated triangle must be a $(3, 3, 3)$-triangle and at least one of the neighbors of the isolated triangle on the 11-face must be of degree 4 or less.

If instead, both of the neighbors of the vertices on the isolated triangle were degree 5 or more, the total contribution of the 2-groups is no more than $\frac{28}{12} + \frac{24}{12} = \frac{52}{12}$ and the weight on the 11-face after discharging would be non-negative.

If the isolated triangle is a $(3, 3, 4)$-, or any other triangle, then the maximum weight discharged is $\frac{12}{12} + 3\left(\frac{24}{12}\right) = \frac{84}{12}$, and the weight on the 11-face is non-negative.

10-Faces

For the weight on a 10-face to be negative, the contribution required by adjacent triangles and vertices of degree 3 must be at least $\frac{73}{12}$. This may occur in one of five configurations, described below.

Adjacent to seven triangles, one 3-group, and two 2-groups:

10.1 Five or more of the triangles must be $(3, 3, 4)$-triangles requiring a weight contribution of $\frac{12}{12}$; hence at least one of the two-groups is composed entirely of $(3, 3, 4)$-triangles.

If only four of the triangles required a weight contribution of $\frac{12}{12}$, then the maximum weight discharged from the 10-face would be $4\left(\frac{12}{12}\right) + 2\left(\frac{10}{12}\right) + \frac{4}{12} = \frac{72}{12}$ and the weight on the 10-face would be non-negative.

Adjacent to six triangles, three 2-groups:

10.2 Four or more of the triangles must be $(3,3,4)$-triangles requiring a weight contribution of $\frac{12}{12}$; hence at least one of the two-groups is composed entirely of $(3,3,4)$-triangles.

If only three of the triangles required a weight contribution of $\frac{12}{12}$, then the maximum weight discharged from the 10-face would be $3\left(\frac{12}{12}\right)+3\left(\frac{10}{12}\right)+\frac{4}{12}=\frac{70}{12}$ and the weight on the 10-face would be non-negative.

Adjacent to six triangles, two 2-groups, and two isolated triangles:

10.3 At least one of the isolated triangles must be a $(3,3,3)$-triangle and one of the neighbors of that triangle on the 10-face must be of degree no more than 4.

If instead, both neighbors of the triangle were of degree 5 or more. This would force one of the 2-groups to require a maximum contribution of $\frac{14}{12}$ and hence the maximum weight discharged from the 10-face would be $\frac{70}{12}$ and the final weight on the 10-face would be non-negative.

Furthermore, if both isolated triangles required a contribution of no more than $\frac{12}{12}$, then the maximum weight discharged from the 10-face would be $\frac{72}{12}$ and the final weight on the 10-face would be non-negative.

Adjacent to five triangles, one 2-group, and three isolated triangles:

10.4 All three of the isolated triangles are $(3,3,3)$-triangles, the vertex not part of any triangle on the 10-face must be of degree 3, and the 2-group must require a contribution of at least $\frac{21}{12}$.

If two or fewer of the isolated triangles require a contribution of $\frac{16}{12}$, then the maximum weight discharged from the 10-face is $\frac{72}{12}$.

Adjacent to five triangles, all isolated:

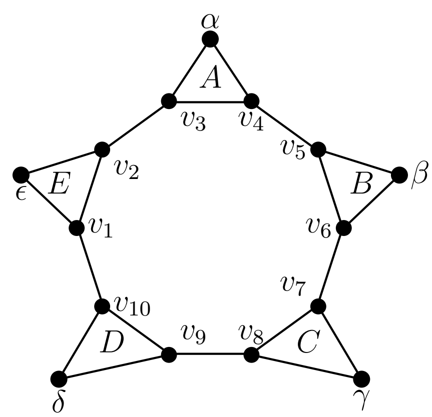

Figure C.1. A 10-Face adjacent to 5 triangles

10.5 In order for the weight on the face to be negative, at least three of the five triangles must be $(3,3,3)$-triangles requiring a contribution of $\frac{16}{12}$ from the 10-face.

If only two of the triangles required a contribution of $\frac{16}{12}$, then the maximum weight discharged from the 10-face would be $\frac{68}{12}$ and the weight on the 10-face would be non-negative.

9-Faces

A 9-face has weight $\frac{60}{12}$ before the discharging. For the weight on the face to become negative after the redistribution of weights, the contributions required by triangles and vertices of degree three must total at least $\frac{61}{12}$. This may occur in several different configurations, each detailed below.

Adjacent to four triangles, one 2-group and two isolated triangles:

9.1 The 2-group requires a contribution of $\frac{22}{12}$ or more.

If the 2-group requires a contribution of $\frac{20}{12}$ or less, then the maximum weight discharged from the 9-face is $\frac{20}{12} + 2\left(\frac{4}{12}\right) + 2\left(\frac{16}{12}\right) = \frac{60}{12}$ and the final weight on the 9-face would be non-negative.

Adjacent to four triangles, each isolated:

9.2 Three of the isolated triangles are $(3,3,3)$-triangles.

If only two of the isolated triangles are $(3,3,3)$-triangles, then the maximum weight discharged from the 9-face is $2\left(\frac{16}{12}\right)+2\left(\frac{12}{12}\right)+\frac{4}{12}=\frac{60}{12}$ and the final weight on the 9-face would be non-negative.

Adjacent to five triangles, one 3-group and two isolated triangles:

9.3 Both isolated triangles are $(3,3,3)$-triangles and the triangles on each end of the 3-group require a contribution of at least $\frac{10}{12}$.

If only one of the isolated triangles is a $(3,3,3)$-triangle, then the maximum weight discharged from the 9-face is $\frac{16}{12}+3\left(\frac{12}{12}\right)+2\left(\frac{4}{12}\right)=\frac{60}{12}$ and the final weight on the 9-face would be non-negative.

Adjacent to five triangles, two 2-groups and one isolated triangle:

9.4 At least one 2-group requires a contribution of at least $\frac{22}{12}$.

If both 2-groups require a contribution of $\frac{20}{12}$ or less, then the maximum weight discharged from the 9-face is $2\left(\frac{20}{12}\right)+\frac{16}{12}+\frac{4}{12}=\frac{60}{12}$ and the final weight on the 9-face would be non-negative.

Adjacent to five triangles, one 2-group and three isolated triangles:

This configuration is illustrated in Figure C.2. In order for the final weight on the 9-face to be negative, at least one of the isolated triangles must be a $(3,3,3)$-triangle. If none of them were $(3,3,3)$-triangles then the maximum weight discharged would be $3\left(\frac{12}{12}\right)+\frac{24}{12}=\frac{60}{12}$ and the final weight on the 9-face would be non-negative.

9.5 Triangle A is a $(3,3,3)$-triangle, $d(v_1)\leq 4$ and the sum of triangles C, D,

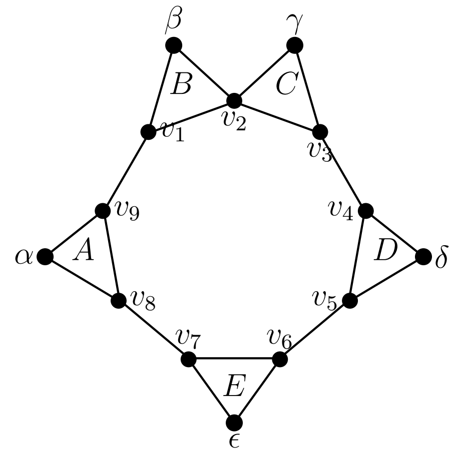

Figure C.2. A 9-face adjacent to five triangles, one 2-group and three isolated triangles.

and E is at least $\frac{41}{60}$.

9.6 Triangle A is a $(3, 3, 3)$-triangles, $d(v_7) \leq 4$ and the sum of triangles B, C, and D is at least $\frac{29}{12}$.

9.7 Triangle D is a $(3, 3, 3)$-triangle, $d(v_3) \leq 4$ and the sum of triangles A, B, and E is at least $\frac{41}{60}$.

9.8 Triangle D is a $(3, 3, 3)$-triangle, $d(v_6) \leq 4$ and the sum of triangles A, B, and C is at least $\frac{29}{60}$.

9.9 Triangle E is a $(3, 3, 3)$-triangle, $d(v_8) \leq 4$ and the sum of triangles B and C is at least $\frac{12}{12}$ ($\frac{20}{12}$ if A and D are $(3, 3, 4)$-triangles).

9.10 Triangle E is a $(3, 3, 3)$-triangle, $d(v_5) \leq 4$ and the sum of triangles B and C is at least $\frac{12}{12}$ (at least $\frac{20}{12}$ if A and D are $(3, 3, 4)$-triangles).

Adjacent to six triangles, one 4-group and two isolated triangles:

9.11 Both isolated triangles are $(3, 3, 3)$-triangles and the 4-group requires a total contribution of at least $\frac{29}{12}$.

Adjacent to six triangles, one 3-group, one 2-group, and one isolated triangle:

9.12 The isolated triangle is a $(3,3,3)$-triangle and the 2-group requires a contribution of at least $\frac{17}{12}$.

9.13 The 2-group requires a contribution of at least $\frac{22}{12}$ and the total contribution required by the other triangles is at least $\frac{39}{12}$.

Adjacent to six triangles: three 2-groups

9.13 At least one of the 2-groups requires a weight contribution of at least $\frac{22}{12}$ and the others require a total contribution of at least $\frac{39}{12}$.

8-faces

An 8-face has weight $\frac{48}{12}$ before the discharging procedure is applied. In order for the 8-face to have negative weight after the redistribution it must appear in one of the configurations listed below, and be adjacent to at least three triangles.

Adjacent to three triangles: one 2-group and one isolated triangle

8.1 The 2-group requires a contribution of $\frac{22}{12}$ and the remaining weight contribution required is at least $\frac{27}{12}$.

If the 2-group requires a contribution of $\frac{20}{12}$ or less, then the maximum weight discharged is $\frac{20}{12} + \frac{16}{12} + 3\left(\frac{4}{12}\right) = \frac{48}{12}$ and the weight on the 8-face is non-negative.

Adjacent to three Triangles: each isolated

In order for the 8-face to have negative weight, at least two of the triangles must require a contribution of at least $\frac{16}{12}$ and at least one of v_5 or v_6 must be degree 3.

8.2 Triangles A, B, and C are $(3,3,3)$-triangles and $d(v_5) = 3$, $d(v_6) \geq 3$

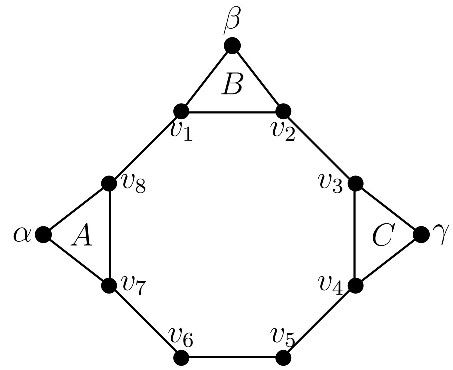

Figure C.3. An 8-face adjacent to 3 triangles.

8.3 Triangles A and B are $(3, 3, 3)$-triangles, triangle C is a $(3, 3, 4)$-triangle and $d(v_5) = d(v_6) = 3$

8.4 Triangles A and C are $(3, 3, 3)$-triangles, triangle B is a $(3, 3, 4)$-triangle and $d(v_5) = d(v_6) = 3$

If only one of the triangles requires a contribution of $\frac{16}{12}$, then the maximum contribution totals no more than $\frac{48}{12}$ and the weight is non-negative.

Note that if three triangles are adjacent to an 8-face as a 3-group, then the maximum weight discharged is $\frac{12}{12} + \frac{12}{12} + \frac{4}{12} + 4\left(\frac{4}{12}\right) = \frac{44}{12}$ and the weight on the 8-face is non-negative.

Adjacent to four triangles: one 3-group and one isolated

8.5 The isolated triangle is a $(3, 3, 3)$-triangle and the remaining contribution required by the vertices and the 3-group is at least $\frac{33}{12}$, hence both of the vertices not on a triangle are of degree 3.

If the isolated triangle requires a contribution of $\frac{12}{12}$ or less, then the final weight on the 8-face is non-negative.

Adjacent to four triangles: two 2-groups

8.6 One of the two groups requires a contribution of $\frac{22}{12}$ or more and the remaining contribution required by the vertices and second 2-group is at least $\frac{27}{12}$.

If both 2-groups require contributions of $\frac{20}{12}$ or less then the maximum weight discharged is $\frac{40}{12} + \frac{4}{12} + \frac{4}{12} = \frac{48}{12}$ and the final weight on the 8-face is non-negative.

Adjacent to four triangles: one 2-group and two isolated triangles

8.7 The 2-group requires a contribution of $\frac{22}{12}$ or more and the remaining contribution required by the isolated triangles and vertex is at least $\frac{27}{12}$

8.8 Both isolated triangle are $(3,3,3)$-triangles, the 2-group requires a contribution of between $\frac{17}{12}$ and $\frac{20}{12}$, forcing the corners of the end triangles to be of degree 3, and the degree of the vertex not part of any triangle is 4 or more.

8.9 Both isolated triangle are $(3,3,3)$-triangles, the 2-group requires a contribution of between $\frac{13}{12}$ and $\frac{20}{12}$ and the degree of the vertex not part of any triangle is 3.

8.10 One isolated triangle is a $(3,3,3)$-triangle, the other requires a contribution of $\frac{12}{12}$ or less, the vertex not on any triangle is of degree 3, and the 2-group requires a contribution of at least $\frac{17}{12}$, forcing the corners of the end triangles to be of degree 3.

If both isolated triangles require a contribution of $\frac{12}{12}$ or less and the 2-group requires a contribution of $\frac{20}{12}$ or less, then the maximum weight discharged from the 8-face is $\frac{48}{12}$ and the final weight on the 8-face is non-negative.

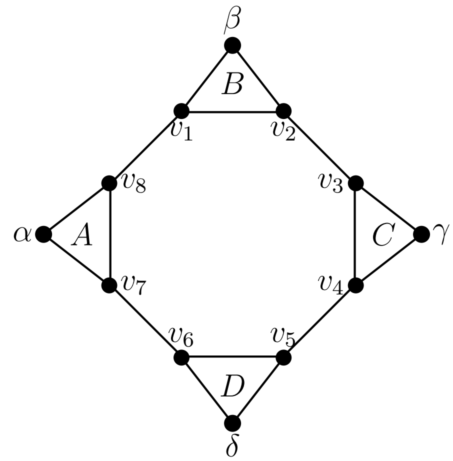

Figure C.4. An 8-face adjacent to four triangles.

Adjacent to four triangles: all isolated triangles

In order for the weight on the 8-face to be negative after discharging, at least one of the triangles must require a contribution of $\frac{16}{12}$.

8.11 Triangles A, B, C, and D are all $(3,3,3)$-triangles.

8.12 Triangles A, B, and C are $(3,3,3)$-triangles, and triangle D requires a contribution of at least $\frac{1}{12}$.

8.13 Triangles A, B, and D are $(3,3,3)$-triangles, and triangle C requires a contribution of at least $\frac{1}{12}$.

8.14 Triangles A and B are $(3,3,3)$-triangles, and triangles C and D require a total contribution of at least $\frac{17}{12}$.

8.15 Triangles A and C are $(3,3,3)$-triangles, and triangles C and D require a total contribution of at least $\frac{17}{12}$, forcing one of v_1, v_2, v_5, or v_6 to be of degree no more than 4.

8.16 Triangle A is a $(3,3,3)$-triangle, and at least two of triangles B, C, or D requires a contribution of $\frac{12}{12}$, and the remaining require a contribution of at least $\frac{10}{12}$

Adjacent to five triangles: a 4-group and one isolated triangle

8.17 The isolated triangle is a $(3,3,3)$-triangle and the 4-group requires a contribution of at least $\frac{29}{12}$, forcing both of the corner vertices of the end triangles of the 4-group to be of degree 3.

If the isolated triangle requires a contribution of no more than $\frac{12}{12}$, then the maximum weight discharged from the face is $\frac{12}{12} + \frac{32}{12} + \frac{4}{12} = \frac{48}{12}$ and the final weight is non-negative.

Adjacent to five triangles: one 3-group and one 2-group

First note that the vertices which are neighbors of each other on the ends of the 3-group and 2-group must both be of degree 4 or less, otherwise the weight on the 8-face after discharging is non-negative.

8.18 The 2-group requires a contribution of $\frac{22}{12}$ or more and the 3-group requires a contribution of at least $\frac{19}{12}$, forcing each of the corner vertices of the end triangles of the 3-group to be of degree no more than 4.

8.19 The 2-group requires a contribution of $\frac{20}{12}$ or less, the vertex not on any triangle has degree 4 or less, and the contribution required by the 3-group is at least $\frac{25}{12}$. This forces at least one end of the 3-group to be a $(3,3,4)$-triangle.

Note that if the 2-group requires a contribution of $\frac{20}{12}$ or less, the vertex not on any triangle has degree 5 or more, and the contribution required by the 3-group is at least $\frac{29}{12}$, which is impossible and the weight on the 8-face is non-negative.

Adjacent to five triangles: one 3-group and two isolated triangles

This configuration is illustrated in Figure C.5. Note that at least of v_8 and v_3 is of degree 3. Throughout the configurations below, we assume that triangle D has the maximum contribution of triangles D and E.

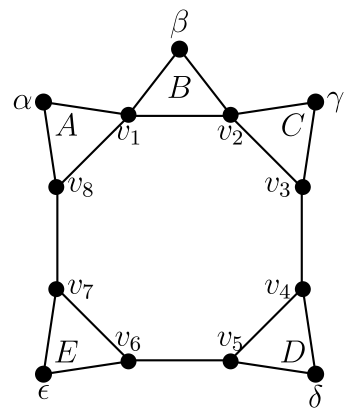

Figure C.5. An 8-face adjacent to five triangles.

8.20 Triangle D is a $(3,3,3)$-triangle and one of $d(v_3) \leq 4$ or $d(v_6) \leq 4$.

8.21 Triangle D is a $(3,3,4)$-triangle, triangle E is a $(3,3,4)$-triangle, and the 3-group requires a contribution of $\frac{25}{12}$. This forces at least one end of the 3-group to be a $(3,3,4)$-triangle as well.

8.22 Triangle D is a $(3,3,4)$-triangle, triangle E requires a contribution of $\frac{10}{12}$ or $\frac{9}{12}$ and the 3-group requires a contribution of $\frac{27}{12}$, forcing both triangle A and triangle C to be $(3,3,4)$-triangles.

If the total contribution required by triangles D and E is $\frac{20}{12}$ or less, then the final weight on the 8-face is non-negative.

Adjacent to five triangles: two 2-groups and one isolated triangle

This configuration is illustrated in Figure C.6.

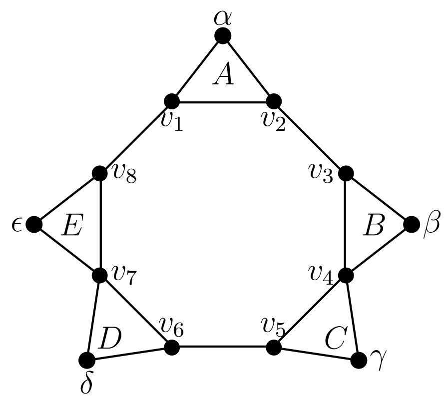

Figure C.6. An 8-face adjacent to five triangles.

8.23 One of the 2-groups requires a contribution of $\frac{22}{12}$ or more and the remaining elements in the configuration require a contribution totaling at least $\frac{27}{12}$.

From this point on, we assume both 2-groups require contributions of no more than $\frac{20}{12}$.

8.24 Triangle A is a $(3, 3, 3)$-triangle and one of $d(v_3) \leq 4$ or $d(v_8) \leq 4$.

Note that if both $d(v_8) \geq 5$ and $d(v_3) \geq 5$, then the weight on the 8-face is non-negative.

8.25 Triangle A is a $(3, 3, 4)$-triangle and both of $d(v_3) \leq 4$ and $d(v_8) \leq 4$.

8.26 Triangle A is a $(3, 3, 4)$-triangle, $d(v_3) \leq 4$, $d(v_8) \geq 5$, and both triangles C and D are $(3, 3, 4)$-triangles.

8.27 All of the triangles require a contribution of $\frac{10}{12}$. This forces triangles A and B to be either both $(3, 4, 4)$-triangles or both $(3, 3, 5)$-triangles and $d(v_6) = 3$.

Adjacent to six triangles: a 5-group and an isolated triangle

8.28 The isolated triangle is a $(3, 3, 3)$-triangle, and one of it's neighbors on the 8-face is of degree no more than 4.

If the isolated triangle is a $(3,3,3)$-triangle and both neighbors on the 8-face are of degree at least 5, then the weight discharged by the 5-group is no more than $\frac{16}{12}$ and the final weight on the 8-face is non-negative. Furthermore, if the isolated triangle requires a contribution of $\frac{12}{12}$ or less, then the total weight discharged from the 8-face is no more than $\frac{12}{12}+\frac{36}{12}$ and the weight on the 8-face is non-negative.

Adjacent to six triangles: a 4-group and a 2-group

8.29 The 2-group requires a contribution of $\frac{22}{12}$ or more and the 4-group requires a contribution of at least $\frac{25}{12}$.

8.30 The 2-group requires a contribution of $\frac{20}{12}$ or less and the 4-group requires a contribution of $\frac{29}{12}$ or more. This forces at least one end triangle of the 4-group to be a $(3,3,4)$-triangle. Furthermore, in order for the weight on the 8-face to be negative, both of the corner vertices on the 2-group must be of degree no more than 4.

Adjacent to six triangles: two 3-groups

8.31 One of the end triangles of the 3-group is a $(3,3,4)$-triangle and its neighbor on the 8-face is of degree 4 or less.

If none of the triangles is a $(3,3,4)$-triangle, then the maximum weight discharged is $\frac{48}{12}$ and the weight on the 8-face is non-negative. If one of the triangles is a $(3,3,4)$-triangle but its neighbor on the 8-face is of degree 5 or more, then the maximum weight discharged is $\frac{36}{12}+\frac{8}{12}+\frac{2}{12}$, and the weight on the 8-face is non-negative.

7-Faces

A 7-face has weight $\frac{36}{12}$ before discharging. In order for it to return with negative weight after the redistribution of weights, the total weight discharged must be at least $\frac{37}{12}$. This may occur in several configurations, each discussed below.

Adjacent to two triangles: one 2-group

7.1 The 2-group requires a contribution of at least $\frac{22}{12}$.

If the 2-group requires a contribution of $\frac{20}{12}$ or less, then the total weight discharged is no more than $\frac{20}{12} + 4\left(\frac{4}{12}\right) = \frac{36}{12}$ and the final weight on the 7-face is non-negative.

Adjacent to two triangles: next to each other

This configuration is illustrated in Figure C.7.

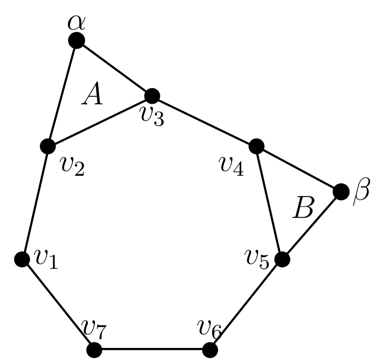

Figure C.7. A 7-face adjacent to two triangles.

In order for the weight on the 7-face to be negative at least one of the triangles must be a $(3, 3, 3)$-triangle.

7.2 Triangles A and B are $(3, 3, 3)$-triangles, $d(v_1) = 3$, $d(v_6) = 3$, and $d(v_7) \geq 3$.

7.3 Triangles A and B are $(3, 3, 3)$-triangles, $d(v_1) = 3$, $d(v_6) \geq 3$, and $d(v_7) = 3$.

7.4 Triangle A is a $(3, 3, 3)$-triangle, triangle B is a $(3, 3, 4)$-triangle, and $d(v_1) = d(v_6) = d(v_7) = 3$

Adjacent to two triangles: on opposite sides

This configuration is illustrated in Figure C.8.

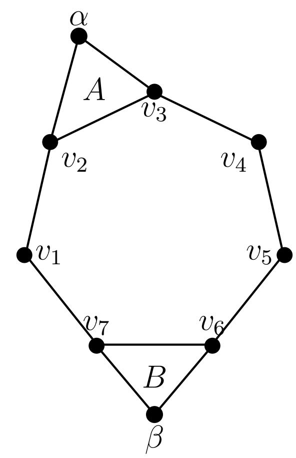

Figure C.8. A 7-face adjacent to two triangles.

In order for the weight on the 7-face to be negative at least one of the triangles must require a contribution of $\frac{16}{12}$

7.5 Triangles A and B are $(3,3,3)$-triangles, $d(v_1) = 3$, $d(v_4) = 3$, and $d(v_7) \geq 3$

7.6 Triangles A and B are $(3,3,3)$-triangles, $d(v_1) = 3$, $d(v_4) \geq 3$, and $d(v_7) = 3$.

7.7 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-triangle, and $d(v_1) = d(v_4) = d(v_7) = 3$.

Adjacent to three triangles: one 3-group

Note that the maximum contribution of any 3-group is $\frac{28}{12}$ and hence the other three vertices must all be of degree 3.

7.8 One end of the 3-group is a $(3,3,4)$-triangle and the three vertices not members of triangles on the 7-face are of degree 3.

Adjacent to three triangles: one 2-group and one isolated triangle

7.9 The 2-group requires a contribution of $\frac{22}{12}$ or more and the remaining contribution required by the isolated triangle and the vertices is at least $\frac{15}{12}$.

7.10 The 2-group requires a contribution of $\frac{20}{12}$, one of the vertices not on a triangle is of degree 3, the other is of degree 5 or more, and the isolated triangle is a $(3, 3, 3)$-triangle.

7.11 The 2-group requires a contribution of $\frac{20}{12}$, both of the vertices not on a triangle are of degree 3, and the isolated triangle requires a contribution of at least $\frac{10}{12}$.

7.12 The 2-group requires a contribution of $\frac{20}{12}$ with triangle A and B both $(3, 3, 5)$-$(3, 4, 4)$-, or one $(3, 4, 5)$- and one $(3, 3, 4)$- triangles. Both of the vertices not on a triangle are of degree 3, and the isolated triangle requires a contribution of at least $\frac{9}{12}$.

7.13 The 2-group requires a contribution of $\frac{20}{12}$ with a $(3, 4, 6)$-triangle and a $(3, 3, 4)$-triangle, the isolated triangle is a $(3, 3, 6)$-triangle, and both vertices not on a triangle are of degree 3.

7.14 The 2-group requires a contribution of $\frac{18}{12}$ or less, the isolated triangle is a $(3, 3, 3)$- or $(3, 3, 4)$-triangle and at least one of the vertices not on a triangle is of degree 3.

Adjacent to three triangles: each isolated

This configuration is illustrated in Figure C.9 below.

In order for the weight on the 7-face to be negative, either all three triangles must require a contribution of $\frac{12}{12}$, or at least one requires a contribution of $\frac{16}{12}$.

7.15 Triangles A, B, and C are $(3, 3, 3)$-triangles, and $d(v_1) \geq 3$.

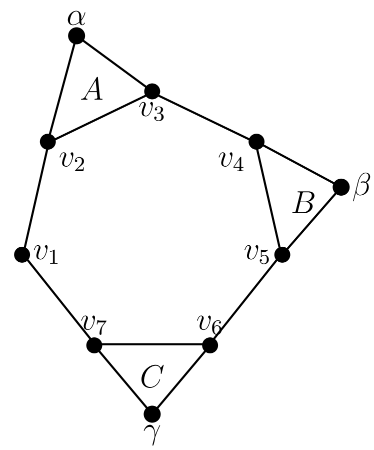

Figure C.9. A 7-face adjacent to three triangles.

7.16 Triangles A and B are $(3,3,3)$-triangles and the contribution required by triangle C and the other vertex is at least $\frac{5}{12}$.

7.17 Triangles A and C are $(3,3,3)$-triangles, and the contribution required by triangle B and the other vertex is at least $\frac{5}{12}$. Note that this forces at least one of v_1, v_4, or v_5 to be of degree 4 or less.

7.18 Triangle A is a $(3,3,3)$-triangle, triangle B is a $(3,3,4)$-, $(3,3,5+)$-, $(3,4,4+)$, or $(3+,4,4+)$-triangle in which $d(v_4) \leq 4$, and the contribution required by triangle C and the other vertex is at least $\frac{9}{12}$.

7.19 Triangle A is a $(3,3,3)$-triangle, $d(v_4) \geq 5$, $d(v_1) \leq 4$, and the total contribution required by triangles B and C is at least $\frac{17}{12}$.

7.20 Triangle A is a $(3,3,3)$-triangle, $d(v_4) \geq 5$, $d(v_1) \geq 5$, and the total contribution required by triangles B and C is at least $\frac{21}{12}$. This forces triangle C to be a $(3,3,4)$-triangle and triangle B to be a $(3,3,5)$-triangle, since v_1 will return at least $\frac{1}{12}$ to the 7-face.

7.21 Triangle B is a $(3,3,3)$-triangle, $d(v_3) \leq 4$, and the contribution required by the other elements in the configuration totals at least $\frac{21}{12}$.

7.22* Triangle B is a $(3,3,3)$-triangle, $d(v_3) \geq 5$, $d(v_6) \geq 5$, and $d(v_1) = 3$. In this case triangles A and C must require a contribution of at least $\frac{17}{12}$.

Adjacent to four triangles: one 4-group

7.23 One triangle on the end of the 4-group requires a contribution of $\frac{12}{12}$ and both of the vertices not part of triangles are of degree 3.

If the 4-group instead requires a maximum contribution of $\frac{28}{12}$, then the total weight discharged is no more than $\frac{36}{12}$ and the final weight on the 7-face is non-negative.

Adjacent to four triangles: one 3-group and one isolated

This configuration is illustrated in Figure C.10.

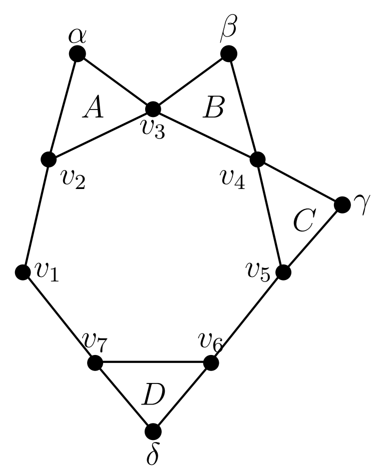

Figure C.10. A 7-face adjacent to four triangles as a 3-group and one isolated triangle.

Note that for the weight on the 7-face to be negative, v_2 or v_5 must have degree 3, since the contribution required by the 3-group must be at least $\frac{17}{12}$. If the contribution required by triangle D is no more than $\frac{12}{12}$, then both v_2 and v_5 are of degree no more than 4. Furthermore, at least two of triangles A, C, and D must require a contribution of $\frac{10}{12}$ or more.

7.24 Triangle D is a $(3,3,3)$-triangle, or a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle in which $d(v_6) = 3$ and the contribution required by the other elements of the configuration is at least $\frac{21}{12}$; this forces $d(v_5) \leq 4$.

7.25 Triangles A and B require a total contribution of $\frac{16}{12}$ or more, triangle D requires a contribution of $\frac{12}{12}$ or less, and the remaining elements of the configuration require a contribution of at least $\frac{9}{12}$.

7.26 Triangles C and B require a total contribution of $\frac{16}{12}$ or more, triangle D requires a contribution of $\frac{12}{12}$ or less, and the remaining elements of the configuration require a contribution of at least $\frac{9}{12}$.

7.27 Triangle A requires a contribution of $\frac{12}{12}$, $d(v_1) \leq 4$, triangle B requires a contribution of $\frac{2}{12}$ or less, and the remaining elements of the configuration require a contribution of at least $\frac{19}{12}$.

7.29 Triangle A requires a contribution of $\frac{12}{12}$, $d(v_1) \geq 5$, triangle B requires a contribution of $\frac{2}{12}$ or less, and the remaining elements of the configuration require a contribution of at least $\frac{23}{12}$, forcing triangles C and D to be either both $(3,3,4)$-triangles, or one $(3,3,5)$-triangle and one $(3,3,3)$-triangle.

7.30 Triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) \leq 4$, and the remaining elements of the configuration require a contribution of at least $\frac{23}{12}$.

7.31 Triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) \geq 5$, and the remaining elements of the configuration require a contribution of at least $\frac{27}{12}$, this forces triangles C and D to be either both $(3,3,4)$-triangles, or one $(3,3,5)$-triangle and one $(3,3,3)$-triangle.

7.32 Triangle A requires a contribution of $\frac{9}{12}$ or less, triangle D is a $(3,3,3)$- or $(3,3,4)$-triangle, and triangle C is a $(3,3,5)$-, $(3,4,4)$- or $(3,3,4)$-triangle.

If Triangle A requires a contribution of $\frac{9}{12}$ or less, and triangle D requires a contribution of $\frac{10}{12}$ or less, then the final weight on the 7-face is non-negative.

Note that, if triangle D requires a contribution of $\frac{9}{12}$ or less, then the final weight on the 7-face is non-negative

Adjacent to four triangles: two 2-groups

This configuration is illustrated in Figure C.11. We assume that the 2-group A and B require the largest contributions.

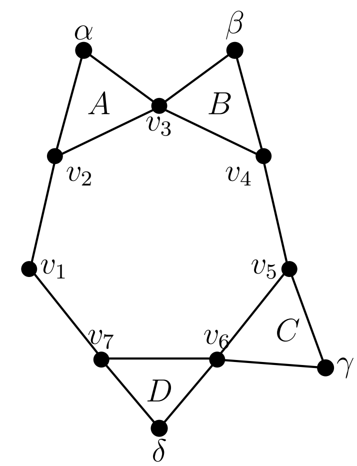

Figure C.11. A 7-face adjacent to four triangles as two 2-groups.

7.33 Triangles A and B require a total contribution of $\frac{22}{12}$ or more and the contribution required by the other elements of the configuration is at least $\frac{15}{12}$.

In all the cases below we assume that any 2-group does not require a contribution totaling more than $\frac{20}{12}$.

7.34 Triangles A and B are both $(3,4,4)$-triangles, or one $(3,4,4)$-triangle and one $(3,4,5)$-triangle, and the remaining elements require a contribution of at least $\frac{17}{12}$.

7.35 Triangle A (or B) is a $(3,3,5)$-triangle and $d(v_1) \leq 4$ (or $d(v_5) \leq 4$) and the contribution required by the remaining elements of the configuration is at least $\frac{23}{12}$.

7.36 Triangles A and B are both $(3,3,6)$-triangles and $d(v_1) = d(v_5) = 3$.

7.37 Triangle A (or B) is a $(4,4,4)$-triangle, triangle B (or A) is a $(3,3,4)$- $(3,4,4)$-, or $(3,4,5)$-triangle, and the contribution required by the other elements of the configuration totals at least $\frac{21}{12}$.

7.38 Triangle A (or B) is a $(3,5,7+)$-triangle, triangle B (or A) is a $(3,3,5)$-triangle, $d(v_1) = 3$, and $d(v_5) = 3$.

Note that any other arrangement in which one of the triangles requires a contribution of $\frac{4}{12}$ results in a final non-negative weight on the 7-face. So in all the future cases, we assume that each triangle must require a contribution of at least $\frac{6}{12}$. This forces both $d(v_4) = d(v_5) = 3$.

7.39* Triangle A is a $(3,3,4)$-triangle $d(v_1) \geq 5$, triangle B is a $(3,4,6)$-triangle, triangles C and D are exactly as triangles B and A.

7.40 Triangle A is a $(3,4,4)$-triangle $d(v_1) \geq 5$, triangle B is a $(3,4,6)$-triangle. Note that $d(v_4) = 3$, otherwise the contribution required by Triangle B is $\frac{2}{12}$ or less and the weight on the 7-face is non-negative. Since v_1 contributes $\frac{2}{12}$ to the 7-face, triangle C must be at a $(3,3,4)$- $(3,4,4)$-, or $(3,3,5)$-triangle.

7.41 Triangle A requires a contribution $\frac{8}{12}$ or less, this forces one of triangles B or C to require a contribution of at least $\frac{10}{12}$ and $d(v_4) \leq 4$ and $d(v_1) \leq 4$.

Note that if triangle A requires a contribution of $\frac{8}{12}$ or less and $d(v_1) \geq 5$, then the total contribution required by the remaining elements of the configuration is at least $\frac{31}{12}$ and will force a mirror image situation to one of the above cases.

Adjacent to four triangles: one 2-group and two isolated triangles

This configuration is illustrated in Figure C.12.

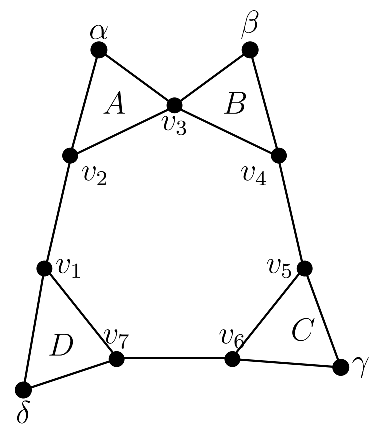

Figure C.12. A 7-face adjacent to four triangles, one 2-group, two isolated.

In order for the weight on the 7-face to be negative, at least three of v_1, v_2, v_4, v_5, v_6, and v_7 must be of degree 3.

7.42 The 2-group requires a contribution of $\frac{22}{12}$ or more and triangles C and D require a contribution totaling $\frac{13}{12}$ or more.

7.43 The 2-group requires a contribution of $\frac{20}{12}$ or less, and $d(v_1) = d(v_5) = 3$.

7.44 The 2-group requires a contribution of $\frac{20}{12}$ or less, and $d(v_1 = 3$ and $d(v_5) = 4$.

7.45 The 2-group requires a contribution of $\frac{20}{12}$ or less, and $d(v_1 = 4$ and $d(v_5) = 4$.

7.46* The 2-group requires a contribution of $\frac{20}{12}$ or less, and $d(v_1) \geq 5$.

Adjacent to five triangles: as a 5-group

7.47 For the weight on the 7-face to be negative, the vertex not part of any triangle on the 7-face must be of degree 3 and one of the triangles at the ends of the 5-group must require a contribution of at least $\frac{12}{12}$.

Adjacent to five triangles: a 4-group and one isolated triangle

This configuration is illustrated in Figure C.13 below. For the weight on the 7-face to be negative, at least one of v_1 or v_7 must be of degree 3 and both of v_2 and v_6 must be of degree 3.

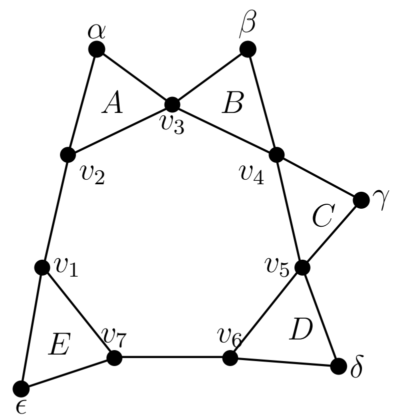

Figure C.13. A 7-face adjacent to five triangles as a 4-group and one isolated triangle.

7.48 Triangle E is a $(3, 3, 3)$-triangle and the contribution required by the 4-group is at least $\frac{21}{12}$.

7.49 Triangle E is a $(3, 3, 4)$-triangle and the contribution required by the 4-group is at least $\frac{25}{12}$

Assume below that triangle E requires a contribution of no more than $\frac{10}{12}$

7.50 Triangles A and D require a contribution of at least $\frac{10}{12}$ and the contribution required by the remaining elements of the configuration totals at least $\frac{17}{12}$.

7.51 Triangles B and C require a total contribution totaling $\frac{8}{12}$ and the contribution required by the remaining elements of the configuration totals at least $\frac{29}{12}$.

Assume below that triangles B and C require a contribution totaling no more than $\frac{6}{12}$. Note that in order for the weight on the 7-face to be negative, they must in fact require a contribution of at least $\frac{5}{12}$ and hence at least one of them is a $(3,4,4)$- or $(4,4,4)$-triangle.

7.52 Triangle A requires a contribution of $\frac{12}{12}$, triangle B requires a contribution of $\frac{4}{12}$, triangle C requires a contribution of $\frac{2}{12}$, triangle D requires a contribution of $\frac{9}{12}$, and triangle E requires a contribution of $\frac{10}{12}$

Adjacent to five triangles: one 3-group and one 2-group

This configuration is illustrated in Figure C.14. For the weight on the 7-face to be negative, at least one of v_1 or v_6, and one of v_2, or v_5, must be of degree 3 and all four of these vertices must have degree no more than 4.

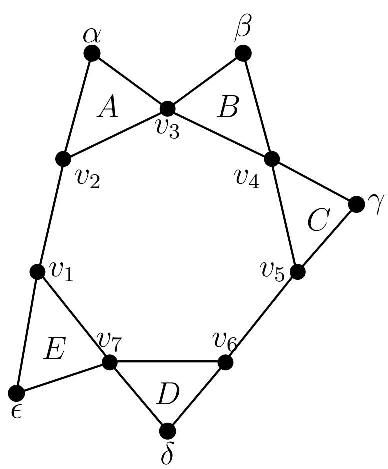

Figure C.14. A 7-face adjacent to five triangles as a 3-group and a 2-group.

7.53 Triangle D requires a contribution of $\frac{10}{12}$ or more and the contribution required by the other elements totals at least $\frac{27}{12}$.

7.54 Triangle E requires a contribution of $\frac{10}{12}$ or more and the contribution required by the other elements totals at least $\frac{27}{12}$.

From this point forward, we assume that the contribution required by the 2-group is no more than $\frac{18}{12}$ and that each of the triangles requires a contribution of $\frac{9}{12}$ or less.

7.55 Triangle A requires a contribution of $\frac{10}{12}$ or more and the contribution required by the other elements totals at least $\frac{27}{12}$.

7.56 Triangle C requires a contribution of $\frac{10}{12}$ or more and the contribution required by the other elements totals at least $\frac{27}{12}$.

From this point forward, we assume that the total contribution required by triangles A and C is no more than $\frac{18}{12}$ and that each of the triangles requires a contribution of $\frac{9}{12}$ or less. Since the contribution required by the 2-group is also a maximum of $\frac{18}{12}$, both of v_2 and v_5 are of degree 3.

7.57 Triangles D and E are $(3, 3, 6)$-triangles, $d(v_2) = d(v_5) = 3$.

Adjacent to six triangles: as a 6-group

7.58 One of the triangles on the end of the 6-group requires a contribution of $\frac{12}{12}$ and the other end requires a contribution of at least $\frac{9}{12}$.

Indeed, if both require a contribution of $\frac{10}{12}$ or less, then the maximum weight discharged is $\frac{20}{12} + \frac{16}{12} = \frac{36}{12}$ and the resulting weight on the 7-face is non-negative.

Note that the case in which the 7-face is adjacent to a 7-group results immediately in a non-negative weight.

6-Faces

A 6-face has weight $\frac{24}{12}$ prior to the discharging. To have negative weight after discharging, it must be adjacent to one, two, three, four, or five triangles.

Adjacent to one triangle

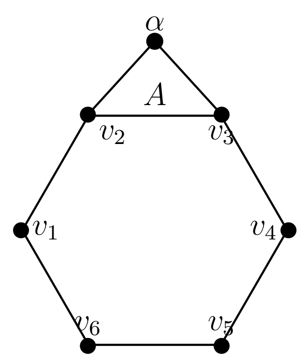

Figure C.15. A 6-face adjacent to one triangle.

In order for the weight on the 6-face to be negative, the triangle must require a contribution of at least $\frac{12}{12}$.

6.1 Triangle A is a $(3,3,3)$-triangle, $d(v_1) = d(v_4) = d(v_5) = 3$, and $d(v_6) \geq 3$.

6.2 Triangle A is a $(3,3,3)$-triangle, $d(v_1) = d(v_5) = d(v_6) = 3$, and $d(v_4) \geq 3$.

6.3 Triangle A is a $(3,3,4)$-triangle, and $d(v_1) = d(v_4) = d(v_5) = d(v_6) = 3$.

Adjacent to two triangles: one 2-group

This configuration is illustrated in Figure C.16. Throughout this list, we assume that triangle A requires the maximum weight of the triangles.

6.4 The 2-group requires a contribution of at least $\frac{22}{12}$ and at least one of v_1, v_5, or v_6 is of degree 3.

Suppose from now on that the 2-group requires a contribution of no more than $\frac{20}{12}$. This forces at least two of v_1, v_5, and v_6 to be of degree 3 in order

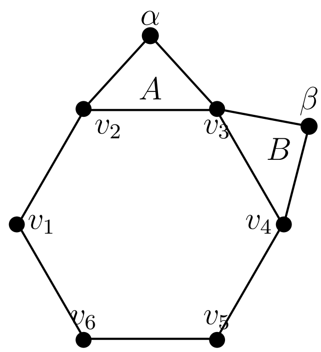

Figure C.16. A 6-face adjacent to a 2-group of triangles.

for the 6-face to have negative weight.

6.5 Triangle A is a $(3, 3, 4)$-triangle, $d(v_1) \leq 4$, and the contribution required by the remaining elements of the 6-face is at least $\frac{19}{12}$

6.6 Triangle A is a $(3, 3, 4)$-triangle and $d(v_1) = 5$. This forces $d(v_5) = d(v_6) = 3$ and triangle B to require a contribution of at least $\frac{8}{12}$.

6.7 Triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) \leq 4$, and the contribution of the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.8 Triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) \geq 5$, and triangle B requires a contribution of at $\frac{10}{12}$, $d(v_6) = d(v_5) = 3$. Note that v_1 contributes $\frac{2}{12}$ to the 6-face and hence triangle B must require at least $\frac{10}{12}$.

6.9 Triangles A and B require contributions of $\frac{9}{12}$, $d(v_1) = d(v_5) = 3$, $d(v_6) \leq 4$.

6.10 Triangles A and B require contributions of $\frac{9}{12}$, $d(v_1) = d(v_6) = 3$, $d(v_5) \leq 4$.

6.11* All of v_1, v_5 and v_6 are of degree 3, and the 2-group requires a contribution of at least $\frac{13}{12}$ but no more than $\frac{16}{12}$.

Adjacent to two triangles: next to each other

This configuration is illustrated in Figure C.17. Note that, for the weight on the 6-face to get negative, both of triangles A and B must require contributions of at least $\frac{6}{12}$ and at least one must require a contribution of at least $\frac{9}{12}$. We assume that triangle A requires the larger contribution without loss of generality.

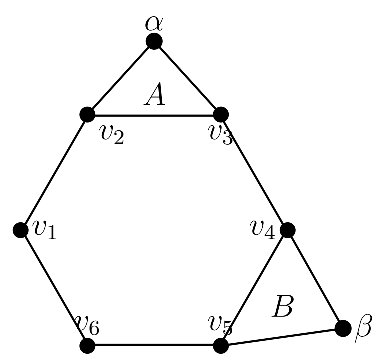

Figure C.17. A 6-face adjacent to two triangles, next to each other.

6.12 Triangle A is a $(3,3,3)$-triangle and $d(v_1) \leq 4$ or $d(v_4) \leq 4$.

6.13* Triangle A is a $(3,3,3)$-triangle, $d(v_1) \geq 5$, $d(v_4) \geq 5$. This forces $d(v_6) = 3$, otherwise the weight on the 6-face is non-negative. Observe that if $d(v_1) \geq 7$, then triangle B requires a contribution of at least $\frac{10}{12}$.

6.14 Triangle A is a $(3,3,4)$-triangle with $d(v_2) = 3$, and $d(v_1) \leq 4$.

6.15 Triangle A is a $(3,3,4)$-triangle with $d(v_2) = 3$, and $d(v_1) \geq 5$. This forces triangle B to be a $(3,3,4)$-triangle as well and $d(v_6) = 3$.

6.16 Triangle A is a $(3,3,4)$-triangle with $d(v_3) = 3$, and $d(v_4) \leq 4$.

6.17* Triangle A is a $(3,3,4)$-triangle with $d(v_3) = 3$, and $d(v_4) \geq 5$. Since triangle B may now require a contribution of no more than $\frac{10}{12}$, at least one of v_1 or v_6 is of degree 3 and the other is of degree no more than 4.

6.18 Triangle A requires a contribution of $\frac{10}{12}$ in which $d(v_2) = 3$, $d(v_1) = d(v_6) = 3$ and the contribution required by triangle B is at least $\frac{8}{12}$.

6.19* Triangle A requires a contribution of $\frac{10}{12}$ in which $d(v_2) \geq 4$, $d(v_1) = d(v_6) = 3$ and the contribution required by triangle B is at least $\frac{8}{12}$.

6.20* Triangle A requires a contribution of $\frac{9}{12}$, $d(v_1) = d(v_6) = 3$, and the contribution required by triangle B is at least $\frac{8}{12}$.

Adjacent to two triangle: on opposite sides

This configuration is illustrated in Figure C.18. Note that, for the weight on the 6-face to get negative, both of triangles A and B must require contributions of at least $\frac{6}{12}$ and at least one must require a contribution of at least $\frac{9}{12}$. We assume that triangle A always requires the maximum contribution of the triangles.

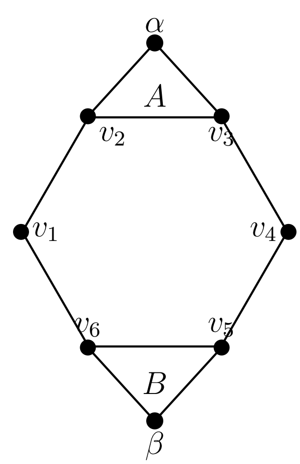

Figure C.18. A 6-face adjacent to two triangles, on opposite sides.

6.21 Triangle A is a $(3, 3, 3)$-triangle and $d(v_1) \leq 4$ or $d(v_4) \leq 4$.

6.22* Triangle A is a $(3, 3, 3)$-triangle, $d(v_1) = 5$, and $d(v_4) \geq 5$. This forces Triangle B to be a $(3, 3, 3)$-triangle as well. Note that, if both of v_1 and v_4 are of degree 6 or more, then the weight on the 6-face is non-negative.

6.23 Triangle A is a $(3,3,4)$-triangle in which $d(v_2) = 3$, $d(v_1) \leq 4$, $d(v_4) = 3$ and triangle B requires a contribution of at least $\frac{9}{12}$.

6.24* Triangle A is a $(3,3,4)$-triangle in which $d(v_2) = 3$, $d(v_1) \geq 5$, $d(v_4) = 3$, this forces triangle B to be a $(3,3,4)$-triangle also.

6.25* Triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) = d(v_4) = 3$, and triangle B requires a contribution of at least $\frac{8}{12}$

6.26* Triangle A requires a contribution of $\frac{9}{12}$, $d(v_1) = d(v_4) = 3$, and triangle B requires a contribution of $\frac{8}{12}$ or $\frac{9}{12}$.

Adjacent to three triangles: one a 3-group

This configuration is illustrated in Figure C.19. Note that, in order for the weight on the 6-face to be negative at least one of triangle A or B requires a contribution of at least $\frac{8}{12}$. We assume that triangle A requires the largest contribution of the triangles in the 3-group.

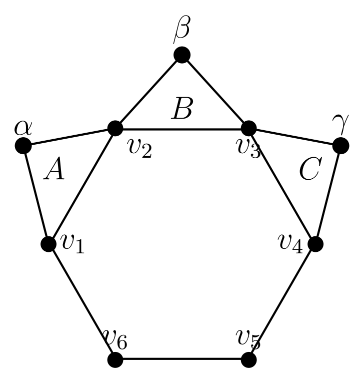

Figure C.19. A 6-face adjacent to a 3-group of triangles.

6.27 Triangle A is a $(3,3,4)$-triangle, $d(v_6) \leq 4$, and the contribution required by the remaining elements of the 6-face is at least $\frac{9}{12}$.

6.28 Triangle A is a $(3,3,4)$-triangle, triangle B is a $(3,4,4)$- or $(4,4,4)$-triangle, and the contribution required by the remaining elements of the 6-face is at least $\frac{9}{12}$.

6.29 Triangle A is a $(3,3,4)$-triangle, $d(v_6) \geq 5$, and triangle B requires a contribution of $\frac{2}{12}$ or less. This forces $d(v_5) = 3$ and triangle C to require a contribution of $\frac{9}{12}$ or more.

6.30 Triangle A requires a contribution of $\frac{10}{12}$, $d(v_6) \leq 4$, and the contribution required by the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.31 Triangle A requires a contribution of $\frac{10}{12}$, triangle B is a $(3,4,4)$- or $(4,4,4)$-triangle, and the contribution required by the remaining elements of the 6-face is at least $\frac{11}{12}$.

Note that if triangle A requires a contribution of $\frac{10}{12}$, triangle B requires a contribution of no more than $\frac{2}{12}$ and $d(v_6) \geq 5$, then the final weight on the 6-face is non-negative, since triangle B cannot require a contribution of more than $\frac{10}{12}$.

6.32* Triangle A requires a contribution of $\frac{9}{12}$. This forces triangle B to require a contribution of no more than $\frac{2}{12}$ and $d(v_5) = d(v_6) = 3$. Triangle C must require a contribution of at least $\frac{6}{12}$.

Note that if $d(v_3) \geq 5$, then triangle B requires no contribution from the 6-face and hence triangle C must require a contribution of at least $\frac{8}{12}$, and must be a $(3,4,5)$-, $(3,3,6+)$-, or $(3,3,7+)$-triangle.

Adjacent to three triangles: one 2-group and one isolated

This configuration is illustrated in Figure C.20.

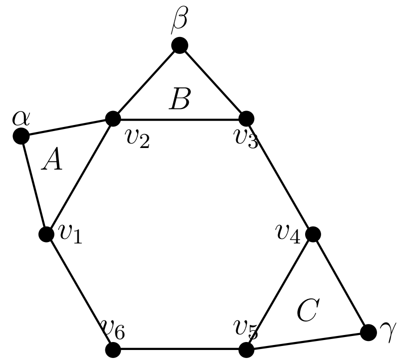

Figure C.20. A 6-face adjacent to a 3-group of triangles.

6.33 The 2-group requires a contribution of $\frac{22}{12}$ or more and the contribution required by the remaining elements of the 6-face is at least $\frac{1}{12}$.

6.34 Triangle C is a $(3,3,3)$-triangle, $d(v_6) \leq 4$ or $d(v_3) \leq 4$, and the contribution required by the remaining elements of the 6-face is at least $\frac{9}{12}$.

From this point forward, we assume that the 2-group requires a contribution of at least $\frac{12}{12}$, but no more than $\frac{20}{12}$; since, if it does not, the configuration must be of [6.24] type.

6.35 Triangle C is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle in which $d(v_4) = 3$, $d(v_3) \leq 4$, and the contribution required by the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.36 Triangle C is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle in which $d(v_5) = 3$, $d(v_6) \leq 4$, and the contribution required by the remaining elements of the 6-face is at least $\frac{11}{12}$.

From this point forward, we assume that if Triangle C is a $(3,3,3)$- $(3,3,4)$-, $(3,3,5)$- or $(3,4,4)$-triangle it does not fall into the configurations above and hence at least one of $d(v_6) \geq 5$ or $d(v_3) \geq 5$.

6.37* The 2-group requires a contribution of $\frac{20}{12}$ and the contribution required by the remaining elements of the configuration is at least $\frac{5}{12}$.

6.38* The 2-group requires a contribution of $\frac{18}{12}$ such that both triangles A and B are $(3,3,6)$-triangles, and the contribution required by the remaining elements in the configuration is at least $\frac{7}{12}$.

6.39 The 2-group requires a contribution of $\frac{18}{12}$ such that triangle A (or B) is a $(3,4,4)$-triangle and triangle B (respectively A) is a $(3,4,5)$-triangle.

6.40 The 2-group requires a contribution of $\frac{18}{12}$ such that the pair of triangles A and B is one of the pairs: $(3,3,4)$- and $(3,4,7+)$-, or $(3,4,4)$- and $(3,4,6)$-, or $(3,3,5)$- and $(3,4,5)$-, $d(v_6) \leq 4$, and the contribution required by triangle C is at least $\frac{3}{12}$.

6.41* The 2-group requires a contribution of $\frac{18}{12}$ such that the pair of triangles A and B is one of the pairs: $(3,3,4)$- and $(3,4,7+)$-, or $(3,4,4)$- and $(3,4,6)$-, or $(3,3,5)$- and $(3,4,5)$-, $d(v_6) \geq 5$, and the contribution required by triangle C is at least $\frac{3}{12}$.

6.42 The 2-group requires a contribution of $\frac{18}{12}$ such that the pair of triangles A and B is one of the ordered pairs: $(3,4,7+)$- and $(3,3,4)$-, or $(3,4,6)$- and $(3,4,4)$-, or $(3,4,5)$- and $(3,3,5)$-, $d(v_4) \leq 4$, and the contribution required by triangle C is at least $\frac{3}{12}$.

6.43* The 2-group requires a contribution of $\frac{18}{12}$ such that the pair of triangles A and B is one of the pairs: $(3,3,4)$- and $(3,4,7+)$-, or $(3,4,4)$- and $(3,4,6)$-, or $(3,3,5)$- and $(3,4,5)$-, $d(v_6) \geq 5$, and the contribution required by triangle C is at least $\frac{3}{12}$.

6.44 The 2-group requires a contribution of $\frac{16}{12}$ such that both A and B are $(3,4,5)$-triangles.

6.45 The 2-group requires a contribution of $\frac{16}{12}$ such that triangle A is a a $(3,3,5)$-triangle, triangle B is a $(3,5,5+)$-triangle, and $d(v_6) \leq 4$.

6.46* The 2-group requires a contribution of $\frac{16}{12}$ such that triangle A is a a $(3,3,5)$-triangle, triangle B is a $(3,5,5+)$-triangle, and $d(v_6) \geq 5$. This forces triangle C to require a contribution of at least $\frac{12}{12}$.

6.47* The 2-group requires a contribution of $\frac{16}{12}$, is not a member of [6.44], [6.45], [6.46], and the contribution required from the remaining elements of the 6-face is at least $\frac{9}{12}$.

6.48* The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,5)$-triangle, triangle B is a $(3,4,7+)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.49* The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,7+)$-triangle, triangle B is a $(3,4,5)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.50* The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,6)$-triangle, triangle B is a $(3,4,7+)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.51* The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,7+)$-triangle, triangle B is a $(3,4,6)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.52 The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,3,5)$-triangle,

triangle B is a $(3,5,7)$-triangle, $d(v_6) \leq 4$, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.53 The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,3,5)$-triangle, triangle B is a $(3,5,7)$-triangle, $d(v_6) \geq 5$. This forces triangle C to require a contribution of $\frac{16}{12}$.

6.54 The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,5,7+)$-triangle, triangle B is a $(3,3,5)$-triangle, $d(v_4) \leq 4$, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.55* The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,5,7+)$-triangle, triangle B is a $(3,3,5)$-triangle, $d(v_4) \geq 5$, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.56* The 2-group requires a contribution of $\frac{14}{12}$ in which one triangle is $(3,4,5)$-triangle and the other is a $(3,5,5)$- or $(3,5,6)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.57 The 2-group requires a contribution of $\frac{13}{12}$ or less in which triangle A is a $(3,4,5)$-triangle, triangle B is a $(4,4,4)$-triangle, and the contribution required by the remaining elements of the 6-face is at least $\frac{12}{12}$.

6.58* The 2-group requires a contribution of $\frac{13}{12}$ or less and is not as configuration [6.57]. The contribution required by the remaining elements of the 6-face is at least $\frac{12}{12}$.

Adjacent to three triangles: all isolated

This configuration is illustrated in Figure C.21. As usual, we split into subcases depending upon the contribution required by triangle A, and assume that triangle A requires the maximum contribution of any of the triangles in the configuration.

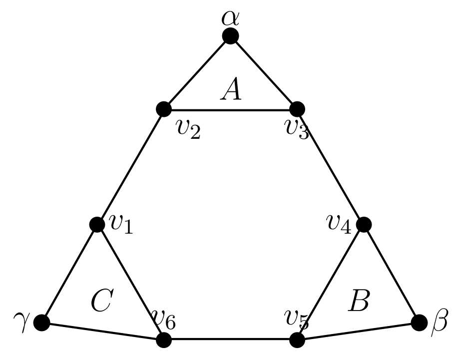

Figure C.21. A 6-face adjacent to three triangles.

6.59 Triangle A is a $(3,3,3)$-triangle and one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$. The contribution required by the remaining elements is at least $\frac{9}{12}$.

6.60* Triangle A is a $(3,3,3)$-triangle, both $d(v_1) \geq 5$, and $d(v_4) \geq 5$ and the contribution required by the remaining elements is at least $\frac{9}{12}$. This forces triangles B and C to be $(3, 3+, 5+)$-triangles.

6.61 Triangle A is a $(3,3,4)$-triangle in which α has degree 4, and one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$. The contribution required by the remaining elements is at least $\frac{13}{12}$.

6.62* Triangle A is a $(3,3,4)$-triangle in which α has degree 4, and both of $d(v_1) \geq 5$ and $d(v_4) \geq 5$. The contribution required by the remaining elements is at least $\frac{13}{12}$. This forces both of $d(v_5) \leq 4$ and $d(v_6) \leq 4$.

6.63 Triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4, $d(v_4) \leq 4$, and the contribution required by the remaining elements is at least $\frac{13}{12}$.

6.64 Triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4, $d(v_4) \geq 5$, and triangle C is a $(3,3,4)$-, $(3,3,5)$-, or $(3,4,4)$-triangle in which one of $d(v_1) = 3$ or $d(v_6) = 3$. Note that, if $d(v_5) \geq 5$, then the contribution required by triangle B is 0, and hence triangle C must require a contribution of $\frac{16}{12}$. So $d(v_5) \leq 4$.

6.65 Triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4, $d(v_4) \geq 5$, and triangle B is a $(3,3,5)$-triangle and $d(v_6) \leq 4$.

6.66 Triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4, $d(v_4) \geq 5$, and triangle B is a $(3,4,5)$-triangle, and triangle C is a $(3,3,6+)$-triangle in which $d(v_1) = d(v_6) = 3$.

6.67* Triangle A is a $(3,3,4)$-triangle in which v_2 is of degree 4, $d(v_4) \geq 5$, and the contribution required by the remaining elements is at least $\frac{13}{12}$. Note that, if $d(v_5) \geq 5$, then the contribution required by triangle B is 0, and hence triangle C must require a contribution of $\frac{16}{12}$. So $d(v_5) \leq 4$.

6.68 Triangle A is a $(3,3,5)$-triangle with $d(\alpha) = 5$, one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$, and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$.

6.69* Triangle A is a $(3,3,5)$-triangle with $d(\alpha) = 5$, both of $d(v_1) \geq 5$ and $d(v_4) \geq 5$, and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$. Note that this forces both $d(v_5) = 3$ and $d(v_6) = 3$.

6.70 Triangle A is a $(3,3,5)$-triangle in which $d(v_2) = 5$, $d(v_4) \leq 4$, and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$.

6.71* Triangle A is a $(3, 3, 5)$-triangle in which $d(v_2) = 5$, $d(v_4) \geq 5$, and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$. Note that this forces both $d(v_5) = 3$ and $d(v_6) = 3$.

6.72 Triangle A is a $(3, 4, 4)$-triangle in which $d(v_3) = 3$, $d(v_4) \leq 4$, and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$.

6.73* Triangle A is a $(3, 4, 4)$-triangle in which $d(v_2) = 5$, $d(v_4) \geq 5$, and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$. Note that this forces both $d(v_5) = 3$ and $d(v_6) = 3$.

6.74* Triangle A is a $(3, 3, 6)$-triangle and the contribution required by the remaining elements is at least $\frac{15}{12}$.

Note that any configuration in which the maximum weight discharged by a triangle is $\frac{8}{12}$ results in non-negative weight on the 6-face.

Adjacent to four triangles: as a 4-group

The configuration is illustrated in Figure C.22. Observe that in order for the final weight on the 6-face to be negative, at least one of triangles A or D must require a contribution of at least $\frac{8}{12}$. As with the previous cases, we assume triangle A requires the maximum weight.

6.75 Triangles B and C require a contribution totaling $\frac{8}{12}$, and the remaining elements require a contribution totaling at least $\frac{17}{12}$.

From this point forward we assume triangles B and C require a contribution totaling $\frac{6}{12}$ or less.

6.76 Triangle A requires a contribution of $\frac{12}{12}$, $d(v_6) \leq 4$, and the remaining elements require a contribution totaling at least $\frac{9}{12}$.

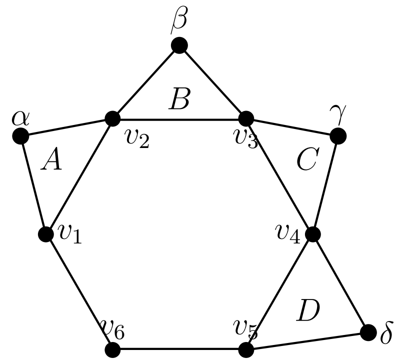

Figure C.22. A 6-face adjacent to a 4-group of triangles.

6.77 Triangle A requires a contribution of $\frac{12}{12}$, triangle B requires a contribution of $\frac{4}{12}$, and the remaining elements require a contribution totaling at least $\frac{9}{12}$.

6.78* Triangle A requires a contribution of $\frac{12}{12}$, $d(v_6) = 5$, and triangle B requires a contribution of no more than $\frac{2}{12}$. This forces triangle D to require a contribution of $\frac{10}{12}$ or more.

6.79 Triangle A requires a contribution of $\frac{10}{12}$, and the remaining elements require a contribution totaling $\frac{15}{12}$. This forces $d(v_6) \leq 4$.

6.80* Triangle A requires a contribution of $\frac{9}{12}$, and the remaining elements require a contribution totaling $\frac{16}{12}$. This forces $d(v_6) = 3$, triangle B to require a contribution of no more than $\frac{2}{12}$, and $d(v_3) = 4$.

6.81* Triangle A requires a contribution of $\frac{8}{12}$. This forces $d(v_6) = 3$, one of triangles B or C to require a contribution of $\frac{4}{12}$, and triangle D to require a contribution of at least $\frac{6}{12}$. These conditions cause $d(v_3) = 4$ and $d(v_5) = 3$.

Adjacent to four triangles: one 3-group and one isolated

The configuration is illustrated in Figure C.23. Note that if both $d(v_1) \geq 5$ and $d(v_4) \geq 5$, then the final weight on the 6-face is non-negative.

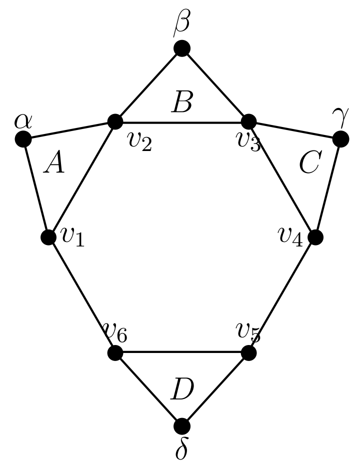

Figure C.23. A 6-face adjacent to a 3-group of triangles and one isolated triangle.

6.82 Triangle D requires a contribution of $\frac{16}{12}$, one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$, and the remaining elements require a contribution of $\frac{9}{12}$ or more.

6.83 Triangle D requires a contribution of $\frac{12}{12}$ in which $d(\delta) = 4$, one of $d(v_1) \leq 4$ or $d(v_4) \leq 4$, and the remaining elements require a contribution of $\frac{13}{12}$ or more.

6.84 Triangle D requires a contribution of $\frac{12}{12}$ in which $d(v_6) = 4$, $d(v_4) \leq 4$, and the contribution required by the other elements is at least $\frac{13}{12}$.

6.85* Triangle D requires a contribution of $\frac{12}{12}$ in which $d(v_6) = 4$, $d(v_4) \geq 5$, and the contribution required by the other elements is at least $\frac{13}{12}$. This forces triangle C to require a contribution of no more than $\frac{2}{12}$ and hence triangle A must require a contribution of at least $\frac{8}{12}$. Note that if triangle A requires a contribution of $\frac{8}{12}$, then triangle B must require a contribution of $\frac{4}{12}$ and hence $d(v_2) = 4$.

From this point forward, we assume that triangle D requires a contribution of $\frac{10}{12}$ or less.

6.86 Triangle A is a $(3,3,4)$-triangle, $d(v_6) \leq 4$ and the contribution required by the other elements is at least $\frac{3}{12}$.

6.87 Triangle A is a $(3,3,4)$-triangle, $d(v_6) = 5$, forcing triangle D to be a $(3,3,5)$-triangle, $d(v_4) \leq 4$, and the contribution required by the other elements is at least $\frac{3}{12}$.

6.88* Triangle A is a $(3,3,4)$-triangle, $d(v_6) = 5$, forcing triangle D to be a $(3,3,5)$-triangle, and $d(v_4) \geq 5$.

6.89* Triangle A is a $(3,3,4)$-triangle, $d(v_6) \geq 6$ and the contribution required by the remaining elements of the 6-face totals at least $\frac{13}{12}$

6.90 Triangle A is a $(3,4,4)$- or $(3,3,5)$-triangle, $d(v_6) \leq 4$ and the contribution required by the other elements is at least $\frac{5}{12}$.

6.91 Triangle A is a $(3,4,4)$- or $(3,3,5)$-triangle, $d(v_6) = 5$, forcing triangle D to be a $(3,3,5)$-triangle, $d(v_4) \leq 4$, and the contribution required by the other elements is at least $\frac{5}{12}$.

6.92* Triangle A is a $(3,4,4)$- or $(3,3,5)$-triangle, $d(v_6) \geq 6$, $d(v_4) \leq 4$, and the contribution required by the other elements is at least $\frac{6}{12}$.

6.93* Triangle A is a $(3,3,6)$-triangle, $d(v_4) \leq 4$, and the contributions required by the other elements is at least $\frac{16}{12}$.

6.94* Triangle A requires a contribution of $\frac{8}{12}$, $d(v_4) \leq 4$, and the contributions required by the other elements total at least $\frac{17}{12}$.

6.95* Triangle A requires a contribution of $\frac{6}{12}$, $d(v_4) \leq 4$, and the contributions required by the other elements total at least $\frac{19}{12}$.

From this point forward, we now assume that triangle D requires a contribution of $\frac{4}{12}$. Each of these configurations force triangle A to require a contribution of at least $\frac{9}{12}$.

6.96* Triangle D is a $(3,4,4)$-, or $(4,4,4+)$-triangle in which $d(v_6)=4$, triangle A requires a contribution of at least $\frac{9}{12}$, and the contribution required by the remaining elements of the 6-face totals at least $\frac{12}{12}$.

6.97 Triangle D is a $(4,4+,4+)$-triangle in which $d(v_6)=4$. This forces both of triangles A and C to require a contribution of $\frac{10}{12}$ or more.

6.98* Triangle D is a $(3,5,7+)$-, $(3,6,5)$-, $(3,6,6)$-, or $(3,4,7+)$-triangle in which $d(v_6)=3$, triangle A or C requires a contribution of at least $\frac{9}{12}$.

Adjacent to four triangles: two 2-groups

This configuration is illustrated in Figure C.24. We proceed assuming that triangle A requires the maximum contribution since the other options are symmetric.

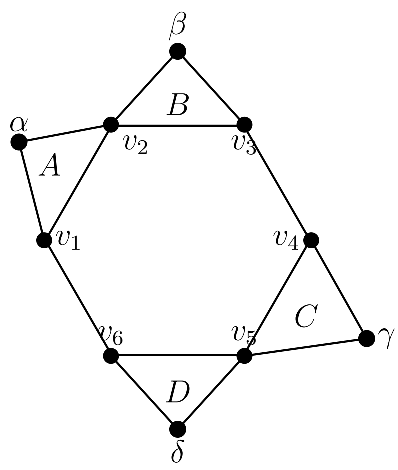

Figure C.24. A 6-face adjacent to two 2-groups.

6.99 One of the 2-groups requires a contribution of $\frac{22}{12}$ or more and the other requires a contribution of at least $\frac{1}{12}$.

6.100 Triangle A is a $(3, 3, 4)$-triangle and $d(v_6) \leq 4$, and the remaining triangles require a contribution totaling $\frac{13}{12}$ or more.

6.101 Triangle A is a $(3, 3, 4)$-triangle, triangle B is a $(3, 4, 5)$-, $(3, 4, 4)$- or $(4, 4, 4)$-triangle, and the remaining 2-group requires a contribution of at least $\frac{5}{12}$.

6.102* Triangle A is a $(3, 3, 4)$-triangle, $d(v_6) \geq 5$, $d(v_3) \leq 4$, and the contribution required by the remaining 2-group is at least $\frac{5}{12}$.

6.103* Triangle A is a $(3, 3, 4)$-triangle, $d(v_6) \geq 5$, $d(v_3) \geq 5$, and the contribution required by the remaining 2-group is at least $\frac{11}{12}$.

6.104 Triangle A is a $(3, 4, 4)$-triangle in which $d(v_1) = 3$, $d(v_6) \leq 4$, and the remaining triangles require a contribution of at least $\frac{15}{12}$.

6.105 Triangle A is a $(3, 4, 4)$-triangle in which $d(v_1) = 3$, triangle B is a $(3, 4, 5)$-, $(3, 4, 4)$- or $(4, 4, 4)$-triangle, and the remaining 2-group requires a contribution of at least $\frac{7}{12}$.

6.106* Triangle A is a $(3, 4, 4)$-triangle in which $d(v_1) = 3$, $d(v_6) \geq 5$, and the remaining triangles require a contribution of at least $\frac{15}{12}$. This forces $d(v_3) \leq 4$, otherwise the weight on the 6-face becomes non-negative.

6.107 Triangle A is a $(3, 3, 5)$-triangle, $d(v_6) \leq 4$, and the remaining triangles require a contribution totaling at least $\frac{15}{12}$.

6.108 Triangles A and B are $(3, 3, 5)$-triangles, $d(v_6) \geq 5$, and the remaining triangles require a contribution totaling at least $\frac{5}{12}$. This forces $d(v_4) \leq 5$.

6.109* Triangle A is a $(3,3,5)$-triangle in which $d(v_1) = 3$, $d(v_6) \geq 5$, and the remaining triangles require a contribution of at least $\frac{15}{12}$. This forces $d(v_3) \leq 4$, otherwise the weight on the 6-face becomes non-negative.

6.110* Triangle A and B are $(3,3,6)$-triangles, and the remaining triangles require a contribution of at least $\frac{7}{12}$.

6.111* Triangle A is a $(3,3,6)$-triangle and the remaining triangles require a contribution of at least $\frac{16}{12}$.

6.112* Triangle A requires a contribution of $\frac{8}{12}$ or less and the remaining triangles require a contribution of at least $\frac{17}{12}$.

Adjacent to five triangles: as a 5-group

This configuration is illustrated in Figure C.25. For the weight on the 6-face to be negative, at least one of v_5 or v_6 must be of degree 3. Furthermore, the weight on one of triangle A or E must be at least $\frac{8}{12}$, we will assume throughout this case that A has the larger weight, as the alternative is symmetric.

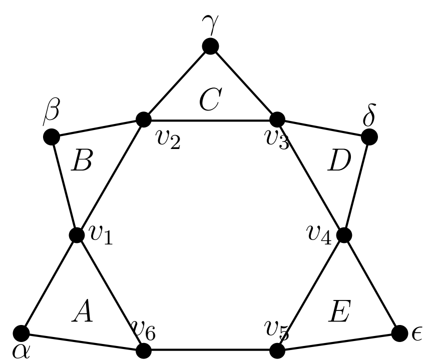

Figure C.25. A 6-face adjacent to five triangles as a 5-group.

6.113 Triangle A requires a contribution of $\frac{12}{12}$. This forces either triangle B to require a contribution of $\frac{4}{12}$ or $d(v_5) \leq 4$, otherwise the final weight on the 6-face is non-negative.

6.114 Triangle A requires a contribution of $\frac{10}{12}$. This forces $d(v_5) \leq 4$, otherwise the final weight on the 6-face is non-negative.

6.115 Triangles B and C (or C and D) require a contribution totaling $\frac{8}{12}$, and the remaining elements on the 6-face require a contribution of at least $\frac{17}{12}$.

From this point forward we assume that together triangles B and C, or C and D require a contribution totaling no more than $\frac{6}{12}$, forcing at least one of them to require a contribution of $\frac{2}{12}$ or less.

6.115* Triangle A requires a contribution of $\frac{9}{12}$. This forces the weight contribution on triangle B to be $\frac{2}{12}$ or less, and therefore $d(v_5) = 3$.

6.116* Triangles A and E require a contribution of $\frac{8}{12}$, triangles B and D require a contribution of $\frac{4}{12}$, and triangle C requires a contribution of $\frac{2}{12}$.

APPENDIX D

Faces at Distance at least 3 from each other in any graph G in $\mathcal{F}$

In each section we present the configurations of faces of size k that are required to be distance at least 3 from each other in the family $\mathcal{F}$. The general configuration is given in a figure and each of the vertices are labeled; specific configurations are then listed by giving the degree of each labeled vertex or the contribution required by the triangles.

7-Faces

We first detail the 7-faces.

Adjacent to three triangles: each isolated

This configuration is illustrated in Figure D.1 below.

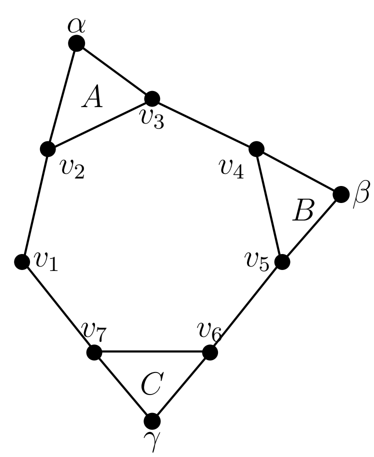

Figure D.1. A 7-face adjacent to three triangles.

7.22* Forbidden configuration: Triangle B is a $(3, 3, 3)$-triangle, $d(v_3) \geq 6$, $d(v_6) \geq 6$, $d(v_1) = 3$.

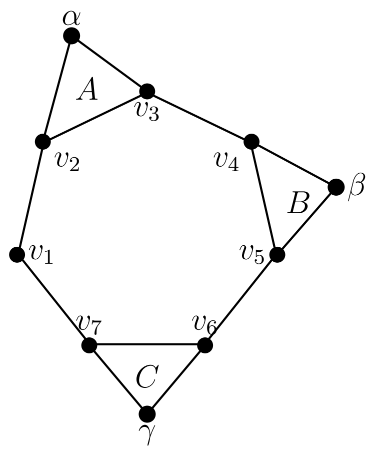

Figure D.2. A 7-face adjacent to three triangles.

Adjacent to four triangles: two 2-groups

This configuration is illustrated in Figure D.3.

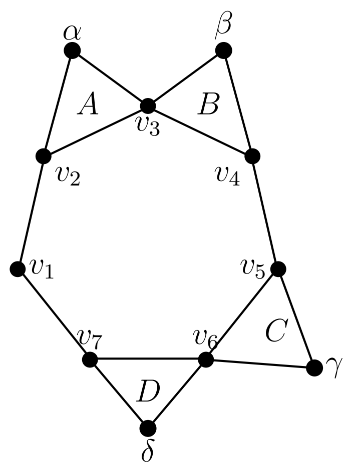

Figure D.3. A 7-face adjacent to four triangles as two 2-groups.

7.39* Forbidden Configuration: triangle A is a $(3, 3, 4)$-triangle $d(v_1) \geq 5$, triangle B is a $(3, 4, 6)$-triangle, triangles C and D are exactly as triangles B and A.

Adjacent to four triangles: one 2-group and two isolated triangles

This configuration is illustrated in Figure D.4.

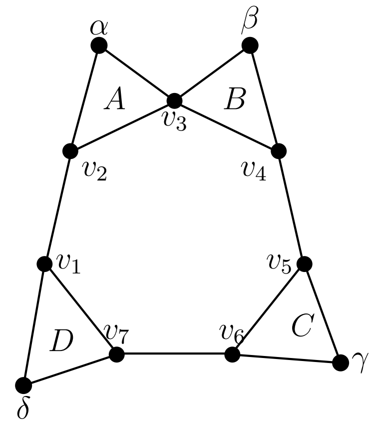

Figure D.4. A 7-face adjacent to four triangles, one 2-group, two isolated.

7.46* Forbidden Configuration: $d(v_1) = 5$, $d(v_2) = 3$, $d(\alpha) = 3, 4$, $d(v_3) = 4, 5, 6$, $d(\beta) = 3, 4$, $d(v_4) = 3$, $d(v_5) = 3$, $d(v_6) = 3$, $d(\gamma) = 6$, $d(v_7) = 3$, $d(\gamma) = 3$.

6-Faces

Multiple 6-faces are required to be distance at least 3 from each other in any graph in $\mathcal{F}$. We deal with each type of configuration separately.

Adjacent to two triangles: one 2-group

This configuration is illustrated in Figure D.5.

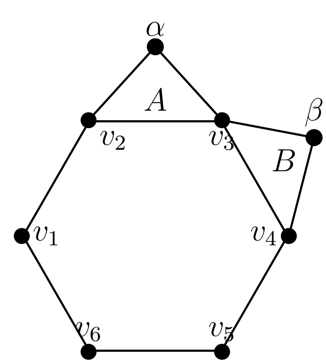

Figure D.5. A 6-face adjacent to a 2-group of triangles.

6.11* Forbidden Configuration: $d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) \geq 7$, $d(v_4) = 3$, $d(v_5) =$

3, $d(v_6) = 3$, $d(\alpha) = 3$, $d(\beta) = 3$.

Adjacent to two triangles: next to each other

This configuration is illustrated in Figure D.6.

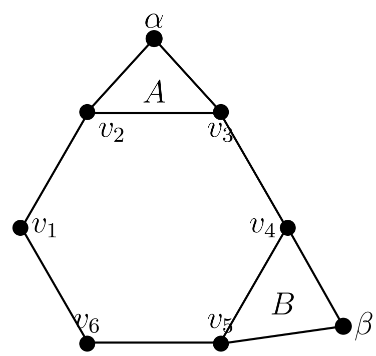

Figure D.6. A 6-face adjacent to two triangles, next to each other.

6.13*a Forbidden Configuration: Triangle A is a $(3, 3, 3)$-triangle, $d(v_1) = 5$, and triangle B is a $(3, 3, 7+)$-triangle.

6.13*b Forbidden Configuration: Triangle A is a $(3, 3, 3)$-triangle, $d(v_1) = 6$, and triangle B is a $(3, 3, 6)$-triangle.

6.17*a Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle with $d(v_3) = 3$, triangle B is a $(3, 3, 6)$-triangle with $d(v_4) = 6$, and only one of $d(v_1) = 3$ or $d(v_6) = 3$, the other is of degree 4.

6.17*b Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle with $d(v_3) = 3$, triangle B requires a contribution of $\frac{8}{12}$ with $d(v_4) \geq 5$, $d(v_1) = 3$, and $d(v_6) = 3$.

6.19* Forbidden Configuration: Triangle A requires a contribution of $\frac{10}{12}$ with $d(v_2) \geq 4$ and $d(v_4) \geq 5$.

6.20* Forbidden Configuration: Triangle A requires a contribution of $\frac{9}{12}$, $d(v_1) = d(v_6) = 3$, and the contribution required by triangle B is at least $\frac{8}{12}$.

Adjacent to two triangle: on opposite sides

This configuration is illustrated in Figure D.7.

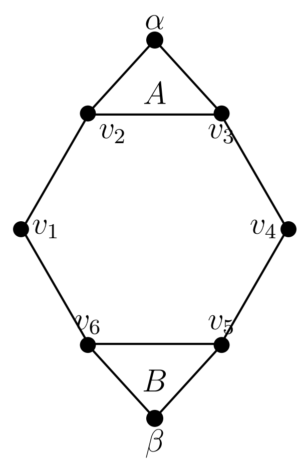

Figure D.7. A 6-face adjacent to two triangles, on opposite sides.

6.22* Forbidden Configuration: Triangle A is a $(3, 3, 3)$-triangle, $d(v_1) = 5$, and $d(v_4) = 6$.

6.24* Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle in which $d(v_2) = 3$ and $d(v_3) = 4$, $d(v_1) \geq 5$, $d(v_4) = 3$, and triangle B is a $(3, 3, 4)$-triangle in which $d(v_6) = 3$ and $d(v_5) = 4$.

6.25* Forbidden Configuration: Triangle A requires a contribution of $\frac{10}{12}$, $d(v_1) = d(v_4) = 3$, and triangle B requires a contribution of at least $\frac{8}{12}$ in which $d(v_5) \geq 6$.

6.26* Forbidden Configuration: Triangle A requires a contribution of $\frac{9}{12}$, $d(v_1) = d(v_4) = 3$, and triangle B requires a contribution of $\frac{8}{12}$ or $\frac{9}{12}$.

Adjacent to three triangles: one a 3-group

This configuration is illustrated in Figure D.8.

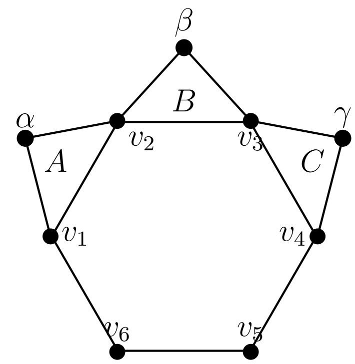

Figure D.8. A 6-face adjacent to a 3-group of triangles.

6.32*a Forbidden Configuration: Triangle A requires a contribution of $\frac{9}{12}$, triangle B requires a contribution of no more than $\frac{2}{12}$, $d(v_5) = d(v_6) = 3$, triangle C is a $(3, 4, 6)$-triangle.

6.32*b Forbidden Configuration: Triangle A requires a contribution of $\frac{9}{12}$, $d(v_3) \geq 5$, $d(v_5) = d(v_6) = 3$, triangle C is not a $(3, 4, 5)$-, $(3, 3, 6+)$-, or $(3, 3, 7+)$-triangle.

Adjacent to three triangles: one 2-group and one isolated

This configuration is illustrated in Figure D.9.

6.37*a Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle, $d(v_6) \geq 5$, triangle B is a $(3, 4, 6)$-triangle, and triangle C requires a contribution of at least $\frac{8}{12}$.

6.37*b Forbidden Configuration: Triangle B is a $(3, 3, 4)$-triangle, $d(v_4) \geq 5$, triangle A is a $(3, 4, 6)$-triangle, and triangle C requires a contribution of at least $\frac{8}{12}$.

6.37*c Forbidden Configuration: Triangles A and B are $(3, 3, 5)$-triangles, $d(v_6) \geq 5$, $d(v_4) \geq 5$, and triangle C requires a contribution of at least $\frac{8}{12}$.

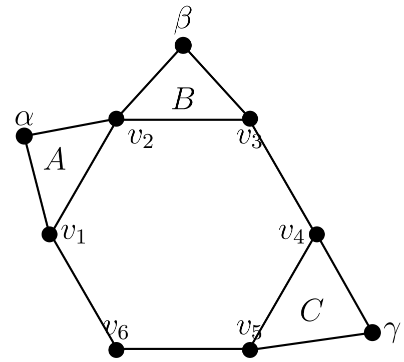

Figure D.9. A 6-face adjacent to a 3-group of triangles.

6.38*a Forbidden Configuration: Triangles A and B are $(3,3,6)$-triangles, $d(v_6) \geq 4$, or $d(v_4) \geq 4$, and triangle C requires a contribution of at least $\frac{8}{12}$.

6.41* Forbidden Configuration: The pair of triangles A and B is one of the pairs: $(3,3,4)$- and $(3,4,7+)$-, or $(3,4,4)$- and $(3,4,6)$-, or $(3,3,5)$- and $(3,4,5)$-, $d(v_6) \geq 5$, and the contribution required by triangle C is at least $\frac{3}{12}$.

6.43* Forbidden Configuration: The pair of triangles A and B is one of the pairs: $(3,3,4)$- and $(3,4,7+)$-, or $(3,4,4)$- and $(3,4,6)$-, or $(3,3,5)$- and $(3,4,5)$-, $d(v_6) \geq 5$, and the contribution required by triangle C is at least $\frac{3}{12}$.

6.46* Forbidden Configuration: Triangle A is a a $(3,3,5)$-triangle, triangle B is a $(3,5,5+)$-triangle, $d(v_6) \geq 5$, triangle C is a $(3,3,4)$-triangle in which $d(v_4) = 4$.

6.47* Forbidden Configuration: The 2-group requires a contribution of $\frac{16}{12}$, is not a member of [6.44], [6.45], [6.46], and the contribution required from the remaining elements of the 6-face is at least $\frac{9}{12}$.

6.48* Forbidden Configuration: The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,5)$-triangle, triangle B is a $(3,4,7+)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.49* Forbidden Configuration: The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,7+)$-triangle, triangle B is a $(3,4,5)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.50* Forbidden Configuration: The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,6)$-triangle, triangle B is a $(3,4,7+)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.51* Forbidden Configuration: The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,4,7+)$-triangle, triangle B is a $(3,4,6)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.55* Forbidden Configuration: The 2-group requires a contribution of $\frac{14}{12}$ and triangle A is a $(3,5,7+)$-triangle, triangle B is a $(3,3,5)$-triangle, $d(v_4) \geq 5$, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.56* Forbidden Configuration: The 2-group requires a contribution of $\frac{14}{12}$ in which one triangle is $(3,4,5)$-triangle and the other is a $(3,5,5)$- or $(3,5,6)$-triangle, and the contribution required from the remaining elements of the 6-face is at least $\frac{11}{12}$.

6.58* The 2-group requires a contribution of $\frac{13}{12}$ or less and is not as configuration [6.57]. The contribution required by the remaining elements of the 6-face is at least $\frac{12}{12}$.

Adjacent to three triangles: all isolated

This configuration is illustrated in Figure D.10.

6.60* Forbidden Configuration: Triangle A is a $(3,3,3)$-triangle, both $d(v_1) \geq 5$,

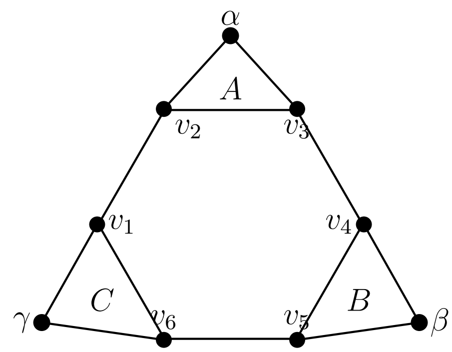

Figure D.10. A 6-face adjacent to three triangles.

and $d(v_4) \geq 5$ and the triangle B is a $(3, 4+, 5)$-, $(3, 3+, 6)$-, or $(3, 3+, 7+)$-triangle.

6.62* Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle in which α has degree 4, and both of $d(v_1) \geq 5$ and $d(v_4) \geq 5$, and neither triangle B nor C is a $(3, 3, 5)$-triangle.

6.67* Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle in which v_2 is of degree 4, $d(v_4) \geq 5$, and the contribution required by the remaining elements is at least $\frac{13}{12}$ and not in one of the configurations [6.63], [6.64], [6.65], or [6.66].

6.69* Forbidden Configuration: Triangle A is a $(3, 3, 5)$-triangle with $d(\alpha) = 5$ both of $d(v_1) \geq 5$ and $d(v_4) \geq 5$, the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$, and neither triangle B nor C is a $(3, 3, 5)$-triangle.

6.71* Forbidden Configuration: Triangle A is a $(3, 3, 5)$-triangle in which $d(v_2) = 5$, $d(v_4) \geq 5$, the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$, neither triangle B nor C is a $(3, 3, 5)$-triangle, and triangle C is not a $(3, 4, 4)$-triangle.

6.73* Forbidden Configuration: Triangle A is a $(3,4,4)$-triangle in which $d(v_2) = 5$, $d(v_4) \geq 5$,the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$, neither triangle B nor C is a $(3,3,5)$-triangle, and triangle C is not a $(3,4,4)$-triangle.

6.74* Forbidden Configuration: Triangle A is a $(3,3,6)$-triangle and the contribution required by the remaining elements of the configuration is at least $\frac{15}{12}$.

Adjacent to four triangles: as a 4-group

The configuration is illustrated in Figure D.11.

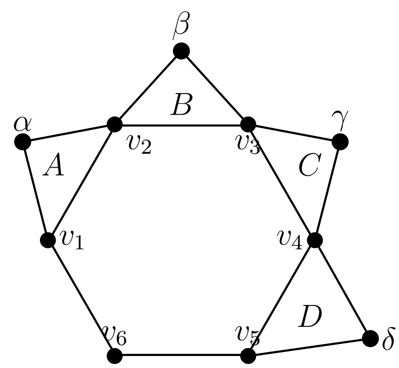

Figure D.11. A 6-face adjacent to a 4-group of triangles.

6.78* Forbidden Configuration: Triangles A and D require a contribution of $\frac{12}{12}$, $d(v_6) = 5$, triangles B and C require a contribution of $\frac{2}{12}$ and both $d(\beta) = 6$ and $d(\gamma) = 6$.

6.80* Forbidden Configuration: Triangle A requires a contribution of $\frac{9}{12}$, $d(v_6) = 3$, triangle B requires a contribution of no more than $\frac{2}{12}$,$d(v_3) = 4$, and triangle C requires a contribution of less than $\frac{4}{12}$.

6.81*a Forbidden Configuration: Triangles A and B are $(3,4,6)$-triangles, $d(v_6) = 3$, one of triangles B or C requires a contribution of $\frac{4}{12}$, and triangle D requires a contribution of at least $\frac{6}{12}$. These conditions cause $d(v_3) = 4$ and $d(v_5) = 3$.

6.81*b Forbidden Configuration: Triangle A is a $(3, 4, 5)$-triangle in which $d(v_2) = 5$, and both $d(\beta) = d(\gamma) = 4$, $d(v_6) = 3$, one of triangles B or C requires a contribution of $\frac{4}{12}$, and triangle D requires a contribution of at least $\frac{6}{12}$. These conditions cause $d(v_3) = 4$ and $d(v_5) = 3$.

6.81*c Forbidden Configuration: Triangle A is a $(3, 3, 7+)$-triangle, $d(v_3) = d(v_4) = 4$, and triangle D is not a $(3, 4, 5)$-triangle.

Adjacent to four triangles: one 3-group and one isolated

The configuration is illustrated in Figure D.12.

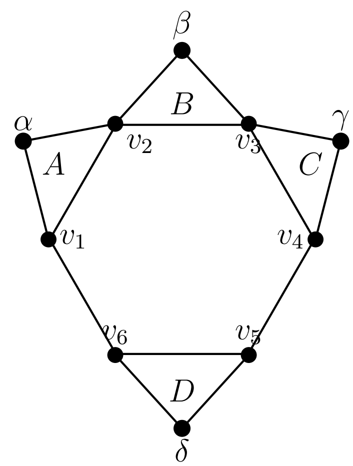

Figure D.12. A 6-face adjacent to a 3-group of triangles and one isolated triangle.

6.85*a Forbidden Configuration: Triangle D requires a contribution of $\frac{12}{12}$, in which $d(v_6) = 4$, $d(v_4) \geq 5$, triangle C requires a contribution of no more than $\frac{2}{12}$, and triangle A is a $(3, 3, 6)$-triangle.

6.85*b Forbidden Configuration: Triangle D requires a contribution of $\frac{12}{12}$, in which

$d(v_6) = 4$, $d(v_4) \geq 5$, triangle C requires a contribution of no more than $\frac{2}{12}$, and triangle A is a $(3,4,6)$-triangle.

6.88* Forbidden Configuration: Triangle A is a $(3,3,4)$-triangle, $d(v_6) = 5$, forcing triangle D to be a $(3,3,5)$-triangle, $d(v_4) \geq 5$, and triangle C requires a contribution of $\frac{2}{12}$ or less.

6.89* Forbidden Configuration: Triangle A is a $(3,3,4)$-triangle, $d(v_6) \geq 6$, and the remaining elements require a contribution of $\frac{13}{12}$ or more.

6.92* Triangle A is a $(3,4,4)$- or $(3,3,5)$-triangle, $d(v_6) \geq 6$, $d(v_4) \leq 4$, and the contribution required by the other elements is at least $\frac{6}{12}$.

6.93*a Forbidden Configuration: Triangle A is a $(3,3,6)$-triangle, $d(v_4) \leq 4$, triangle D is a $(3,3,6)$-triangle or requires a contribution of $\frac{8}{12}$ or less, and the contributions required by the other elements total at least $\frac{7}{12}$.

6.93*b Forbidden Configuration: Triangle A is a $(3,3,6)$-triangle, $d(v_4) \leq 4$, triangle D is a $(3,3,6)$-triangle or requires a contribution of $\frac{8}{12}$ or less, and the contributions required by the other elements total at least $\frac{7}{12}$.

6.94* Forbidden Configuration: Triangle A requires a contribution of $\frac{8}{12}$, $d(v_4) \leq 4$, and triangle D is a $(3,3,6)$-triangle or requires a contribution of $\frac{8}{12}$ or less, and the contributions required by the other elements total at least $\frac{7}{12}$.

6.95* Forbidden Configuration: Triangle A requires a contribution of $\frac{6}{12}$, $d(v_4) \leq 4$, and triangle D is a $(3,3,6)$-triangle or requires a contribution of $\frac{8}{12}$ or less, and the contributions required by the other elements total at least $\frac{9}{12}$.

6.96* Triangle D is a $(3,4,4)$-, or $(4,4,4+)$-triangle in which $d(v_6) = 4$, triangle A requires a contribution of exactly $\frac{9}{12}$, and the contribution required by the remaining elements of the 6-face totals at least $\frac{12}{12}$.

6.98*a Forbidden Configuration: Triangle D is a $(3,5,7+)$-, $(3,6,5)$-, $(3,6,6)$-, or $(3,4,7+)$-triangle in which $d(v_6) = 3$, triangle A is a $(3,3,6)$-triangle and $d(\beta) \geq 4$.

6.98*b Forbidden Configuration: Triangle D is a $(3,5,7+)$-, $(3,6,5)$-, $(3,6,6)$-, or $(3,4,7+)$-triangle in which $d(v_6) = 3$, triangle A is a $(3,3,6)$-triangle and triangle C is not a $(3,3,4)$-, $(3,4,4)$-, or $(4,4,4)$-triangle.

6.98*c Forbidden Configuration: Triangle D is a $(3,5,7+)$-, $(3,6,5)$-, $(3,6,6)$-, or $(3,4,7+)$-triangle in which $d(v_6) = 3$, triangle A is requires a contribution of $\frac{8}{12}$ or less.

Adjacent to four triangles: two 2-groups

This configuration is illustrated in Figure D.13.

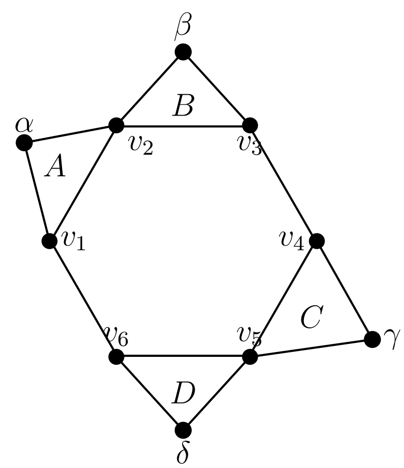

Figure D.13. A 6-face adjacent to two 2-groups.

6.102* Forbidden Configuration: Triangle A is a $(3,3,4)$-triangle, $d(v_6) \geq 5$, $d(v_3) \leq 4$, and triangle C is not a $(3,3,4)$-, $(3,4,4)$-, or $(3,3,5)$-triangle in which $d(v_4) = 3$.

6.103* Forbidden Configuration: Triangle A is a $(3, 3, 4)$-triangle, $d(v_6) \geq 5$, $d(v_3) \geq 5$, and the contribution required by the remaining 2-group is at least $\frac{11}{12}$.

6.106* Forbidden Configuration: Triangle A is a $(3, 4, 4)$-triangle in which $d(v_1) = 3$, $d(v_6) \geq 5$, and triangle C is not a $(3, 3, 4)$-, $(3, 4, 4)$-, or $(3, 3, 5)$-triangle in which $d(v_4) = 3$.

6.109* Forbidden Configuration: Triangle A is a $(3, 3, 5)$-triangle in which $d(v_1) = 3$, $d(v_6) \geq 5$, and triangle C is not a $(3, 3, 4)$-, $(3, 4, 4)$-, or $(3, 3, 5)$-triangle in which $d(v_4) = 3$.

6.110* Forbidden Configuration: Triangle A and B are $(3, 3, 6)$-triangles, $d(v_4) \geq 4$ or $d(v_6) \geq 4$.

6.111* Forbidden Configuration: Triangle A is a $(3, 3, 6)$-triangle and the remaining triangles require a contribution of at least $\frac{16}{12}$.

6.112* Forbidden Configuration: Triangle A requires a contribution of $\frac{8}{12}$ or less, and at least one of triangle A and B is not a $(3, 4, 5)$-triangle and the remaining triangles require a contribution of at least $\frac{17}{12}$.

Adjacent to five triangles: as a 5-group

This configuration is illustrated in Figure D.14.

6.115* Forbidden Configuration: Triangle A requires a contribution of $\frac{9}{12}$, Triangle E is a $(3, 3, 6)$-triangle, triangle C requires a contribution of $\frac{2}{12}$ and triangle D requires a contribution of $\frac{4}{12}$.

6.116* Forbidden Configuration: Triangles A and E are $(3, 4, 6)$-triangles, triangles B and D require a contribution of $\frac{4}{12}$, and triangle C requires a contribution of $\frac{2}{12}$.

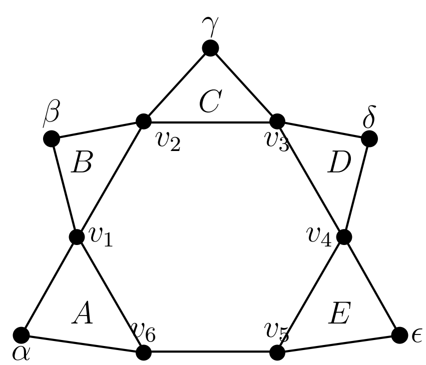

Figure D.14. A 6-face adjacent to five triangles as a 5-group.

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