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Inversion Applied to the Common Equations of the Conic Sections

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1 Definition.

If on a line joining the points P, and O, (the centre of a circle), a point P′ be so placed that the rectangle OP·OP′ be equal to the square on the radius of the circle, P′ is said to be the inverse point to P, O is called the centre of inversion, and the circle, the circle of inversion.

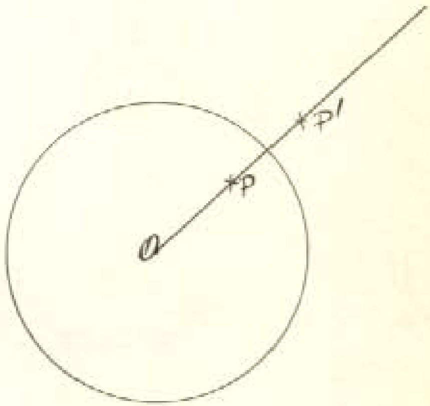

Fig. 1.

Problem.

The geometrical constructions of the inverses to straight lines and circles are familiar, and the present paper will deal wholly with analytical proofs. First, we will inquire into the behavior of the equations of the straight line, when the inversion substitutions are applied then we will go on to the common equations of the second degree, noting the shape of the curves when inverted.

Derivation of substitution formulae.

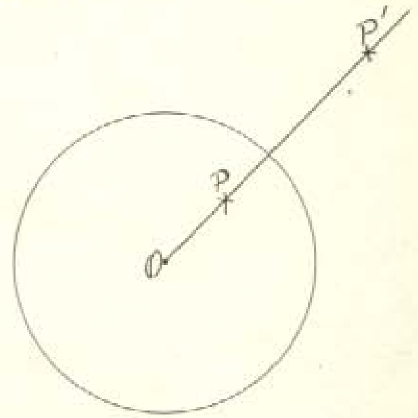

Fig. 2.

Let O, the origin, be the centre of inversion, and on any line, $y = mx$, passing through the origin, take the point P, and its inverse P′. Let the coordinates of P be (x_1, y_1), and of P′, (x_2, y_2). Substituting (x_1, y_1) in

$$y = mx, \tag{1}$$

we have

$$y_1 = m x_1,$$

$$m = \frac{y_1}{x_1},$$

and placing this value for m in (1),

$$x_1 y = x y_1. \tag{2}$$

Substituting (x_2, y_2) for (x, y),

$$x_1 y_2 = x_2 y_1, \tag{3}$$

$$y_2 = \frac{x_2 y_1}{x_1}.$$

By hypothesis,

$$OP \cdot OP' = r^2. \tag{4}$$

By Geometry,

$$OP = \sqrt{x_1^2 + y_1^2} ,$$

$$OP' = \sqrt{x_2^2 + y_2^2} .$$

Substituting these values in(4),

$$\sqrt{x_1^2 + y_1^2}\cdot\sqrt{x_2^2 + y_2^2} = r^2 .$$

Squaring,

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = r^2 .$$

Substituting value for y_2,

$$(x_1^2 + y_1^2)(x_2^2 + \frac{x_2^2 y_1^2}{x_1^2}) = r^4 .$$

Simplifying,

$$x_2^2(x_1^2 + y_1^2) = r^4 x_1^2 ,$$

and solving,

$$x_2 = \frac{r^2 x_1^2}{x_1^2 + y_1^2} ,$$

$$y_2 = \frac{r^2 y_1^2}{x_1^2 + y_1^2} ,$$

which gives us the coordinates of any inverse point, in terms of the coordinates of the original point, and the radius of the circle of inversion.

Denoting the coordinates of any inverse point by (x_1, y_1), the substitution formulae become,

$$x = \frac{r^2 x_1}{x_1^2 + y_1^2} , \quad y = \frac{r^2 y_1}{x_1^2 + y_1^2} .$$

2 The Straight Line.

The intercept equation and the equation of its inverse.

Taking the intercept equation,

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad (1)$$

and applying the substitution formulae, we have

$$\frac{\dfrac{r^2x_1}{x_1^2+y_1^2}}{a} + \frac{\dfrac{r^2y_1}{x_1^2+y_1^2}}{b} = 1.$$

Simplifying,

$$abx_1^2 + aby_1^2 - br^2x_1 - ar^2y_1 = 0. \qquad (2)$$

This equation (2), the inverse to equation (1), represents a circle passing through the origin, for the equation has no absolute term.

Dividing (2) by ab, adding one fourth the squares of the coefficients of x and y, and re-writing, we obtain

$$(x_1 - \frac{br^2}{ab})^2 + (y - \frac{r^2}{b})^2 = \frac{b^2r^4}{4a^2b^2} + \frac{r^4}{4b^2},$$

therefore the centre is at the point $(\frac{r^2}{a}, \frac{r^2}{b})$.

Solving for y in (1),

$$y = -\frac{bx}{a} + b,$$

and equation

$$y = \frac{ax}{b}, \tag{4}$$

being perpendicular to it, after substituting $\left(\frac{r^2}{a}, \frac{r^2}{b}\right)$, and simplifying, becomes

$$\frac{r^2}{b} = \frac{r^2}{b},$$

therefore the centre of circle (2), the inverse to line (1), is on line (4), which is perpendicular to line (1), and passes through $(0, 0)$, the centre of inversion.

Inverse to line passing through the centre of inversion.

Substituting the inversion formulae in the equation of a line through the origin,

$$y = mx, \tag{1}$$

we have,

$$\frac{r^2 y_1}{x_1^2 + y_1^2} = \frac{m r^2 x_1}{x_1^2 + y_1^2}.$$

Omitting subscripts,

$$y = mx, \tag{2}$$

hence line (1) inverts into itself.

Explanation of Figure (1).

This figure represents a system of orthogonal lines parallel to the axes, and the inverse to the system. The origin coincides with the centre of inversion.

In the original figure, (at the left), the point U is at infinity and in the inverse figure(at the right), the point O is at infinity.

The axes invert into themselves, and the other lines invert into circles which pass through the centre of inversion.

3 The Circle.

The inverse to the general circle.

Taking the general equation,

$$x^2 + y^2 + 2\mathcal{G}x + 2\mathcal{F}y + C = 0 , \tag{1}$$

and applying the substitution formulae, we have

$$\frac{r^4x_1^2}{(x_1^2+y_1^2)^2} + \frac{r^4y_1^2}{(x_1^2+y_1^2)^2} + \frac{2\mathcal{G}r^2x_1}{(x_1^2+y_1^2)} + \frac{2\mathcal{F}r^2y_1}{x_1^2+y_1^2} + C = 0 .$$

Simplifying, we have

$$Cx_1^2 + Cy_1^2 + 2\mathcal{G}r^2x_1 + 2\mathcal{F}r^2y_1 + r^4 = 0 , \tag{2}$$

which is the equation of a circle.

Hence the general circle inverts into a circle.

Dividing by C in (2), and adding $\frac{\mathcal{G}^2r^4}{C^2} + \frac{\mathcal{F}^2r^4}{C^2}$,

$$x_1^2 + y_1^2 + \frac{2\mathcal{G}r^2x_1}{C} + \frac{2\mathcal{F}r^2y_1}{C} + \frac{r^4}{C} + \frac{\mathcal{G}^2r^4}{C^2} + \frac{\mathcal{F}^2r^4}{C^2} = \frac{\mathcal{G}^2r^4}{C^2} + \frac{\mathcal{F}^2r^4}{C^2} .$$

Transposing and combining,

$$(x_1 + \frac{\mathcal{G}r^2}{C}) + (y_1 + \frac{\mathcal{F}r^2}{C}) + \frac{r^4}{C} - \frac{\mathcal{G}^2r^4}{C^2} - \frac{\mathcal{F}^2r^4}{C^2} = 0 \quad ;$$

therefore the centre of circle (2), the inverse to circle (1), is $(\frac{\mathcal{G}r^2}{C} , \frac{\mathcal{F}r^2}{C})$. (4)

The radius of circle (2) is $\frac{r^4}{C}(1 - \frac{\mathcal{G}^2}{C} - \frac{\mathcal{F}^2}{C})$.

Combining in (1),

$$(x+G)^2+(y+F)^2 = G^2+F^2-C,$$

therefore the centre of the circle to be inverted is (G,F),

$$y = \frac{F}{G}x.$$

represents the line joining the centre of inversion (0,0) and the centre of the circle (1).

Substituting (x , y) equals (4), we have

$$\frac{Fr^2}{C} = \frac{Fr^2}{C}.$$

Hence, the inverse of any circle is another circle whose centre lies on the same straight line as the centre of inversion and the centre of the circle to be inverted.

The inverse to a circle passing through the centre of inversion.

The equation of this circle,

$$x^2+y^2+2Gx+2Fy=0 \quad , \tag{5}$$

differs from (1) only in that C equals zero; and substituting this value in (2), the equation of the inverse to the general circle, we have

$$2Gr^2x_1+2Fr^2y+r^4=0 \tag{6}$$

which represents a straight line.

Therefore a circle which passes through the centre of inversion inverts into a straight line.

Explanation of Figure 2.

The figure to be inverted (at the left) represents a combination of a system of two-point circles and a system of limiting-pointcircles, the former having one point at the centre of inversion, and the other point on the circle of inversion, the latter being orthogonal to the former. The radical axis of the common point circles is the X-axis, and the radical axis of the limiting-point circles is the represented by the line aGb. At the right is the inverse to this figure. The common point circles invert into straight lines,(for the original circles pass through the centre of inversion), and pass through the point K, (for that point is on the circle of inversion).

The limiting-point circles invert into concentric circles with their centres on the circle of inversion at the point where the inverses to the common point circles intersect.

4 The Parabola.

Its equation referred to the vertex at the centre of inversion, and the equation of its inverse.

Let us take the common equation of the parabola,

$$y^2 = 4ax,$$

and substitute $x = a + x_1$, $y = y_1$, and omitting subscripts, we have

$$y^2 = 4a(a+x), \tag{1}$$

the equation with the focus at the origin.

Applying the substitution formulae, we have

$$\frac{r^4 y_1^2}{(x_1^2+y_1^2)^2} = 4a\left(\frac{r^4 x_1}{x_1^2+y_1^2} + a\right).$$

Simplifying,

$$4a^2y_1^4 + y_1^2(8a^2x_1^2 + 4ar^2x_1 - r^4) + 4ar^2x_1^3 + 4a^2x_1^4 = 0. \tag{2}$$

Equation (2), the inverse to (1), represents a cardioid curve with its cusp at the origin.

Explanation of Figure 3.

This figure represents a system of confocal parabolas, where the focus is at the origin, and the inverse, a system of orthogonal cardioids extending along the axis of X.

5 The Ellipse.

The central equation and the equation of its inverse.

Applying the substitution formulae to a form of the central equation,

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \,, \qquad (1)$$

we have

$$\frac{\dfrac{r^4 y}{(x_1^2+y_1^2)^2}}{b^2} + \frac{\dfrac{r^4 x_1^2}{(x_1^2+y_1^2)^2}}{a^2} = 1 \,.$$

Simplifying,

$$a^2b^2x_1^4 + x_1^2(2a^2b^2y_1^2 - b^2r^4) + a^2b^2y_1^4 - a^2r^4y_1^2 = 0 \,. \qquad (2)$$

This equation (2), the inverse to (1), represents a dumb-bell-shaped curve with its narrowest coinciding with the axis of X.

Explanation of Ellipses in Figure 4.

At the left in this figure a system of confocal ellipses and hyperbolas is represented. The inverses to the ellipses may be easily recognized in the dumbbell-shaped curves at the right. As the a's and b's increase, the ellipses approach the circle and similarly their inverses approach the circle.

6 The Hyperbola.

The central equation and the equation of its inverse.

When the substitution formulae are applied to the form of the central equation,

$$a^2y^2 - b^2x^2 = -a^2b^2, \qquad (1)$$

we have

$$\frac{a^2r^4y_1^2}{(x_1^2+y_1^2)^2} - \frac{b^2r^4x_1^2}{(x_1^2+y_1^2)^2} \quad -a^2b^2.$$

Simplifying,

$$a^2b^2x_1^4 + x_1^2(2a^2b^2y_1^2 - b^2r) + a^2r^4y_1^2 + a^2b^2r^4 = 0 \quad . \qquad (2)$$

This equation (2), the inverse to (1), represents a lemniscate along the axis of X.

Explanation of Hyperbolas in Figure 4.

This figure, representing confocal ellipses and hyperbolas, at the left, shows the hyperbola inverses, at the right, as lemniscates along the X-axis.

As the ellipses and hyperbolas cut each other orthogonally in the original figure, their inverses also cut each other orthogonally.

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