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BIOGRAPHIES

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ABSTRACT

GPS receivers convert the measured pseudoranges from the visible GPS satellites into an estimate of the position and clock offset of the receiver. For various reasons receivers might only track and process a subset of the visible satellites. It would be desired, of course, to use the best subset.

In general selecting the best subset is a combinatorics problem; selecting m objects from a choice of n allows for $\binom{n}{m}$ potential subsets. And since the GDOP performance criterion is nonlinear and non-separable, finding the best subset is a brute force procedure; hence, a number of authors have described sub-optimal algorithms for choosing satellites.

This paper revisits this problem, especially in the context of multiple GNSS constellations, for the GDOP and PDOP criteria. Included are a discussion of optimum constellations (based upon parallel work of these authors on achievable lower bounds to GDOP and PDOP), musings on how the non-separableness of DOP makes it impossible to rank order the satellites, and a review/discussion of subset selection algorithms. Our long term goal is the development of better selection algorithms for multi-constellation GNSS.

INTRODUCTION

GNSS receivers convert the measured pseudoranges into an estimate of the position and the clock offset of the receiver.

The typical implementation of the solution algorithm is an iterative, linearized least squares method [1]. Assuming that pseudoranges from m satellites are measured the "direction cosines" matrix $\mathbf{G}$ is formed. Using an East, North, and Up coordinate frame this matrix is of the form

$$\mathbf{G} = \begin{bmatrix} e_1 & n_1 & u_1 & 1 \\ e_2 & n_2 & u_2 & 1 \\ \vdots & \vdots & & \\ e_m & n_m & u_m & 1 \end{bmatrix} \tag{1}$$

in which (e_k, n_k, u_k) is the unit vector toward the k^{th} satellite from the assumed receiver position. This matrix is then used to form the pseudoinverse to solve the overdetermined equations. Since the pseudoranges themselves are noisy, the resulting solution is random. To describe the accuracy of the solution it is common to describe it statistically via the error covariance matrix, equal to the inverse of $\mathbf{G}^T\mathbf{G}$. Rather than considering the individual elements of the covariance matrix it is common to reduce it to a scalar parameter. The most used reduction is the GDOP (Geometric Dilution of Precision), the square root of the trace of this covariance matrix

$$\text{GDOP} = \sqrt{\text{trace}\left\{(\mathbf{G}^T\mathbf{G})^{-1}\right\}}$$

equivalently, the square root of the sum of the variances of these four estimates. Other popular measures of performance are HDOP (Horizontal DOP), PDOP (Position DOP), and TDOP (Time DOP).

For multiple constellations that are not synchronized these definitions are extended by appending additional columns to Eq. (1) to account for different clock biases. For L constellations $\mathbf{G}$ is of the form

$$\mathbf{G} = \begin{bmatrix} e_{1,1} & n_{1,1} & u_{1,1} & 1 & 0 & 0 & \dots & 0 \\ & \vdots & & \vdots & & & \vdots & \\ e_{1,m_1} & n_{1,m_1} & u_{1,m_1} & 1 & 0 & 0 & \dots & 0 \\ e_{2,1} & n_{2,1} & u_{2,+1} & 0 & 1 & 0 & \dots & 0 \\ e_{2,2} & n_{2,2} & u_{2,2} & 0 & 1 & 0 & \dots & 0 \\ & \vdots & & \vdots & & & \vdots & \\ e_{L,m_L} & n_{L,m_L} & u_{L,m_L} & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

(the first of the two subscripts is the constellation number, 1 to L; the second is the satellite number within the k^{th} constellation, 1 to m_k) and the GDOP is still

$$\text{GDOP} = \sqrt{\text{trace}\left\{(\mathbf{G}^T\mathbf{G})^{-1}\right\}}$$

but now includes the variances of $L+3$ variables, three for the receiver's position and those of the L clock biases. The inclusion of the variances of the L clock biases makes GDOP a poor choice when comparing subsets of satellites; for example, the best subset for GDOP might limit itself to a poorer geometric choice of satellite locations just to eliminate the estimation of an extra clock bias. A similar performance metric that resolves this problem is Position DOP (PDOP), dependent only on the position variances. Defining

$$\mathbf{H} = \left(\mathbf{G}^T\mathbf{G}\right)^{-1}$$

then PDOP is a function of the first three diagonal terms

$$\text{PDOP} \equiv \sqrt{\text{trace}\left\{\mathbf{H}_{[1,1]} + \mathbf{H}_{[2,2]} + \mathbf{H}_{[3,3]}\right\}}$$

the variances of the three position variables of the estimate.

In some instances a receiver cannot process all of the visible satellites. For example, the issue might be

- that the receiver physically cannot track all of the potential signals – this might be a hardware limitation (a fixed number of channels) or the desire to minimize power usage. We note that modern, and expensive, GNSS receivers can track in excess of 100 signals.
- that the receiver is using corrections from some augmentation system and that the bandwidth of the correction channel is insufficient to provide information for all of the visible satellites (see, for example, [2]).

In such a case the question arises: "If only m of the n visible satellites can be processed, which m should they be?" This problem has been considered in the literature, primarily for one constellation. If the receiver limits its attention to the 15 or so visible GPS satellites a brute force comparison of all subsets is possible (for $n = 15$, $\binom{15}{m} < 6,500$ for all $3 < m < 15$, well within modern computational range).

The advent of other constellations such as Galileo, and Beidou exacerbates this problem. For example, with 35 visible satellites (such as frequently occurs with GPS, GLONASS, and Galileo) brute force comparison is no longer viable (e.g. with $n = 35$, $\binom{35}{20} > 3$ billion, impossible for real time usage). We note that ad-hoc and sub-optimum algorithms for subset selection have been developed and are reviewed below; however, these have been developed under the assumption of a single constellation and do not exploit any characteristics of the multi-constellation problem.

The longer term goal of these authors is to develop effective algorithms for choosing between many satellites from multiple constellations; the current paper describes some of our early thinking on this topic. It begins with a review of some parallel work of these authors on lower bounds to GDOP and PDOP performance; specifically, we consider the characteristics of optimum satellite sets for multiple constellations. This discussion includes several examples. Next, we revisit the concept of ranking individual satellites, arguing that the such methods cannot work and that sub-optimum methods *must* be explored. We then review the sub-optimum selection algorithms developed to date. Finally, we suggest possible directions for future study.

BOUNDS ON DOP

In a parallel effort these authors have developed tight lower bounds to GDOP and PDOP for multi-constellation GNSS [3]. In brief:

- The bounds are valid for single or multiple constellations.
- Both GDOP and PDOP are lower bounded.
- The bounds assume that all satellites are above some selected mask angle α with $\alpha \geq 0°$.
- The resulting PDOP bound is a function of *only* the total number of satellites m and the mask angle α; the GDOP bound adds the number of constellations. For best GDOP performance the numbers of satellites per constellation should be equal (essentially this is due to the fact that each constellation's clock variance is included in GDOP and equalizing the satellite counts minimizes the sum of those variances).

- Constellations that achieve the bounds are described.

Most relevant to the work here is this last bullet; constructions of optimum constellations. For a mask angle of 0° the best constellations for both GDOP and PDOP consists of approximately 30% of the satellites directly overhead and the remaining 70% distributed about the horizon in a "balanced" pattern (as the mask angle increases above 0° the fraction of satellites at zenith increases as well and the remaining satellites move up to being balanced at the mask angle).

In [3] balance is described in terms of the components of the unit vectors to the m satellites. For satellites from a single constellation balance includes conditions on the East, North, and Up components of each satellites's position

$$\sum_{k=1}^{m} e_k = \sum_{k=1}^{m} n_k = 0 \tag{2}$$

conditions on the squares of those components

$$\sum_{k=1}^{m} e_k^2 = \sum_{k=1}^{m} n_k^2 = \frac{m}{2} \tag{3}$$

and conditions on the products of those components

$$\sum_{k=1}^{m} e_k n_k = \sum_{k=1}^{m} e_k u_k = \sum_{k=1}^{m} n_k u_k = 0 \tag{4}$$

In [4] it was suggested that constellations resulting in small GDOP might consist of $m - p$ satellites evenly spaced in azimuth at the horizon and the remaining p directly overhead, for some integer p, $0 < p < m$

$$e_k = \begin{cases} \sin\left(\frac{2\pi k}{m-p} + \phi\right) & ; \quad k = 1, 2, \ldots m-p \\ 0 & ; \quad k = m-p+1, \ldots m \end{cases}$$

$$n_k = \begin{cases} \cos\left(\frac{2\pi k}{m-p} + \phi\right) & ; \quad k = 1, 2, \ldots m-p \\ 0 & ; \quad k = m-p+1, \ldots m \end{cases}$$

$$u_k = \begin{cases} 0 & ; \quad k = 1, 2, \ldots m-p \\ 1 & ; \quad k = m-p+1, \ldots m \end{cases}$$

in which ϕ is an arbitrary rotation in azimuth. Such a constellation does, in fact, meet the balance conditions in Eqs. (2), (3), and (4) as long as $m - p \geq 3$ (this can be shown by judicious use of Lagrange's trigonometric identities).

As an example, let $m = 10$. For minimum GDOP or PDOP the constellation should consist of 3 satellites at zenith and 7 distributed about the horizon as shown in

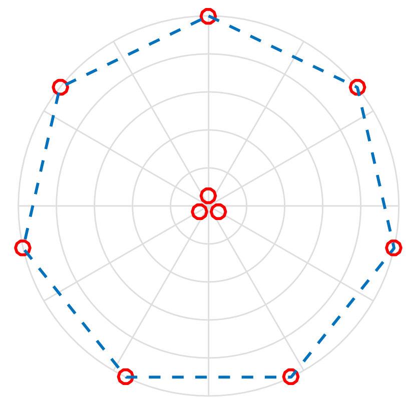

Figure 1: An $m = 10$ constellation that achieves minimum GDOP and PDOP.

Figure 1 (the horizon satellites can be rotated by any arbitrary angle without violating balance; the three at zenith are spread out in this image for visibility). As the mask angle increases this picture is unchanged for GDOP (except that the horizon satellites move up in elevation) until the mask angle goes above 21°. At that point the optimum constellation changes to 4 satellites at zenith and 6 at the mask angle; for PDOP this change occurs at a mask angle of 20°.

Interestingly, other $m = 10$ constellations exist that meet the balance conditions. While they still have approximately 30% of the satellites at zenith the placement of the 7 at the horizon can also consist of two grouping, 3 in a triangle and 4 in a square, each group with an arbitrary angle of rotation; see Figure 2 for an example.

For larger m additional optimum constellations result. Consider $m = 13$ so that 30% is 4 satellites at zenith and the remaining 9 at the horizon. Possible locations

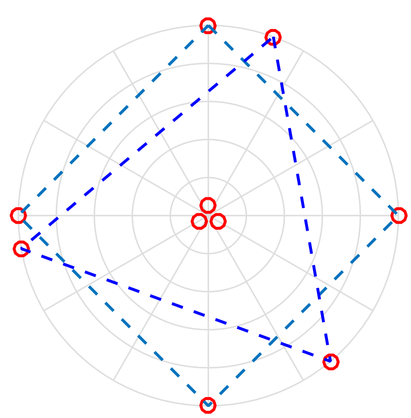

Figure 2: Another $m = 10$ constellation that achieves minimum GDOP and PDOP.

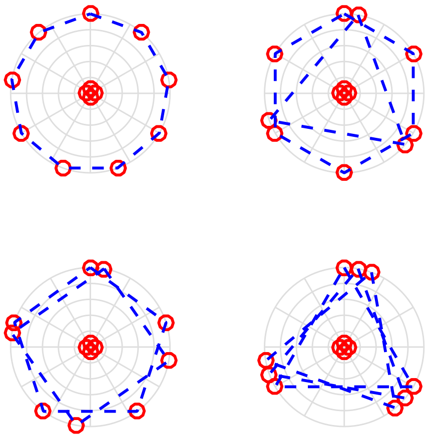

Figure 3: Contenders for the optimum $m = 13$ constellation.

for the horizon satellites include:

- 9 locations each separated by 40°.
- 6 satellites separated by 60° and another 3 separated by 120°. The two groups can have arbitrary rotation, including three paired locations!
- 5 satellites separated by 72° and the remaining 4 separated by 90°, again with arbitrary rotation.
- Three groups of 3 satellites with 120° spacing.

Figure 3 shows several possibilities. All of these are balanced, so achieve the GDOP and PDOP bounds. In fact, for any set of $0.7m$ satellites at the horizon expand the count into the sum of q integers, p_k, with each $p_k \geq 3$

$$0.7m = \sum_{k=1}^{q} p_k$$

With this expansion, q groups, each with p_k uniformly spaced satellites, will combine to yield a balanced constellation with minimum GDOP and PDOP.

Another natural question to ask is: "Do real subset selections look like these optimum sets?" To assess this GPS satellite locations were collected for a 24 hour period and brute force computation of the best subsets of size 7 were found ($m = 7$ was chosen as 30% of m is 2 satellites at zenith). From these results Figures 4 and 5 show examples which yielded low GDOP. In both figures the red circles identify the positions of the $m = 14$ satellites; the blue X's mark those included in the best subset of size 7. Note that both subsets include the two satellites highest in elevation. Also, the other 5 are in, roughly, a pentagonal pattern. For comparison Figure 6 shows a case in which the resulting GDOP was poor. While the two highest elevation satellites are included in the best subset, the underlying set of visible satellites is not rich enough to allow for a pentagonal pattern for the other 5.

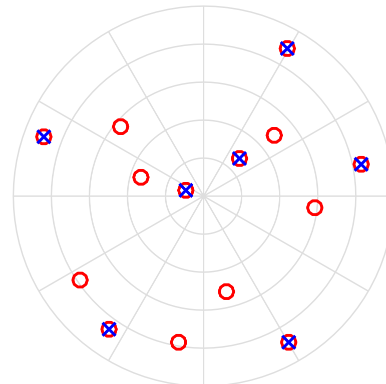

Figure 4: An optimum $m = 7$ example mimicking the optimum configuration.

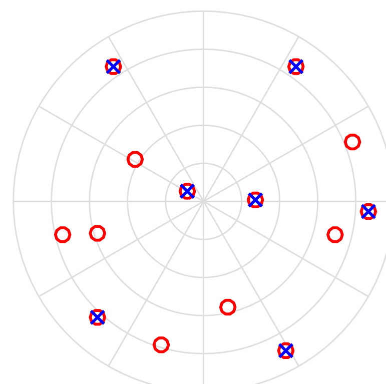

Figure 5: Another example with near optimum geometry.

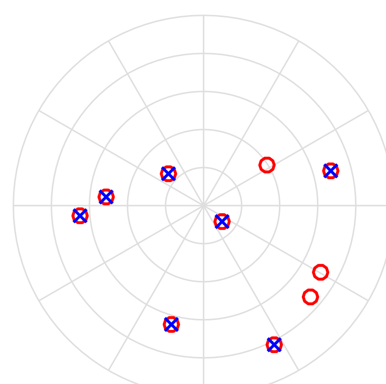

Figure 6: An $m = 7$ example with poor geometry of the visible satellites and, hence, poor performance.

Finally, returning to the results in [3], for multiple constellations of satellites the minimum DOP results if similar conditions hold: approximately 30% of the satellites *from each constellation* at zenith; the remaining 70% at the horizon exhibiting balance including the constraints in Eq. (2) *for each constellation* and the constraints in Eqs. (3) and (4) *across the combined constellations*.

LACK OF RANKING

It is natural to ask if the satellites can be definitively "ranked" in terms of their contribution toward GDOP; if so, this would allow an ordering of the satellites and the selection process would become merely a truncation of the list. An equivalent way to think of this is to ask: "Are smaller sized selections proper subsets of the larger sized selections?" Assuming that such an ordering would work, [5] attempted to develop the ranking function.

While displaying tremendous symmetry, the non-linear and non-separable nature of the expression for GDOP (and PDOP) make an analytic exploration of the possibility of such a ranking a difficult task. Further, for many example constellations, a brute force computation of the best selections for all potential sizes does demonstrate the proper subset property, suggesting that a ranking of the satellites might exist. Unfortunately, counterexamples do occur; it is possible to construct a constellation of satellites such that the optimum satellite selections of smaller sizes are *not* proper subsets of the larger selections.

For example, consider the set of 7 satellites shown in Figure 7: satellites 1-4 are located on the horizon at the corners of the compass (East, West, North, and South), satellites 5 and 6 are at high (80°) elevation in the sky along the East/West direction, and satellite 7 is directly overhead. Table 1 compares the best constellations of smaller sizes. The optimum selections are computed via brute force; the proper subset selections are the result of a greedy search. Red font is used in the table to indicate satellites lost to the next smaller level; blue indicates satellites being reintroduced. Specifically we note that the best 5 satellite selection is *not* a proper subset of the best set of size 6; however, the "ranking" does recover by producing the best selection of size 4. Further, the loss in GDOP performance is only 1%. Anecdotally we note that this loss is typically quite small.

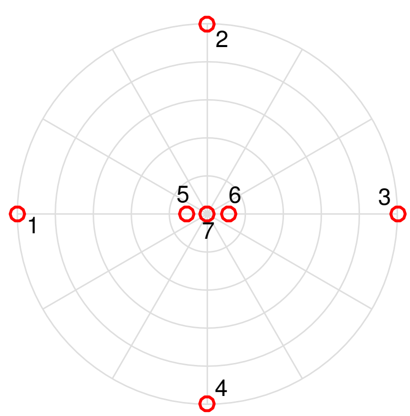

Figure 7: Counterexample #1 – constructed.

Table 1: Subsets for the constructed constellation.

size	optimum	proper subsets	GDOP
7	1,2,3,4,5,6,**7**	1,2,3,4,5,6,**7**	identical
6	1,2,3,4,**5**,**6**	1,2,3,4,5,**6**	identical
5	1,2,**3**,4,**7**	1,2,**3**,4,5	1% higher
4	1,2,4,**5**	1,2,4,5	identical

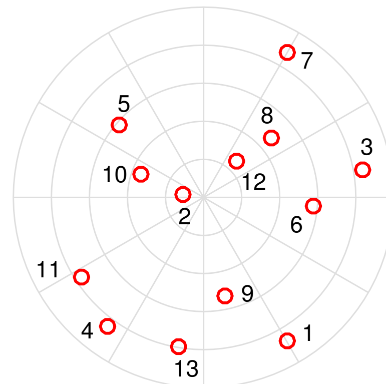

Figure 8: Full sky view of the second counterexample.

Counterexamples to the concept of ranking also occur naturally. To demonstrate this GPS satellite locations were collected for a 24 hour period (once per minute for a set of 1440 constellations) and for each measurement two sets of subset selections (from 4 to the total number of visible satellites) were computed: the optimum set (done by brute force) and the proper subsets as developed by a greedy algorithm. Of the 1440 trials 783 or 54% resulted is some mismatch at one of the constellation sizes. One such counterexample of 13 satellites is shown in Figure 8.

Figure 9 compares the GDOP for the optimum (red) and greedy (blue) selections based on the satellite locations in Figure 8. The top subfigure is the actual GDOP value versus the number of satellites in the selection; we note that the curves are quite close. To highlight the differences the lower subfigure shows the

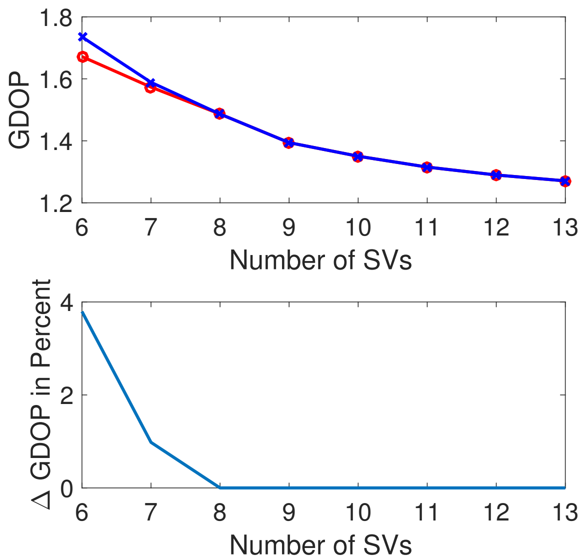

Figure 9: GDOP of the second counterexample.

percentage difference in GDOP for the two methods. We notice that the greedy approach is fine for reducing from 14 to 8 satellites, but is suboptimum for 7 or fewer satellites.

Figure 10 directly compares the selections for the two methods. The columns in this tabular graphic represent each of the 13 visible satellites, the rows indicate the number of satellites in the selection, and red circles and blue X's indicate inclusion in the best and greedy selection, respectively. For a ranked approach to work, whenever a satellite is removed from the subset (i.e. as loss of an X as we go down a column) it cannot be reintroduced without violating the concept of ranking. In this example we see that satellites 1, 6, and 11 reappear in the optimum selection.

Finally, Figures 11 and 12 compare the sky views of the two approaches for the selection of size 7; the red circles show the full set of 14 visible satellites while the blue X's show the included satellites .

The summary is that a ranking of satellites does not always work for GDOP; PDOP is similar.

SELECTION ALGORITHMS

Multiple authors have presented sub-optimum satellite selection procedures; a number of these employ alternative performance measures beyond GDOP. These include volume of the polytope formed by the satellites [6, 7] and cosines of the angles between pairs of satellites [8–10].

The sub-optimum algorithms that employ GDOP or

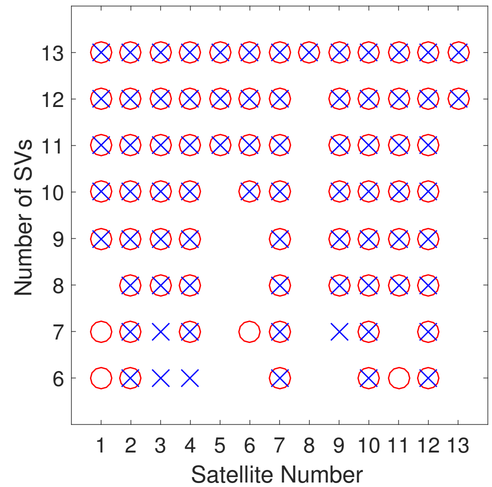

Figure 10: Satellite inclusion for the second counterexample.

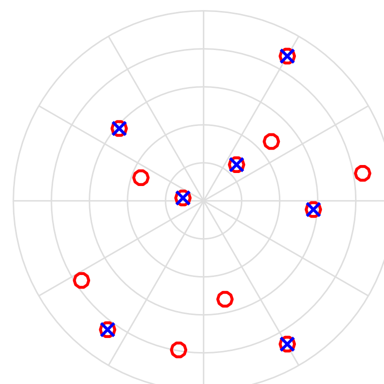

Figure 11: Sky views of the second counterexample: optimum selection of 7 satellites.

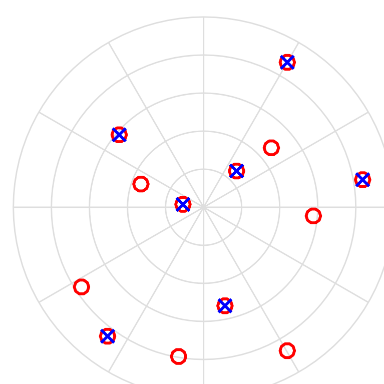

Figure 12: Sky views of the second counterexample: selection using proper subsets.

PDOP as the performance criterion tend to be greedy algorithms (one such method was employed to compute the examples of the previous section). For example, to generate a subset of size m [11] suggests starting with the full set of n satellites and iterating the following steps:

1. Assuming that the current subset consists of k satellites compute k DOPs, one for each proper subset of $k-1$ satellites.

2. Of these k values identify and remove that satellite which results in the smallest increase in DOP.

3. If $k-1>m$ repeat.

This algorithm is greedy in that it makes an optimum choice at each step although the result might not be the global optimum. Specifically, a poor (but still locally optimum) choice at one step might lead to a globally sub-optimum solution for future iterations. It is noted in [11], and observed in the examples above, that the loss to the global optimum for small constellations appears to be small; however, there is no guarantee that this is true for larger numbers of satellites. In a similar way it is possible to grow the subset greedily from the best set of 4 [12].

Another sub-optimum algorithm suggests starting with a subset of size m (and one could discuss how to select this initial set!) and then iterate in a greedy fashion – growing the subset to $m+1$ satellites by adding the most helpful (with respect to DOP) of the unused satellites and then shrinking back down to m by removing the least helpful one, denoted a "revolving door" method [13] .

Simulated annealing has also been proposed as a way to implement a DOP-based satellite selection algorithm [14].

Finally, an algorithm could try to mimic the minimum DOP constellation from the bound discussion. Recall that this best constellation is a combination of satellites at zenith (30%) plus the remainder (70%) at the horizon. The algorithm would then choose high elevation satellites to match the number expected to be at zenith and then attempt to place the remaining satellites at the horizon following "balance." This is attempted in [4] in which the horizon satellites were selected to be as uniformly spaced as possible. We conjecture that our looser definition of balance might make this an easier task. For example, instead of attempting to find 6 satellites at 60° spacing we could look for two sets of 3 satellites at 120° spacing. Further, the results on minimum DOP also suggest how to extend this method to multiple constellations.

CONCLUSIONS/FUTURE WORK

The sections above make several points:

- That the problem of selecting a subset of the visible GNSS satellites is still an important problem.

- That with multiple constellations and the GDOP/PDOP performance criterion, a brute force approach to selecting the best subset is infeasible from a complexity perspective.

- If the underlying set of visible satellites is rich enough then the best subset contains a mix of a few high elevation satellites and the remainder portraying geometric balance at low elevation.

Our future work is to combine the understanding of optimum constellations to improve suboptimum subset selection algorithms for multiple constellations.

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