2015

20. Alternating Currents

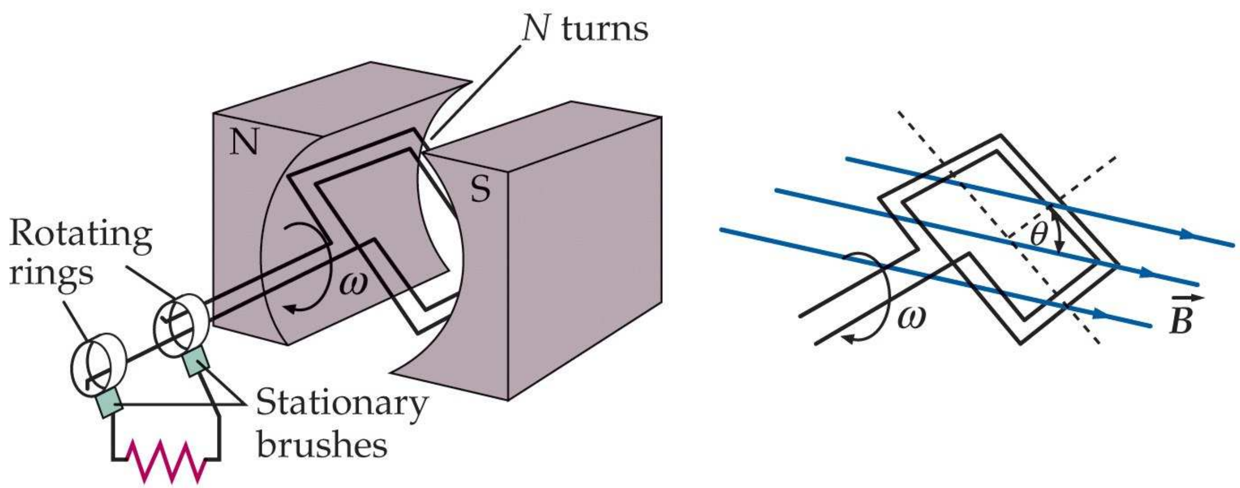
Gerhard Müller
University of Rhode Island, gmuller@uri.edu

Abstract
Part twenty of course materials for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.
Some of the slides contain figures from the textbook, Paul A. Tipler and Gene Mosca. *Physics for Scientists and Engineers*, 5th/6th editions. The copyright to these figures is owned by W.H. Freeman. We acknowledge permission from W.H. Freeman to use them on this course web page. The textbook figures are not to be used or copied for any purpose outside this class without direct permission from W.H. Freeman.

Alternating Current Generator

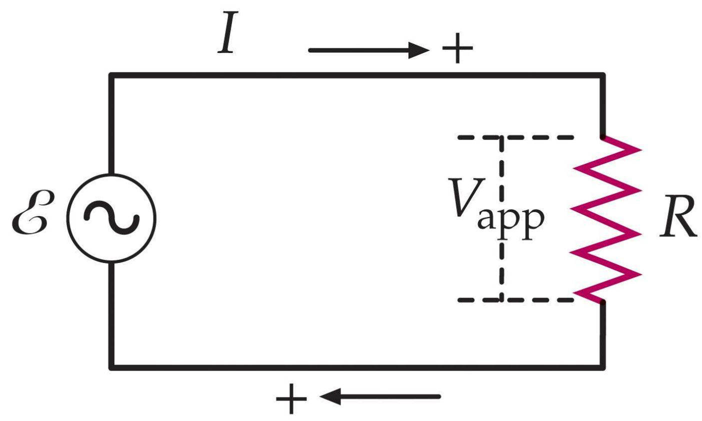
Coil of N turns and cross-sectional area A rotating with angular frequency ω in uniform magnetic field $\vec{B}$.

- Angle between area vector and magnetic field vector: $\theta = \omega t$.
- Flux through coil: $\Phi_B = NBA\cos(\omega t)$.
- Induced EMF: $\mathcal{E} = -\dfrac{d\Phi_B}{dt} = \mathcal{E}_{max}\sin(\omega t)$ with amplitude $\mathcal{E}_{max} = NBA\omega$.
- U.S. household outlet values:
  - $\mathcal{E}_{max} = 120\text{V}\sqrt{2} \simeq 170\text{V}$
  - $f = 60\text{Hz}, \quad \omega = 2\pi f \simeq 377\text{rad/s}.$



Single Device in AC Circuit: Resistor

Voltage of ac source : $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$
Current through device: $I = I_{max} \cos(\omega t - \delta)$

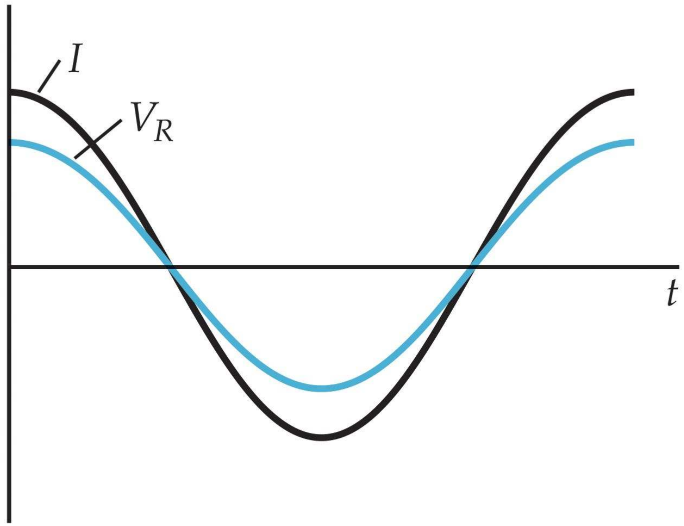


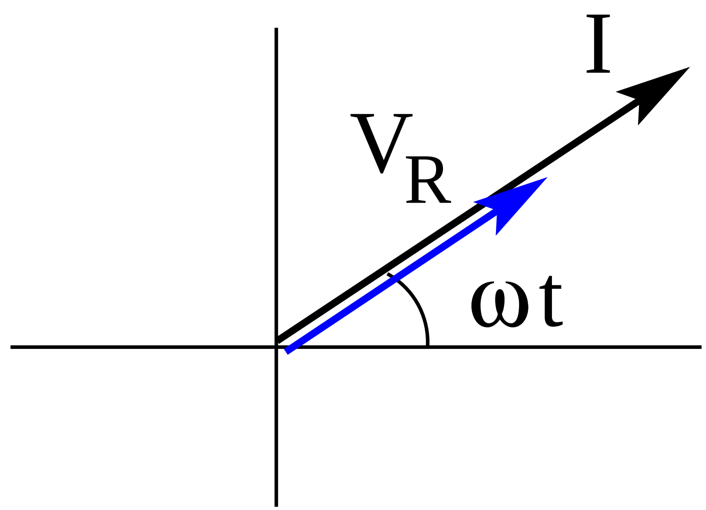
Resistor

$$V_R = RI = \mathcal{E}_{max} \cos \omega t \;\Rightarrow\; I = \frac{\mathcal{E}_{max}}{R} \cos \omega t$$

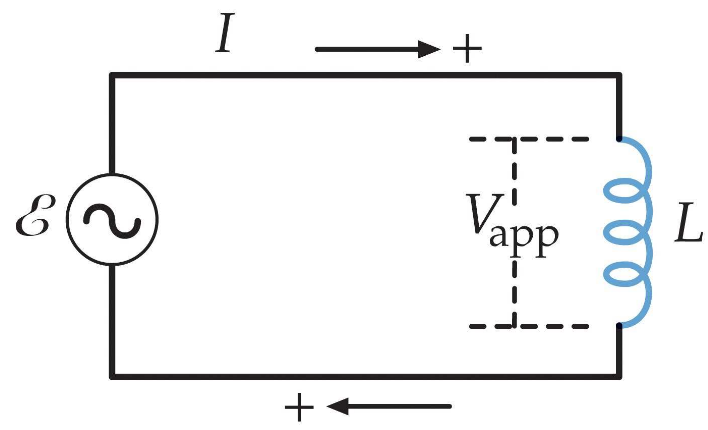
amplitude: $I_{max} = \dfrac{\mathcal{E}_{max}}{R}$, phase angle: $\delta = 0$

impedance: $X_R \equiv \dfrac{\mathcal{E}_{max}}{I_{max}} = R$ (resistance)





Voltage of ac source : $\mathcal{E} = \mathcal{E}_{max} \cos\omega t$
Current through device: $I = I_{max} \cos(\omega t - \delta)$

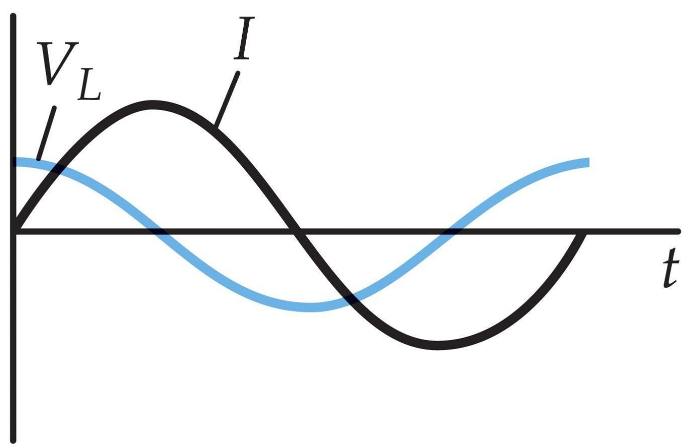


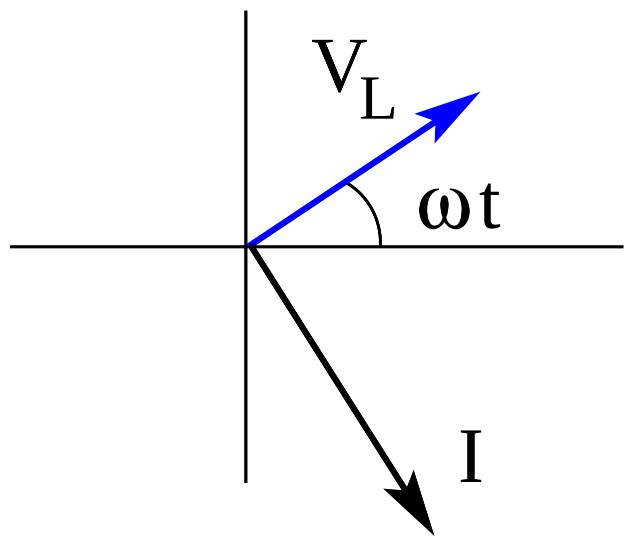
Inductor

$$V_L = L\frac{dI}{dt} = \mathcal{E}_{max} \cos\omega t \;\Rightarrow\; I = \frac{\mathcal{E}_{max}}{\omega L}\sin(\omega t)$$

amplitude: $I_{max} = \dfrac{\mathcal{E}_{max}}{\omega L}$, phase angle: $\delta = \dfrac{\pi}{2}$

impedance: $X_L \equiv \dfrac{\mathcal{E}_{max}}{I_{max}} = \omega L$ (inductive reactance)

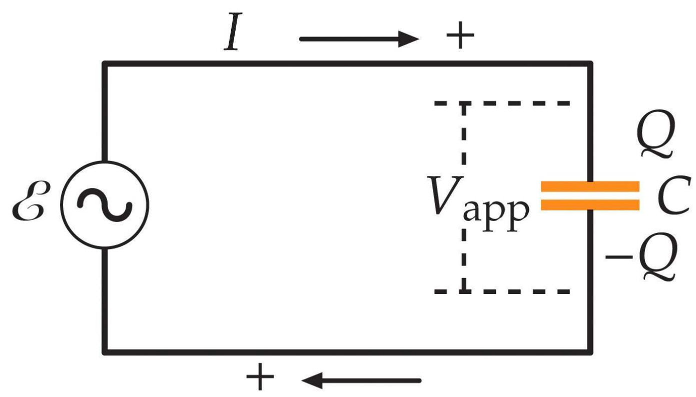




Single Device in AC Circuit: Capacitor

Voltage of ac source : $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$
Current through device: $I = I_{max} \cos(\omega t - \delta)$

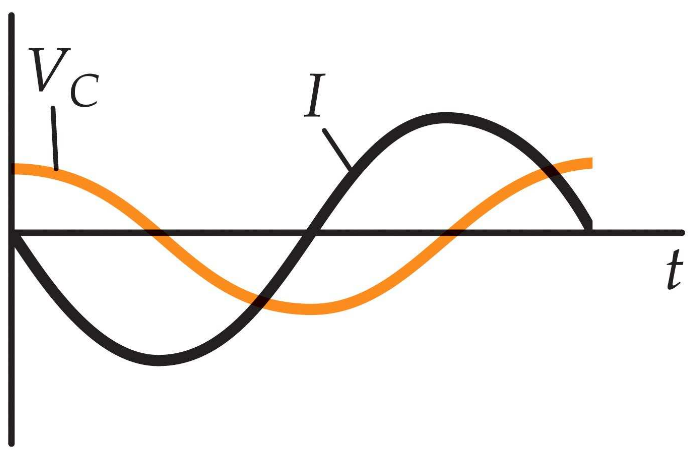
Capacitor

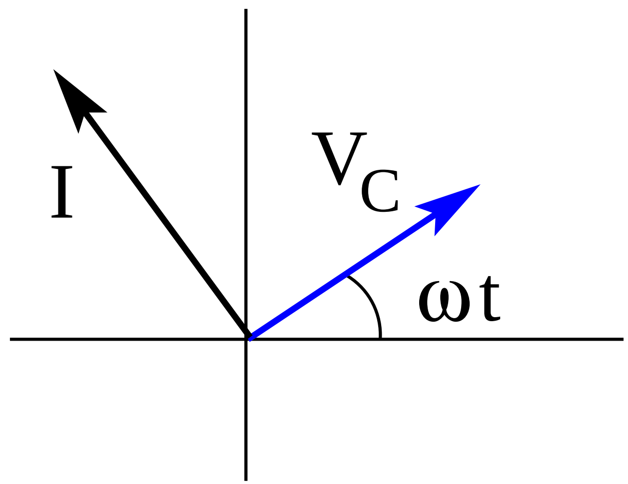


$$V_C = \frac{Q}{C} = \mathcal{E}_{max} \cos \omega t \;\Rightarrow\; I = \frac{dQ}{dt} = -\omega C \mathcal{E}_{max} \sin(\omega t)$$

amplitude: $I_{max} = \omega C \mathcal{E}_{max}$, phase angle: $\delta = -\frac{\pi}{2}$

impedance: $X_C \equiv \frac{\mathcal{E}_{max}}{I_{max}} = \frac{1}{\omega C}$ (capacitive reactance)





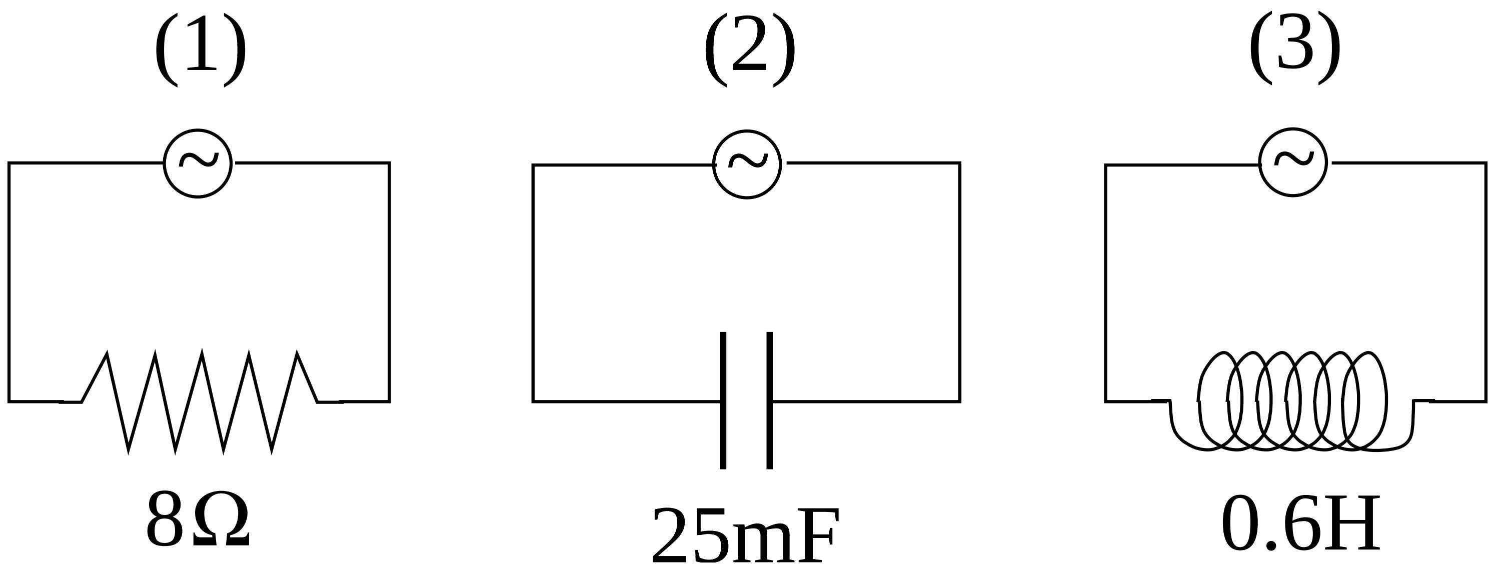
Single Device in AC Circuit: Application (1)

The ac voltage source $\mathcal{E} = \mathcal{E}_{max} \sin \omega t$ has an amplitude of $\mathcal{E}_{max} = 24$V and an angular frequency of $\omega = 10$rad/s.

In each of the three circuits, find

(a) the current amplitude I_{max},

(b) the current I at time $t = 1$s.



Single Device in AC Circuit: Application (2)

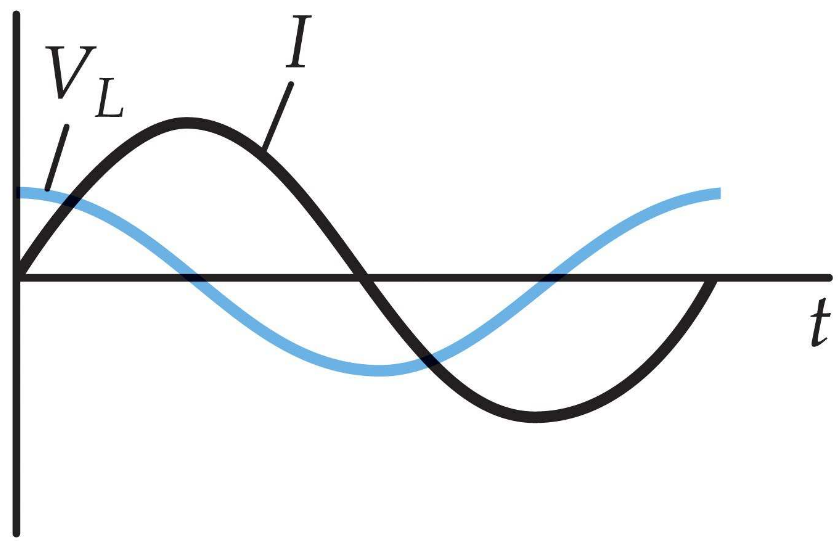
Consider an ac generator $\mathcal{E}(t) = \mathcal{E}_{max}\cos(\omega t)$, $\mathcal{E}_{max} = 25$V, $\omega = 377$rad/s connected to an inductor with inductance $L = 12.7$H.

(a) Find the maximum value of the current.

(b) Find the current when the emf is zero and decreasing.

(c) Find the current when the emf is -12.5V and decreasing.

(d) Find the power supplied by the generator at the instant described in (c).



Single Device in AC Circuit: Application (3)

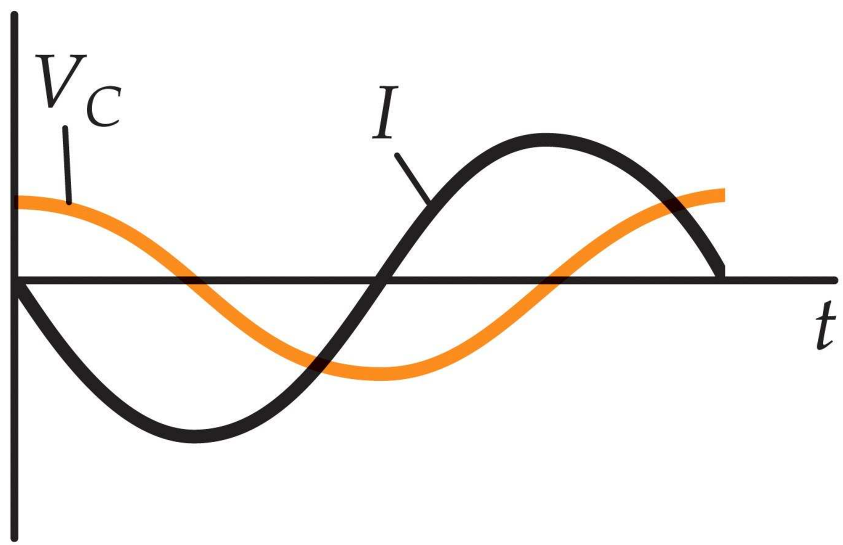
Consider an ac generator $\mathcal{E}(t) = \mathcal{E}_{max}\cos(\omega t)$, $\mathcal{E}_{max} = 25$V, $\omega = 377$rad/s connected to a capacitor with capacitance $C = 4.15\mu$F.

(a) Find the maximum value of the current.

(b) Find the current when the emf is zero and decreasing.

(c) Find the current when the emf is -12.5V and increasing.

(d) Find the power supplied by the generator at the instant described in (c).



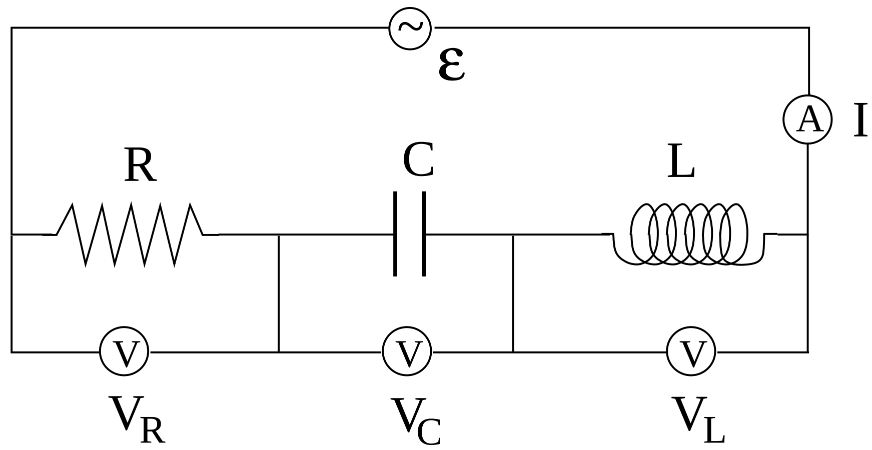
RLC Series Circuit (1)

Applied alternating voltage: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Resulting alternating current: $I = I_{max} \cos(\omega t - \delta)$

Goals:

- Find I_{max}, δ for given $\mathcal{E}_{max}, \omega$.
- Find voltages V_R, V_L, V_C across devices.



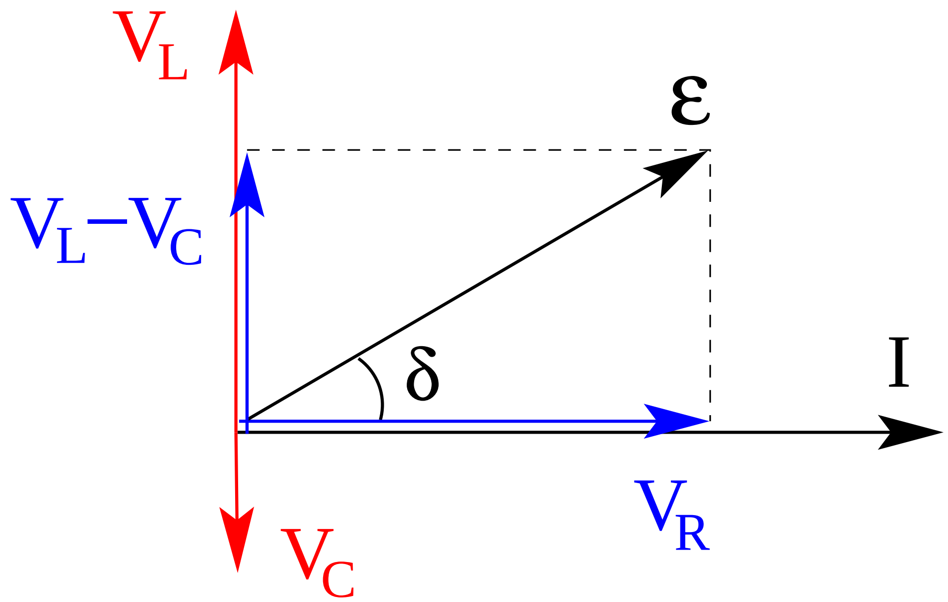
Loop rule: $\mathcal{E} - V_R - V_C - V_L = 0$

Note:

- All voltages are time-dependent.
- In general, all voltages have a different phase.
- V_R has the same phase as I.

RLC Series Circuit (2)

Phasor diagram (for $\omega t = \delta$):



Voltage amplitudes:

- $V_{R,max} = I_{max} X_R = I_{max} R$
- $V_{L,max} = I_{max} X_L = I_{max} \omega L$
- $V_{C,max} = I_{max} X_C = \dfrac{I_{max}}{\omega C}$

Relation between $\mathcal{E}_{max}$ and I_{max} from geometry:

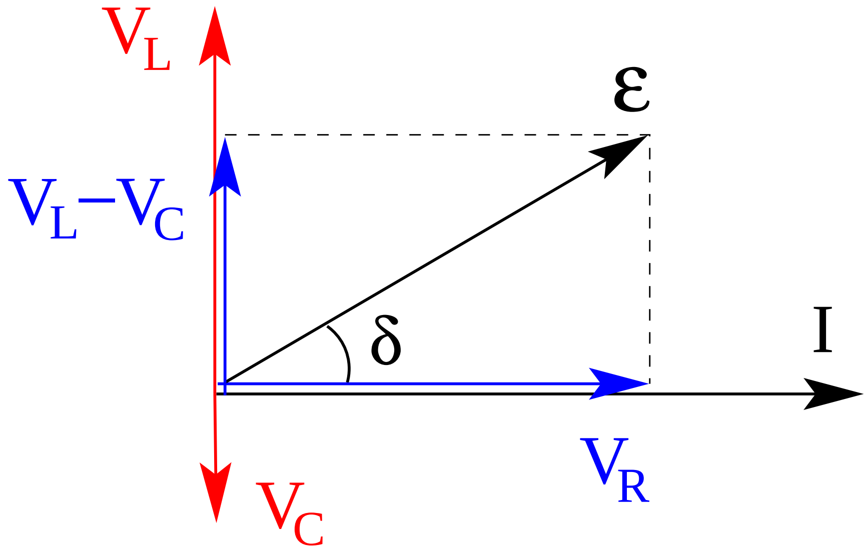
$$\begin{aligned} \mathcal{E}^2_{max} &= V^2_{R,max} + (V_{L,max} - V_{C,max})^2 \\ &= I^2_{max} \left[R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2 \right] \end{aligned}$$

RLC Series Circuit (3)

Impedance: $Z \equiv \dfrac{\mathcal{E}_{max}}{I_{max}} = \sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}$

Current amplitude and phase angle:

- $I_{max} = \dfrac{\mathcal{E}_{max}}{Z} = \dfrac{\mathcal{E}_{max}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}$
- $\tan\delta = \dfrac{V_{L,max} - V_{C,max}}{V_{R,max}} = \dfrac{\omega L - 1/\omega C}{R}$



Voltages across devices:

- $V_R = RI = RI_{max}\cos(\omega t - \delta) = V_{R,max}\cos(\omega t - \delta)$
- $V_L = L\dfrac{dI}{dt} = -\omega L I_{max}\sin(\omega t - \delta) = V_{L,max}\cos\left(\omega t - \delta + \dfrac{\pi}{2}\right)$
- $V_C = \dfrac{1}{C}\displaystyle\int I dt = \dfrac{I_{max}}{\omega C}\sin(\omega t - \delta) = V_{C,max}\cos\left(\omega t - \delta - \dfrac{\pi}{2}\right)$

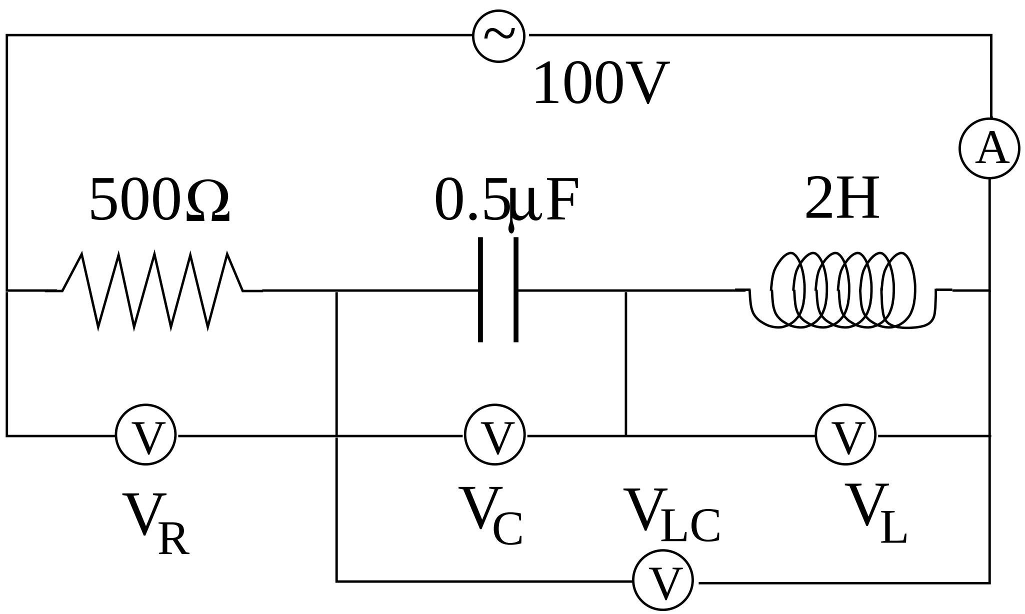
AC Circuit Application (1)

In this RLC circuit, the voltage amplitude is $\mathcal{E}_{max} = 100$V.

Find the impedance Z, the current amplitude I_{max},
and the voltage amplitudes V_R, V_C, V_L, V_{LC}

(a) for angular frequency is $\omega = 1000$rad/s,

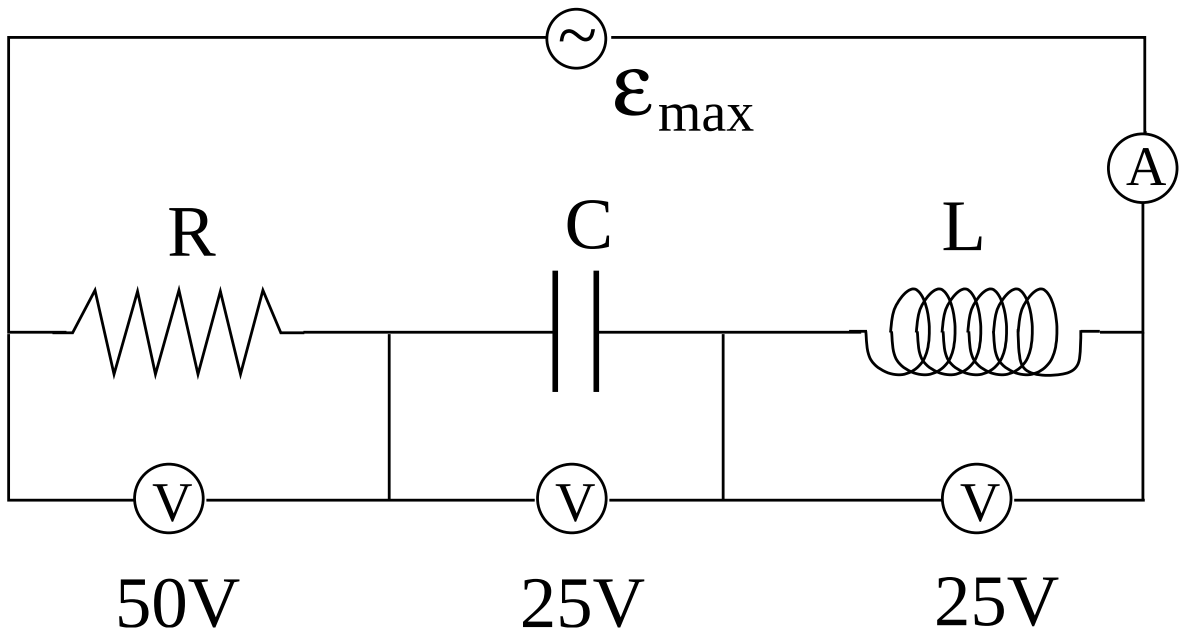
(b) for angular frequency is $\omega = 500$rad/s.



AC Circuit Application (2)

In this RLC circuit, we know the voltage amplitudes V_R, V_C, V_L across each device, the current amplitude $I_{max} = 5$A, and the angular frequency $\omega = 2$rad/s.

- Find the device properties R, C, L and the voltage amplitude $\mathcal{E}_{max}$ of the ac source.



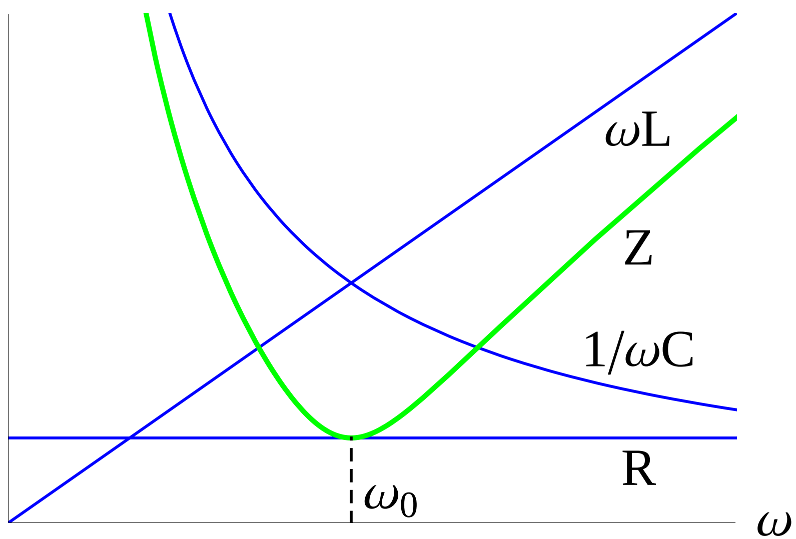
Impedances: RLC in Series (1)

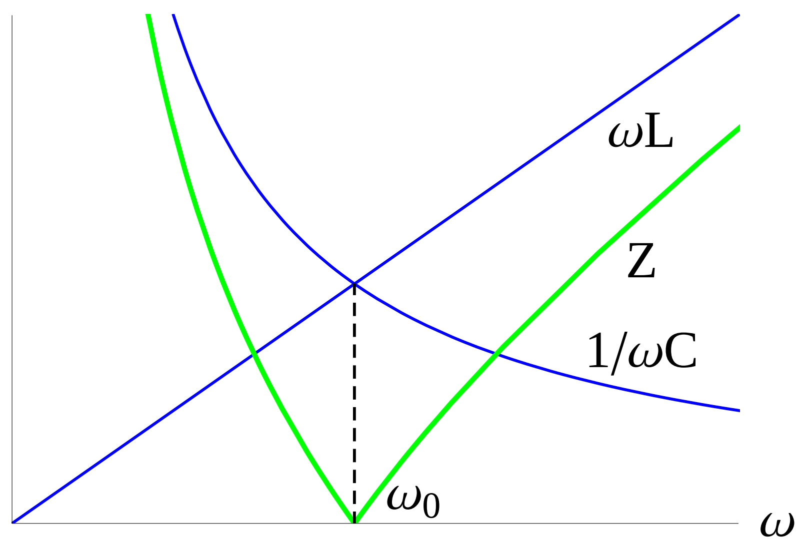
$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

resonance at $\omega_0 = \frac{1}{\sqrt{LC}}$

limit $R \to 0$

$$Z = \left|\omega L - \frac{1}{\omega C}\right|$$





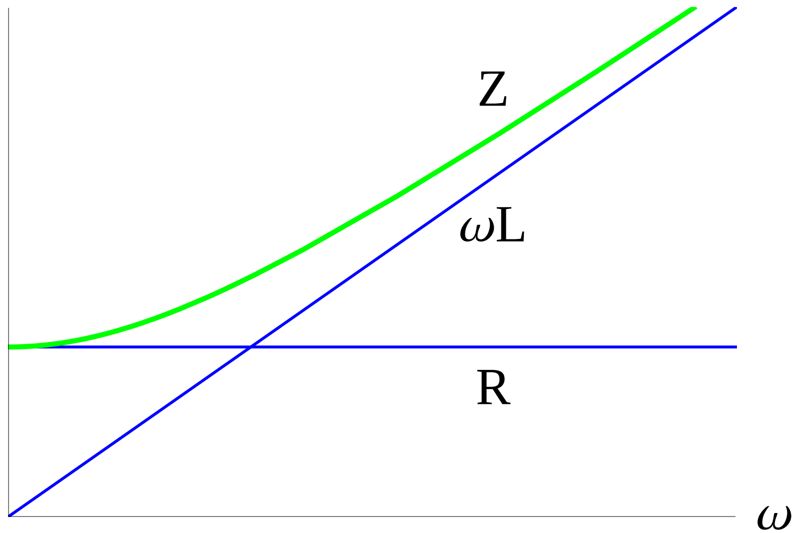
Impedances: RLC in Series (2)

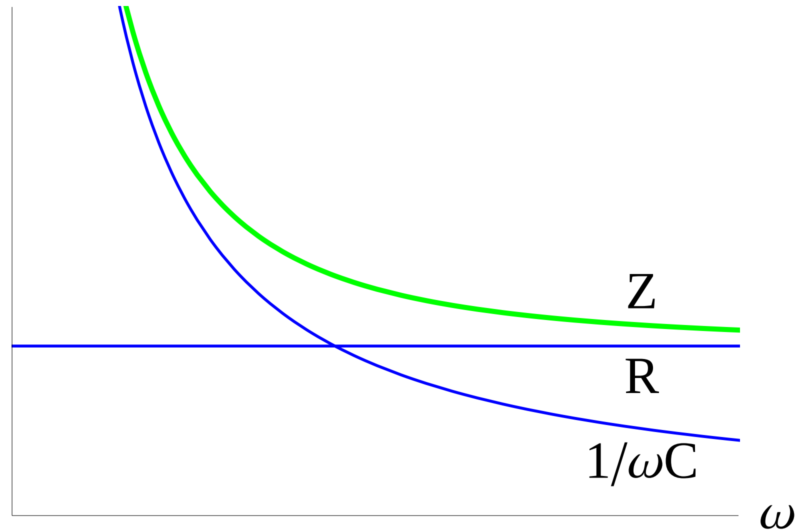
limit $C \to \infty$

$$Z = \sqrt{R^2 + (\omega L)^2}$$

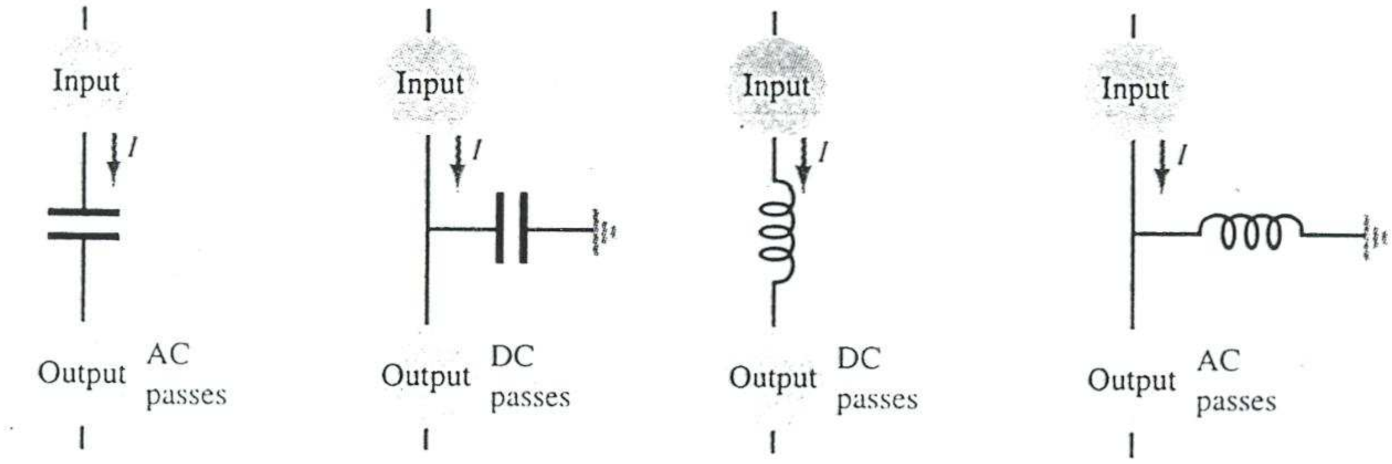
limit $L \to 0$

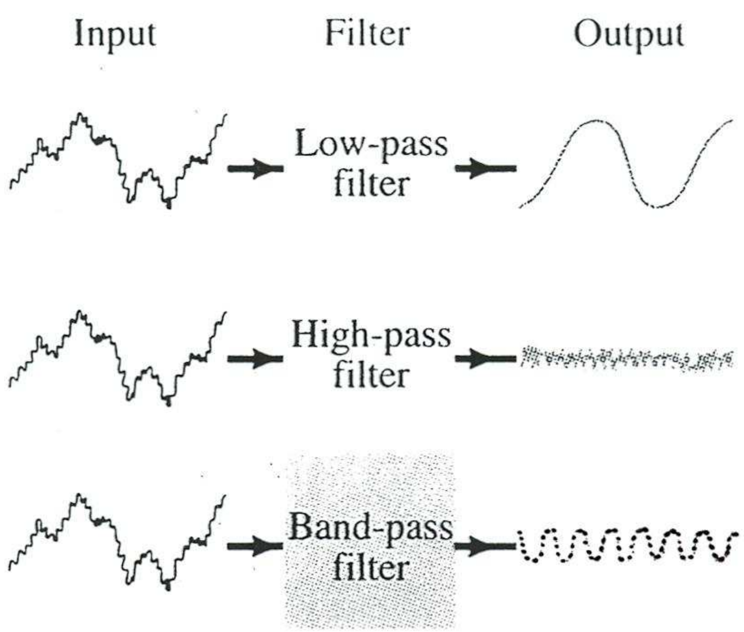
$$Z = \sqrt{R^2 + \frac{1}{(\omega C)^2}}$$





Filters

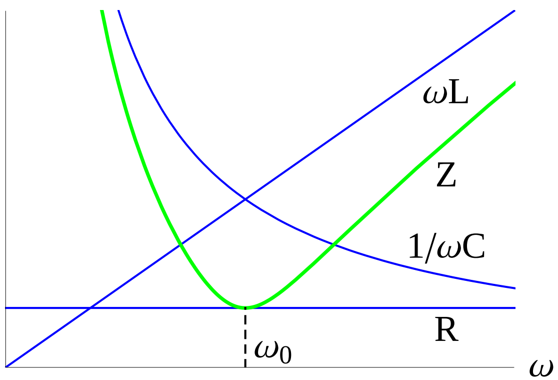

Input
I
Output
AC passes
Input
I
Output
DC passes
Input
I
Output
DC passes
Input
I
Output
AC passes


Input
Filter
Output
Low-pass filter
High-pass filter
Band-pass filter

RLC Series Resonance (1)

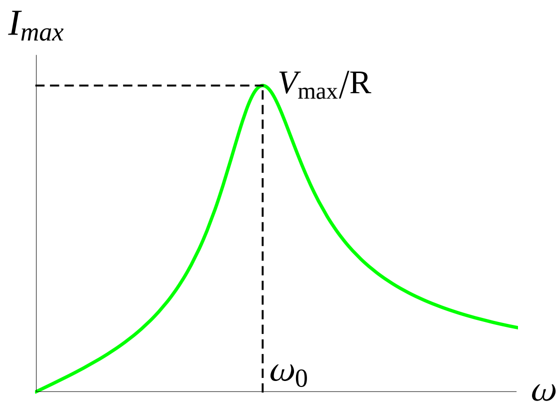
impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$



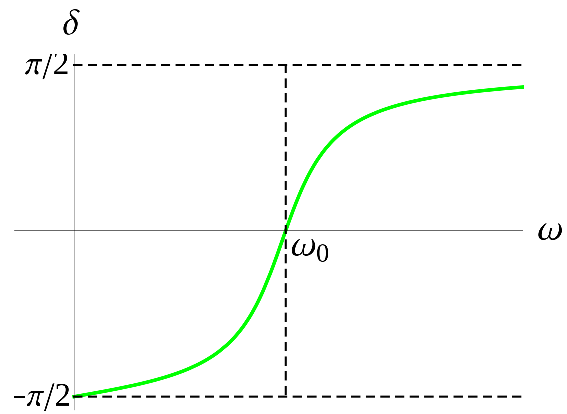
current

$$I_{max} = \frac{V_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$



phase angle

$$\delta = \frac{\omega L - 1/\omega C}{R}$$

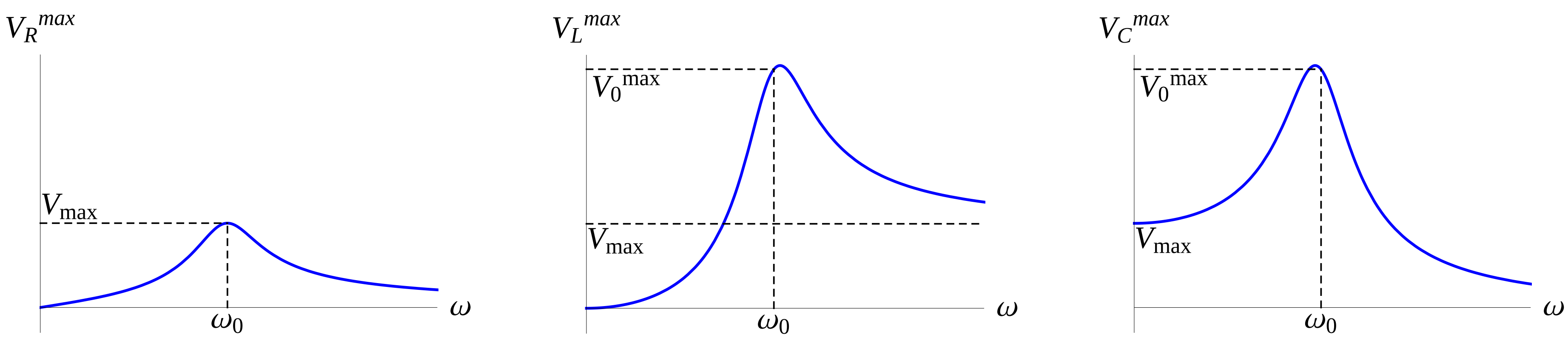


resonance angular frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

RLC Series Resonance (2)

resistor	inductor	capacitor
$V_R^{max} = I_{max} R$	$V_L^{max} = I_{max} \omega L$	$V_C^{max} = I_{max} / \omega C$



- relaxation times: $\tau_{RC} = RC, \quad \tau_{RL} = L/R$
- angular frequencies: $\omega_L = \dfrac{\omega_0}{\sqrt{1 - (\omega_0\, \tau_{RC})^2/2}}, \quad \omega_C = \omega_0 \sqrt{1 - (\omega_0\, \tau_{RC})^2/2}$
- voltages: $V_0^{max} = V_{max}\, \omega_0\, \tau_{RL}, \quad V_L^{max}(\omega_L) = V_C^{max}(\omega_C) = \dfrac{V_0^{max}}{\sqrt{1 - (\omega_0\, \tau_{RC})^2/4}}$

RLC Parallel Circuit (1)

Applied alternating voltage: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$

Resulting alternating current: $I = I_{max} \cos(\omega t - \delta)$

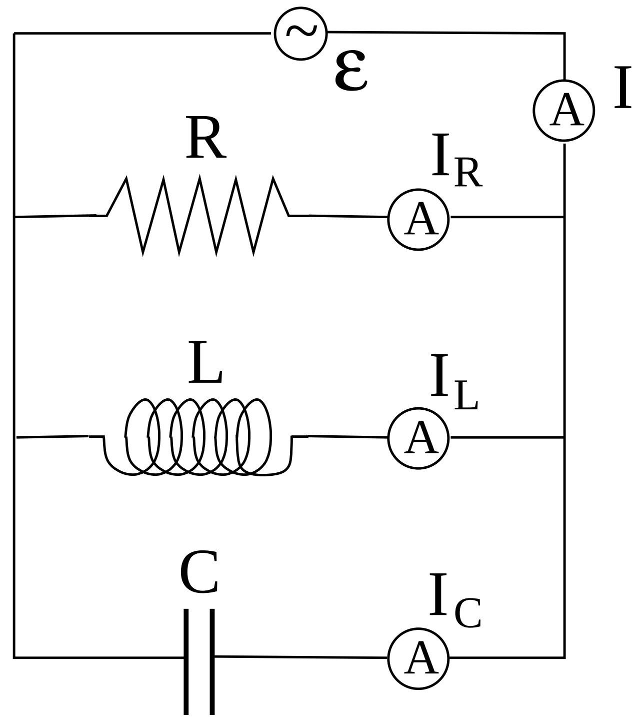
Goals:

- Find I_{max}, δ for given $\mathcal{E}_{max}, \omega$.
- Find currents I_R, I_L, I_C through devices.

Junction rule: $I = I_R + I_L + I_C$

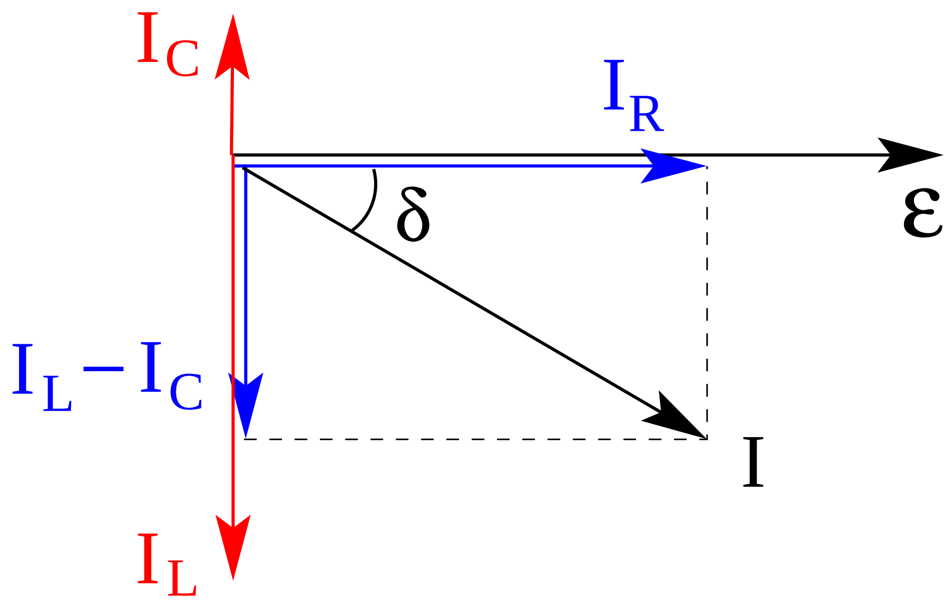
Note:

- All currents are time-dependent.
- In general, each current has a different phase
- I_R has the same phase as $\mathcal{E}$.



RLC Parallel Circuit (2)

Phasor diagram (for $\omega t = \delta$):



Current amplitudes:

- $I_{R,max} = \frac{\mathcal{E}_{max}}{X_R} = \frac{\mathcal{E}_{max}}{R}$
- $I_{L,max} = \frac{\mathcal{E}_{max}}{X_L} = \frac{\mathcal{E}_{max}}{\omega L}$
- $I_{C,max} = \frac{\mathcal{E}_{max}}{X_C} = \mathcal{E}_{max}\omega C$

Relation between $\mathcal{E}_{max}$ and I_{max} from geometry:

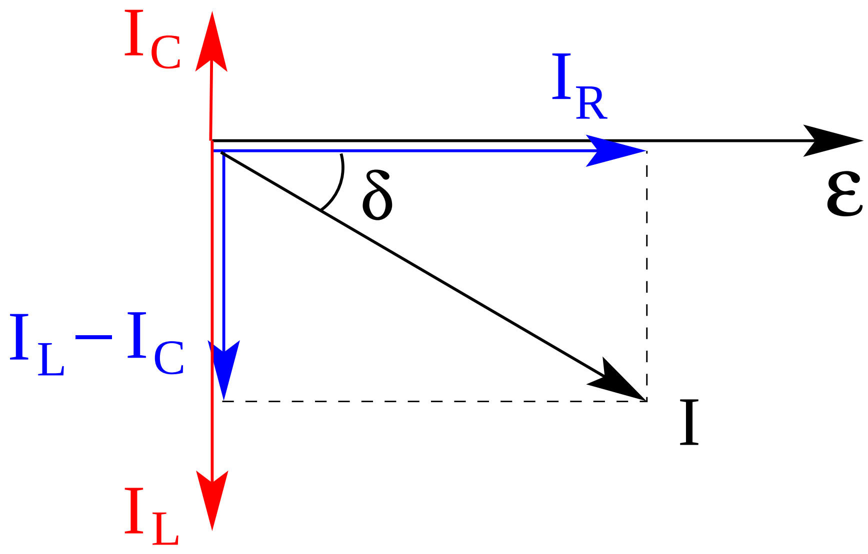
$$\begin{aligned} I_{max}^2 &= I_{R,max}^2 + (I_{L,max} - I_{C,max})^2 \\ &= \mathcal{E}_{max}^2 \left[\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C \right)^2 \right] \end{aligned}$$

RLC Parallel Circuit (3)

Impedance: $\frac{1}{Z} \equiv \frac{I_{max}}{\mathcal{E}_{max}} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}$

Current amplitude and phase angle:

- $I_{max} = \frac{\mathcal{E}_{max}}{Z} = \mathcal{E}_{max}\sqrt{\frac{1}{R^2} + \left(\frac{1}{\omega L} - \omega C\right)^2}$
- $\tan\delta = \frac{I_{L,max} - I_{C,max}}{I_{R,max}} = \frac{1/\omega L - \omega C}{1/R}$



Currents through devices:

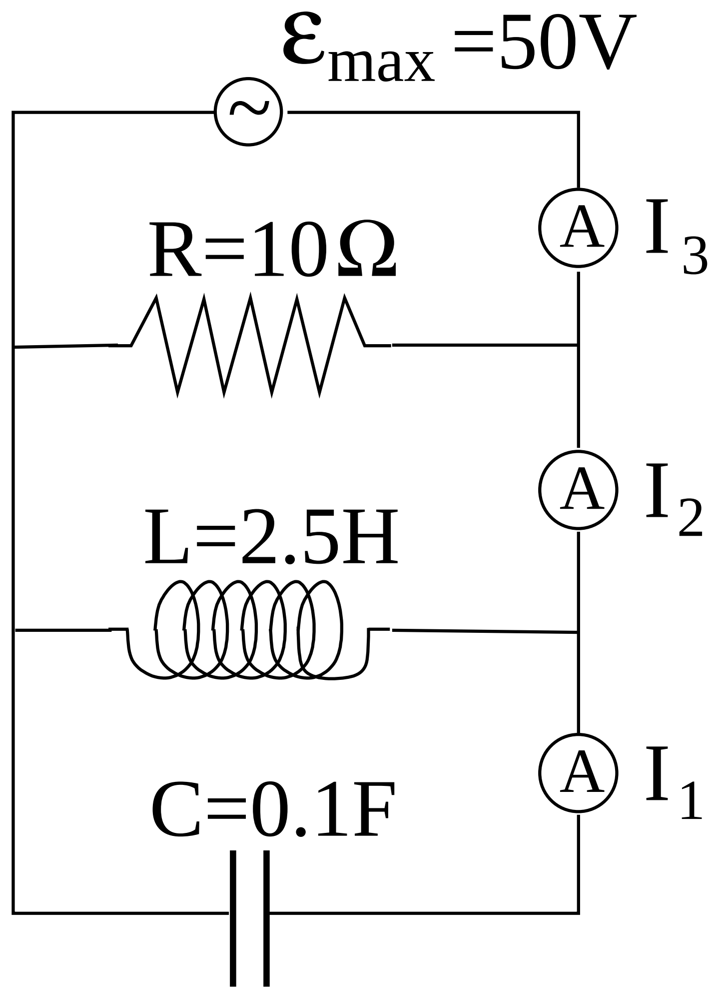
- $I_R = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}_{max}}{R}\cos(\omega t) = I_{R,max}\cos(\omega t)$
- $I_L = \frac{1}{L}\int \mathcal{E} dt = \frac{\mathcal{E}_{max}}{\omega L}\sin(\omega t) = I_{L,max}\cos\left(\omega t - \frac{\pi}{2}\right)$
- $I_C = C\frac{d\mathcal{E}}{dt} = -\omega C\mathcal{E}_{max}\sin(\omega t) = I_{C,max}\cos\left(\omega t + \frac{\pi}{2}\right)$

AC Circuit Application (3)

Find the current amplitudes I_1, I_2, I_3

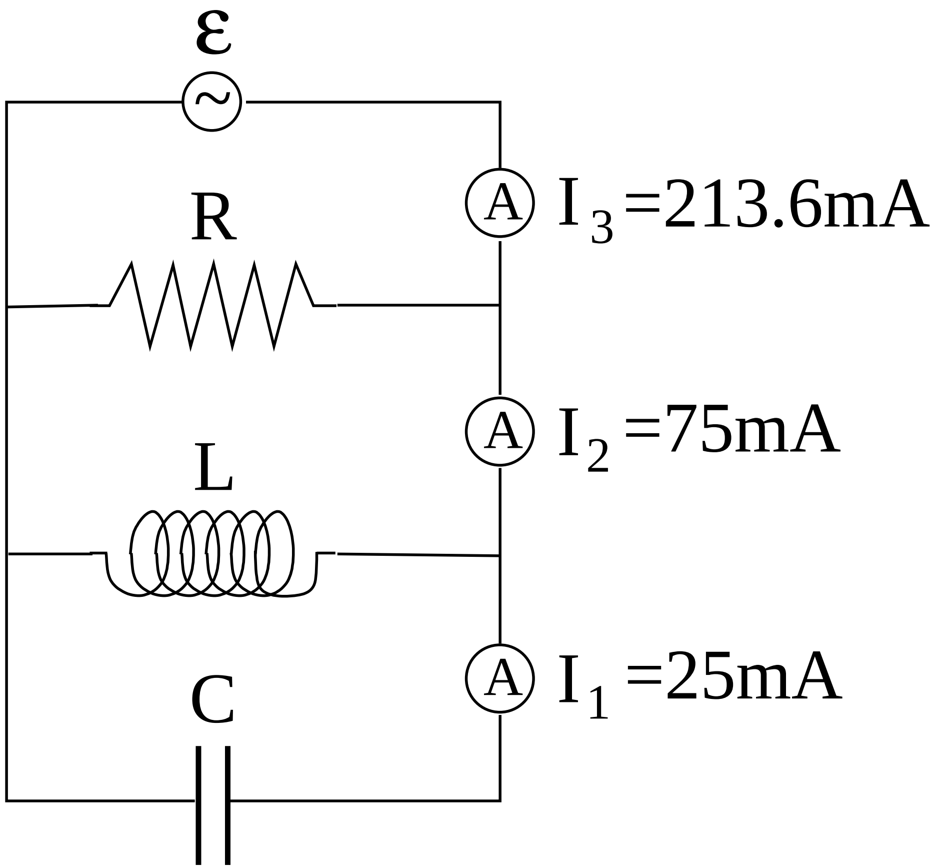
(a) for angular frequency $\omega = 2$rad/s,

(b) for angular frequency $\omega = 4$rad/s.



AC Circuit Application (4)

Given the current amplitudes I_1, I_2, I_3 through the three branches of this RLC circuit, and given the amplitude $\mathcal{E}_{max} = 100$V and angular frequency $\omega = 500$rad/s of the ac source, find the device properties R, L, C.



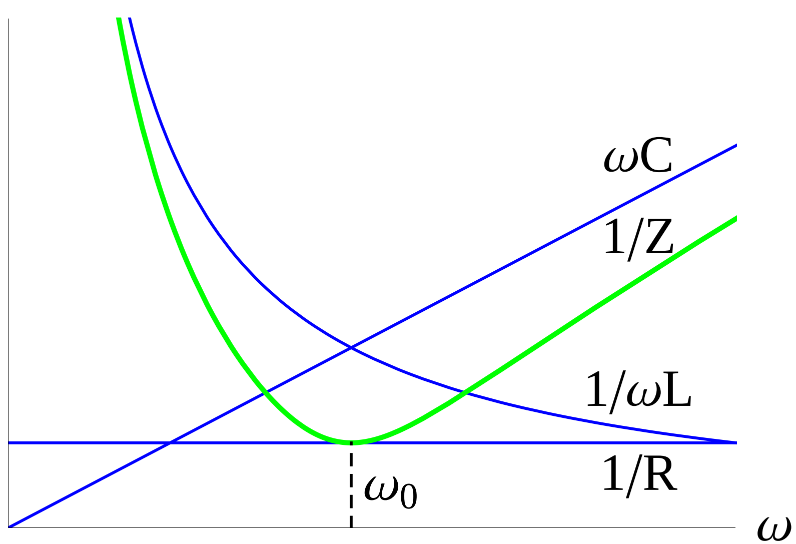
Impedances: RLC in Parallel (1)

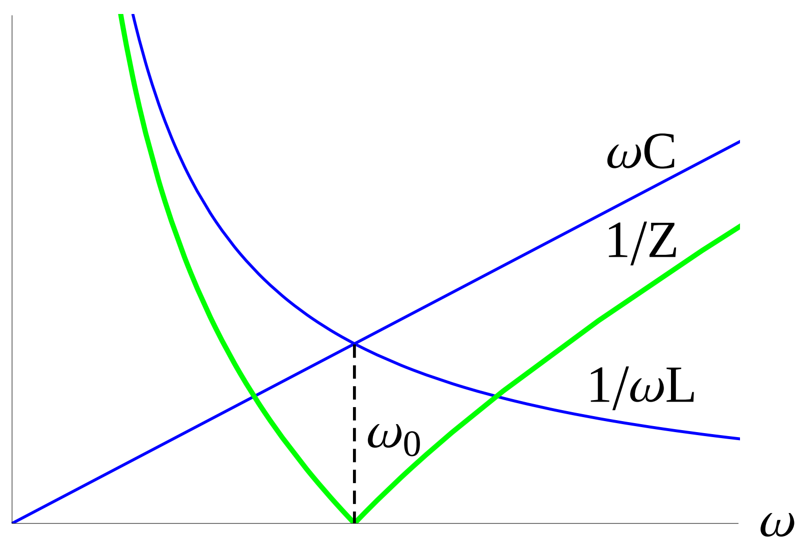
$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

resonance at $\omega_0 = \frac{1}{\sqrt{LC}}$

limit $R \to \infty$

$$\frac{1}{Z} = \left|\omega C - \frac{1}{\omega L}\right|$$





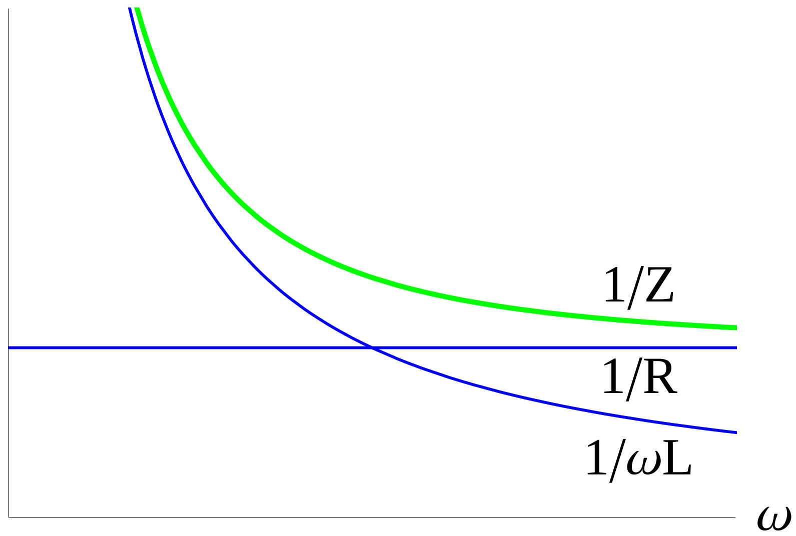
Impedances: RLC in Parallel (2)

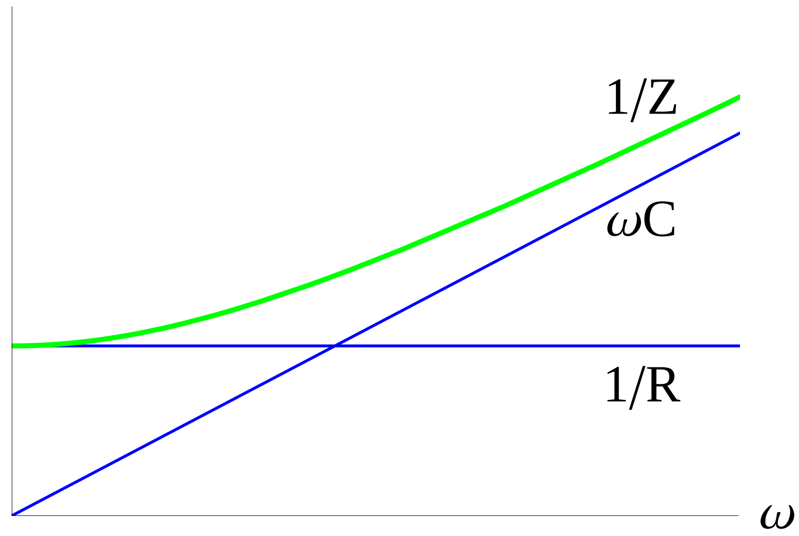
limit $C \to 0$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \frac{1}{(\omega L)^2}}$$

limit $L \to \infty$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + (\omega C)^2}$$

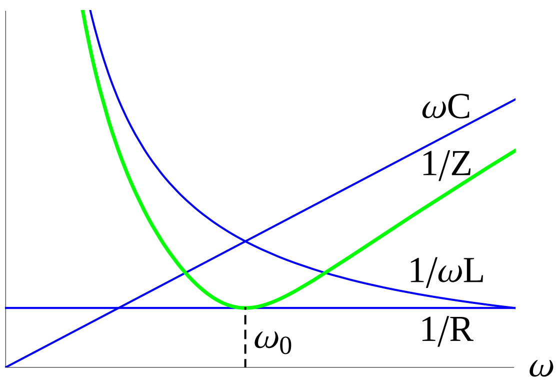




RLC Parallel Resonance (1)

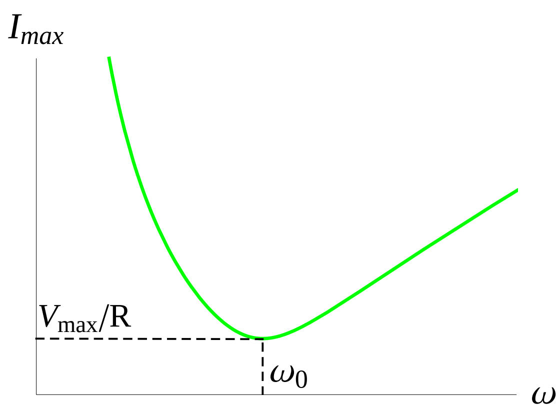
impedance

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$



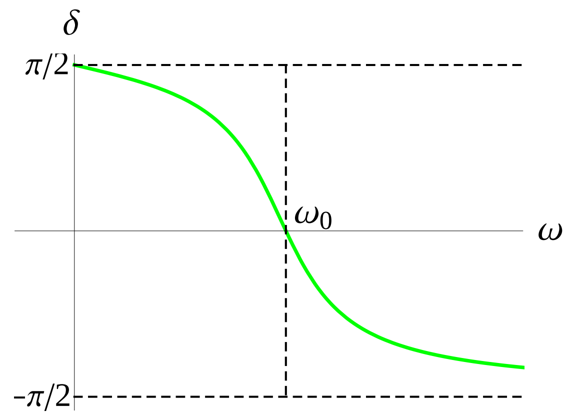
current

$$I_{max} = \frac{V_{max}}{Z}$$



phase angle

$$\delta = \frac{1/\omega L - \omega C}{1/R}$$

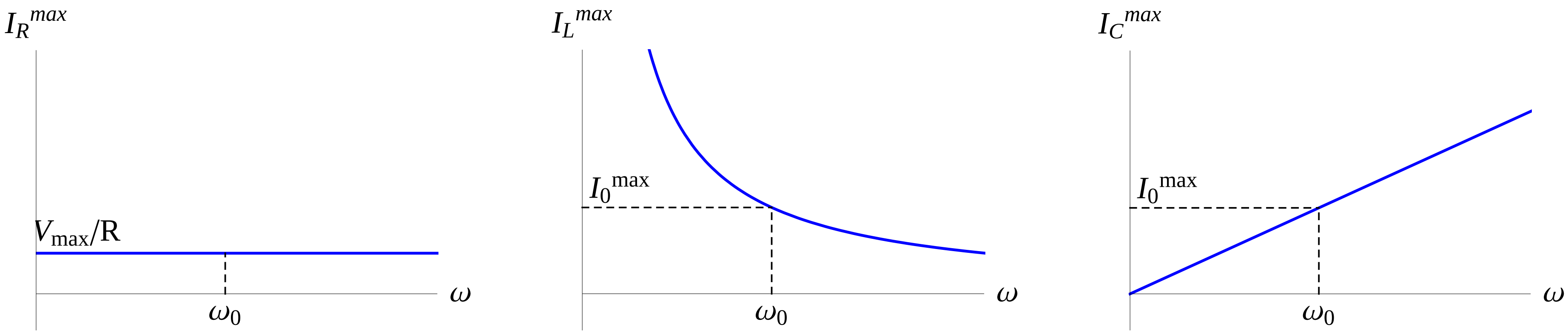


resonance angular frequency:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

RLC Parallel Resonance (2)

resistor	inductor	capacitor
$I_R^{max} = V_{max}/R$	$I_L^{max} = V_{max}/\omega L$	$I_C^{max} = V_{max}\,\omega C$



currents at resonance:

$$I_R^{max} = \frac{V_{max}}{R}, \quad I_L^{max} = I_C^{\rm max} = I_0^{max} = V_{max}\sqrt{\frac{C}{L}}.$$

Power in AC Circuits

Voltage of ac source: $\mathcal{E} = \mathcal{E}_{max} \cos \omega t$
Current through circuit: $I = I_{max} \cos(\omega t - \delta)$

Instantaneous power supplied: $P(t) = \mathcal{E}(t) I(t) = [\mathcal{E}_{max} \cos \omega t][I_{max} \cos(\omega t - \delta)]$

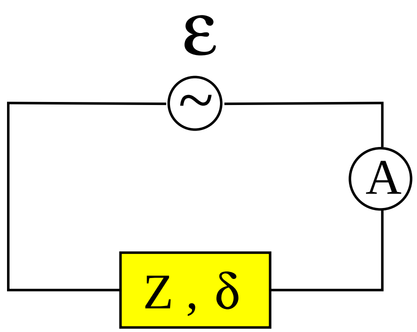
Use $\cos(\omega t - \delta) = \cos \omega t \cos \delta + \sin \omega t \sin \delta$

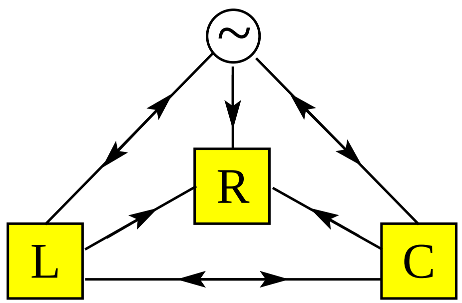
$$\Rightarrow \; P(t) = \mathcal{E}_{max} I_{max} [\cos^2 \omega t \cos \delta + \cos \omega t \sin \omega t \sin \delta]$$

Time averages: $[\cos^2 \omega t]_{AV} = \frac{1}{2}, \quad [\cos \omega t \sin \omega t]_{AV} = 0$

Average power supplied by source: $P_{AV} = \frac{1}{2} \mathcal{E}_{max} I_{max} \cos \delta = \mathcal{E}_{rms} I_{rms} \cos \delta$

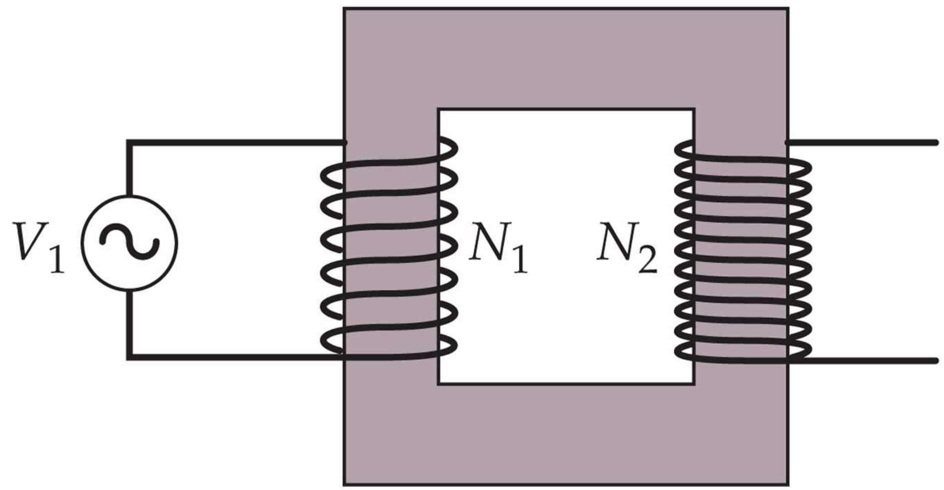
Power factor: $\cos \delta$





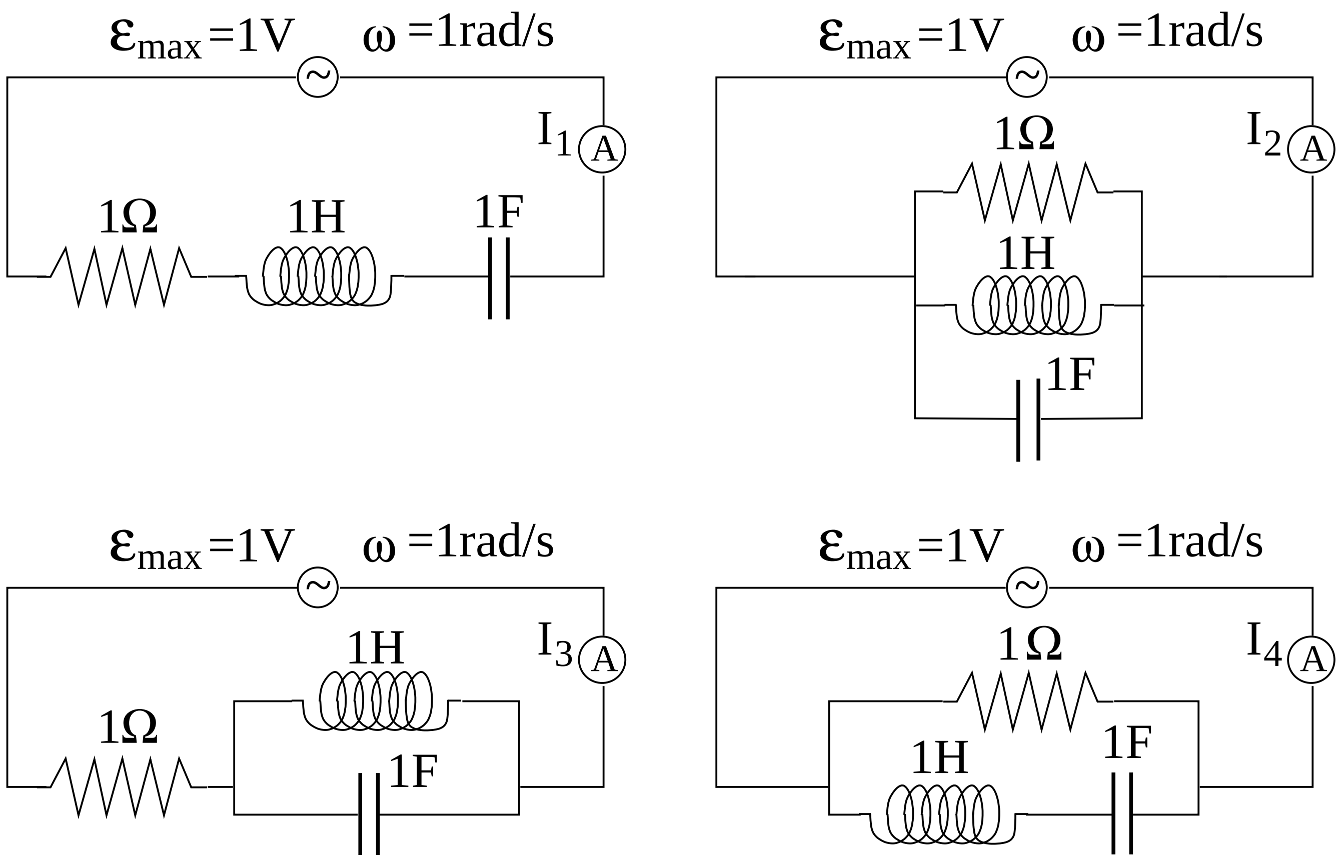
Transformer

- Primary winding: N_1 turns
  $$V_1(t) = V_1^{(rms)} \cos(\omega t), \quad I_1(t) = I_1^{(rms)} \cos(\omega t - \delta_1)$$
- Secondary winding: N_2 turns
  $$V_2(t) = V_2^{(rms)} \cos(\omega t), \quad I_2(t) = I_2^{(rms)} \cos(\omega t - \delta_2)$$
- Voltage amplitude ratio: $\dfrac{V_1^{(rms)}}{V_2^{(rms)}} = \dfrac{N_1}{N_2}$
- Power transfer: $V_1^{(rms)} I_1^{(rms)} \cos\delta_1 = V_2^{(rms)} I_2^{(rms)} \cos\delta_2$



AC Circuit Application (5)

Find the current amplitudes I_1, I_2, I_3, I_4 in the four RLC circuits shown.



AC Circuit Application (6)

Consider an RLC series circuit with inductance $L = 88m$H, capacitance $C = 0.94\mu$F, and unknown resistance R.
The ac generator $\mathcal{E} = \mathcal{E}_{max}\sin(\omega t)$ has amplitude $\mathcal{E}_{max} = 24$V
and frequency $f = 930$Hz. The phase angle is $\delta = 75°$.

(a) Find the resistance R.

(b) Find the current amplitude I_{max}.

(c) Find the maximum energy U_L^{max} stored in the inductor.

(d) Find the maximum energy U_C^{max} stored in the capacitor.

(e) Find the time t_1 at which the current has its maximum value I_{max}.

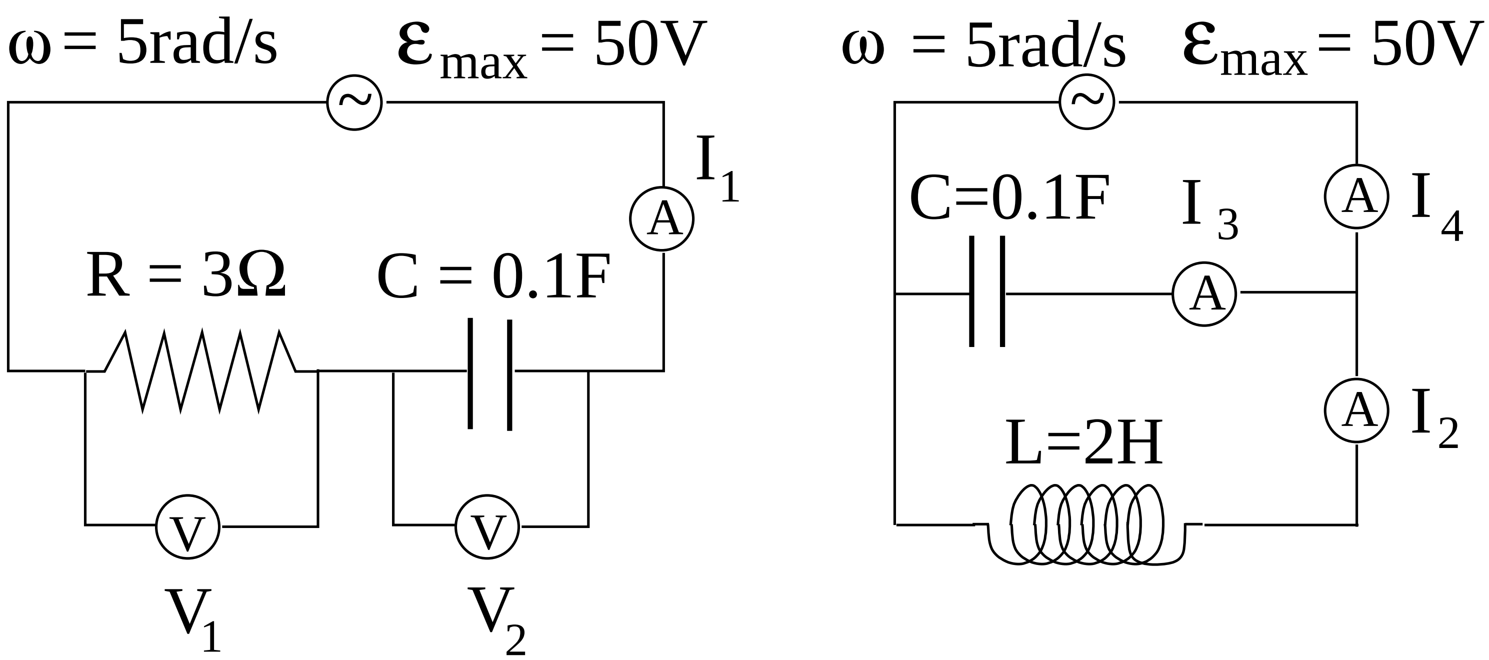
(f) Find the time t_2 at which the charge on the capacitor has its maximum value Q_{max}.

AC Circuit Application (7)

Consider the two ac circuits shown.

(a) In the circuit on the left, determine the current amplitude I_1 and the voltage amplitudes V_1 and V_2.

(b) In the circuit on the right, determine the current amplitudes I_2, I_3, and I_4.

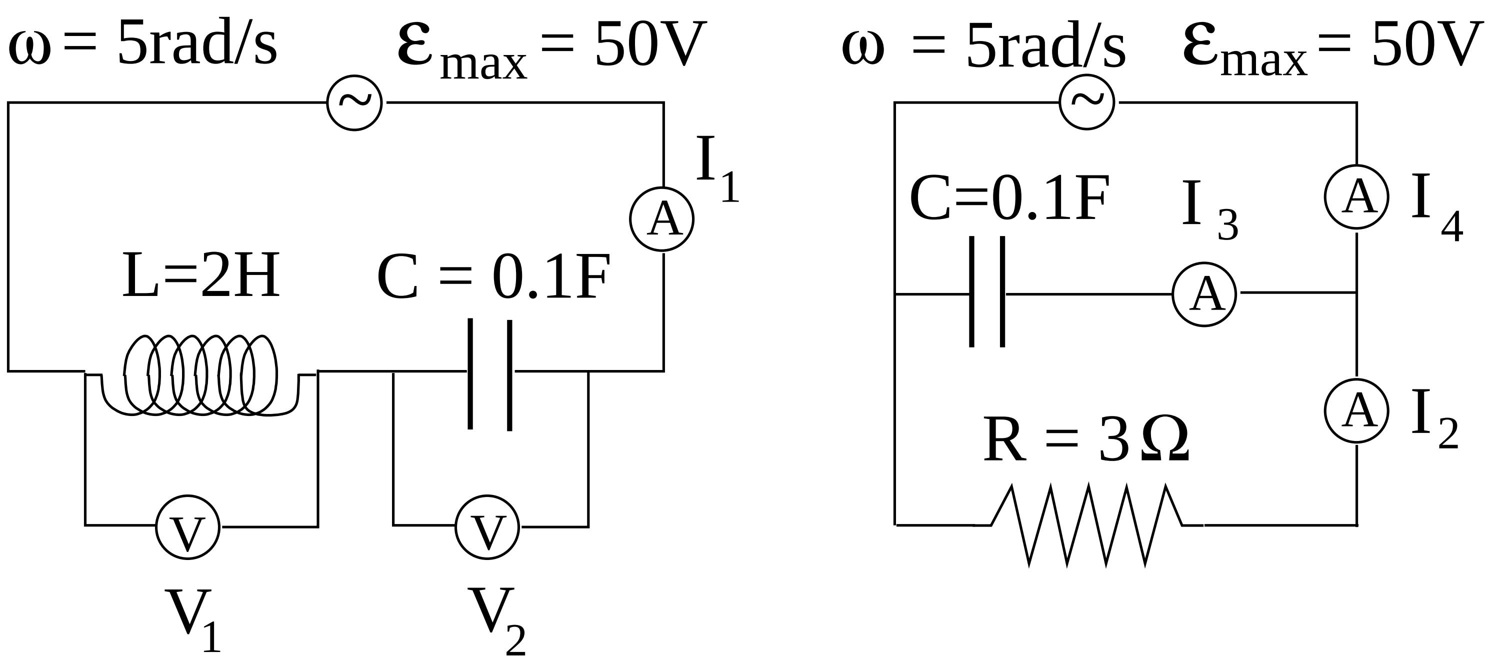


AC Circuit Application (8)

Consider the two ac circuits shown.

(a) In the circuit on the left, determine the maximum value of current I_1 and the maximum value of voltages V_1 and V_2.

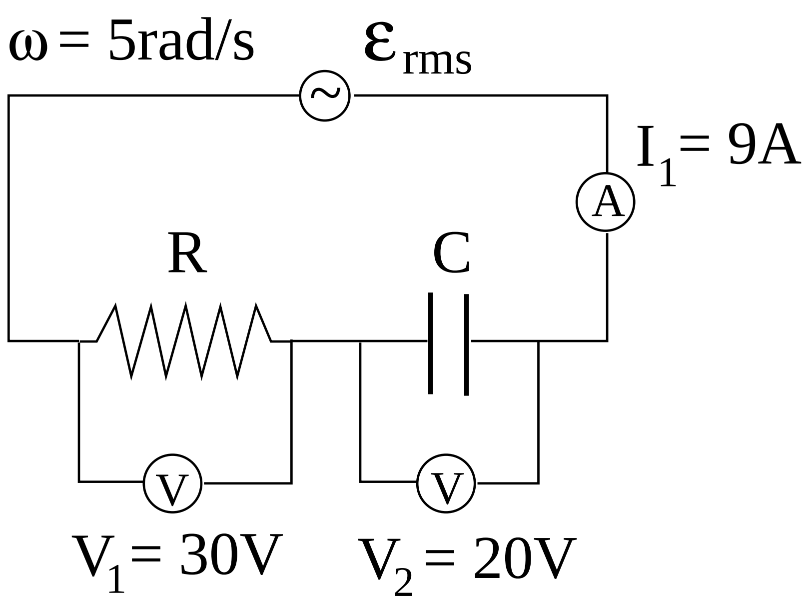
(b) In the circuit on the right, determine the maximum value of currents I_2, I_3, and I_4.

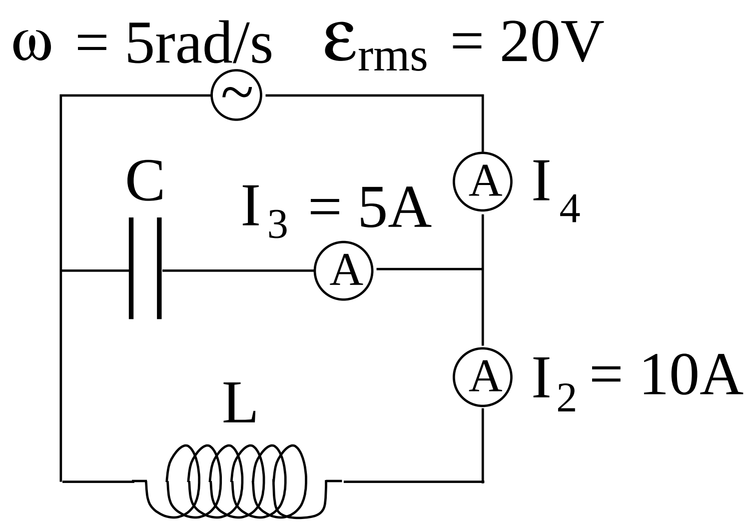


In the two ac circuits shown the ammeter and voltmeter readings are rms values.

(a) In the circuit on the left, find the resistance R of the resistor, the capacitance C of the capacitor, the impedance Z of the two devices combined, and the voltage $\mathcal{E}_{rms}$ of the power source.

(b) In the circuit on the right, find the capacitance C of the capacitor, the inductance L of the inductor, the impedance Z of the two devices combined, and the rms value of the current I_4.





AC Circuit Application (10)

In the two ac circuits shown the ammeter and voltmeter readings are rms values.

(a) In the circuit on the left, find the capacitance C of the capacitor, the inductance L of the inductor, the impedance Z of the two devices combined, and the voltage $\mathcal{E}_{rms}$ of the power source.

(b) In the circuit on the right, find the capacitance C of the capacitor, the resistance R of the resistor, the impedance Z of the two devices combined, and the rms value of the current I_4.