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19. LC and RLC Oscillators

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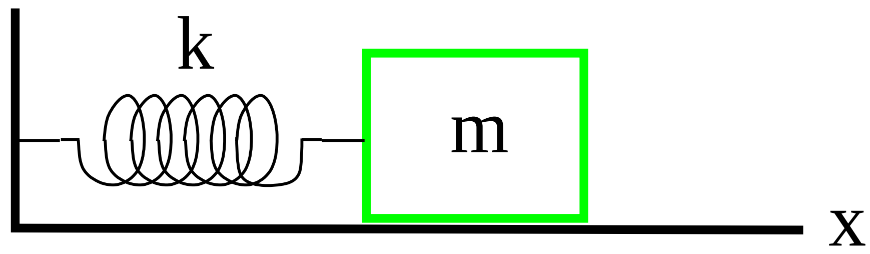
Abstract
Part nineteen of course materials for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.
Some of the slides contain figures from the textbook, Paul A. Tipler and Gene Mosca. *Physics for Scientists and Engineers*, $5^{th}/6^{th}$ editions. The copyright to these figures is owned by W.H. Freeman. We acknowledge permission from W.H. Freeman to use them on this course web page. The textbook figures are not to be used or copied for any purpose outside this class without direct permission from W.H. Freeman.

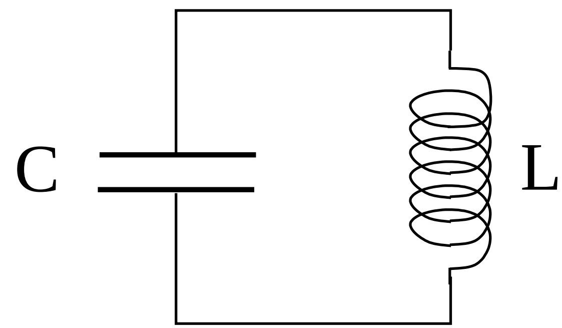
Mechanical Oscillator

- law of motion: $F = ma, \quad a = \frac{d^2x}{dt^2}$
- law of force: $F = -kx$
- equation of motion: $\frac{d^2x}{dt^2} = -\frac{k}{m}x$
- displacement: $x(t) = x_{max}\cos(\omega t)$
- velocity: $v(t) = -\omega x_{max}\sin(\omega t)$
- angular frequency: $\omega = \sqrt{\frac{k}{m}}$
- kinetic energy: $K = \frac{1}{2}mv^2$
- potential energy: $U = \frac{1}{2}kx^2$
- total energy: $E = K + U = \text{const.}$



Electromagnetic Oscillator (LC Circuit)

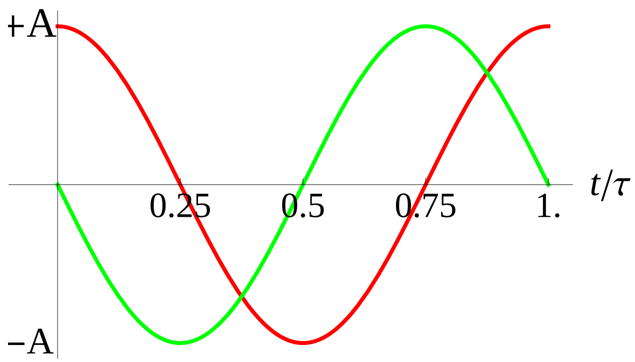
- loop rule: $\frac{Q}{C} + L\frac{dI}{dt} = 0,\ I = \frac{dQ}{dt}$
- equation of motion: $\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$
- charge on capacitor: $Q(t) = Q_{max}\cos(\omega t)$
- current through inductor: $I(t) = -\omega Q_{max}\sin(\omega t)$
- angular frequency: $\omega = \frac{1}{\sqrt{LC}}$
- magnetic energy: $U_B = \frac{1}{2}LI^2$ (stored on inductor)
- electric energy: $U_E = \frac{Q^2}{2C}$ (stored on capacitor)
- total energy: $E = U_B + U_E = \text{const.}$



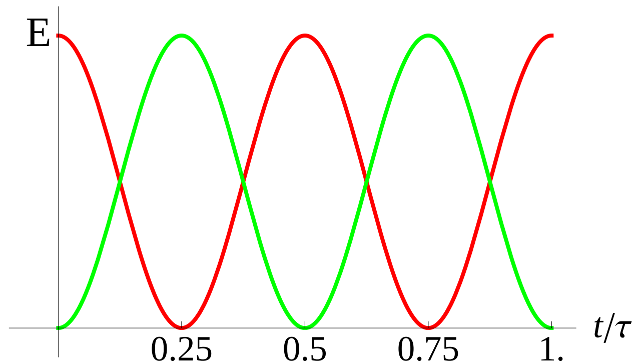
Mechanical vs Electromagnetic Oscillations

mechanical oscillations

- position: $x(t) = A\cos(\omega t)$ [red]
- velocity: $v(t) = -A\sin(\omega t)$ [green]
- period: $\tau = \frac{2\pi}{\omega}, \quad \omega = \sqrt{\frac{k}{m}}$



- potential energy: $U(t) = \frac{1}{2}kx^2(t)$ [r]
- kinetic energy: $K(t) = \frac{1}{2}mv^2(t)$ [g]
- total energy: $E = U(t) + K(t) = \text{const}$



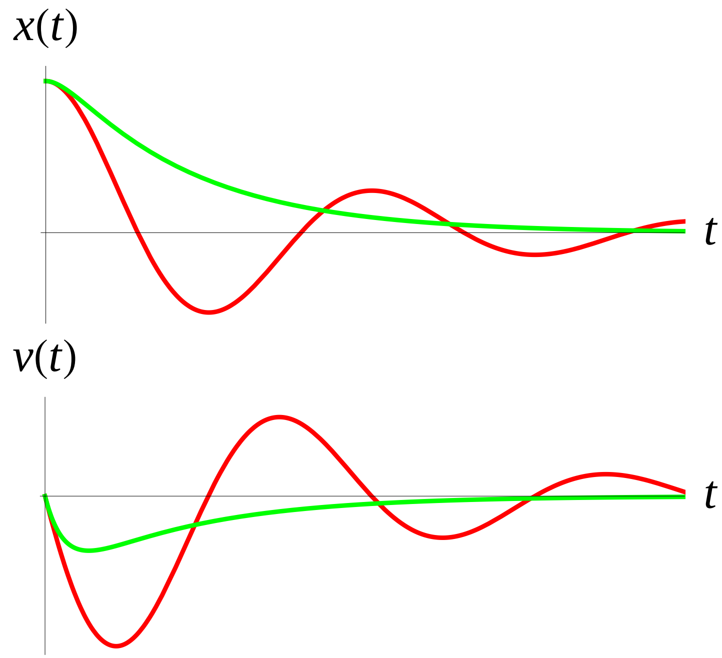
electromagnetic oscillations

- charge: $Q(t) = A\cos(\omega t)$ [red]
- current: $I(t) = -A\sin(\omega t)$ [green]
- period: $\tau = \frac{2\pi}{\omega}, \quad \omega = \frac{1}{\sqrt{LC}}$

- electric energy: $U_E(t) = \frac{1}{2C}Q^2(t)$ [r]
- magnetic energy: $U_B(t) = \frac{1}{2}LI^2(t)$ [g]
- total energy: $E = U_E(t) + U_B(t) = \text{const}$

Mechanical Oscillator with Damping

- law of motion: $F = ma, \quad a = \frac{d^2x}{dt^2}$
- law of force: $F = -kx - bv, \quad v = \frac{dx}{dt}$
- equation of motion: $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$



Solution for initial conditions $x(0) = A,\ v(0) = 0$:

(a) **underdamped motion**: $b^2 < 4km$

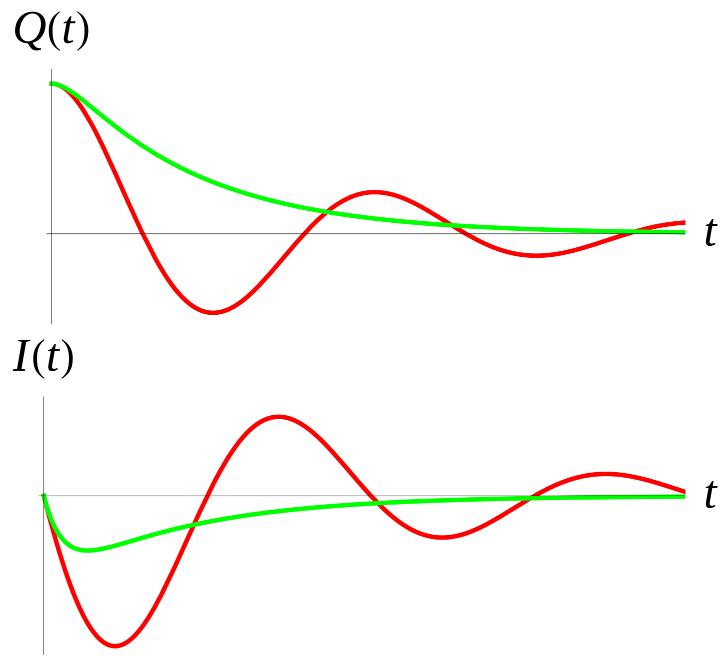
$$x(t) = Ae^{-bt/2m}\left[\cos(\omega' t) + \frac{b}{2m\omega'}\sin(\omega' t)\right] \quad \text{with} \quad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

(b) **overdamped motion**: $b^2 > 4km$

$$x(t) = Ae^{-bt/2m}\left[\cosh(\Omega' t) + \frac{b}{2m\Omega'}\sinh(\Omega' t)\right] \quad \text{with} \quad \Omega' = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

Damped Electromagnetic Oscillator (RLC Circuit)

- loop rule: $RI + L\frac{dI}{dt} + \frac{Q}{C} = 0,\; I = \frac{dQ}{dt}$
- equation of motion: $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0$



Solution for initial conditions $Q(0) = Q_{max},\; I(0) = 0$:

(a) **underdamped motion**: $R^2 < \frac{4L}{C}$

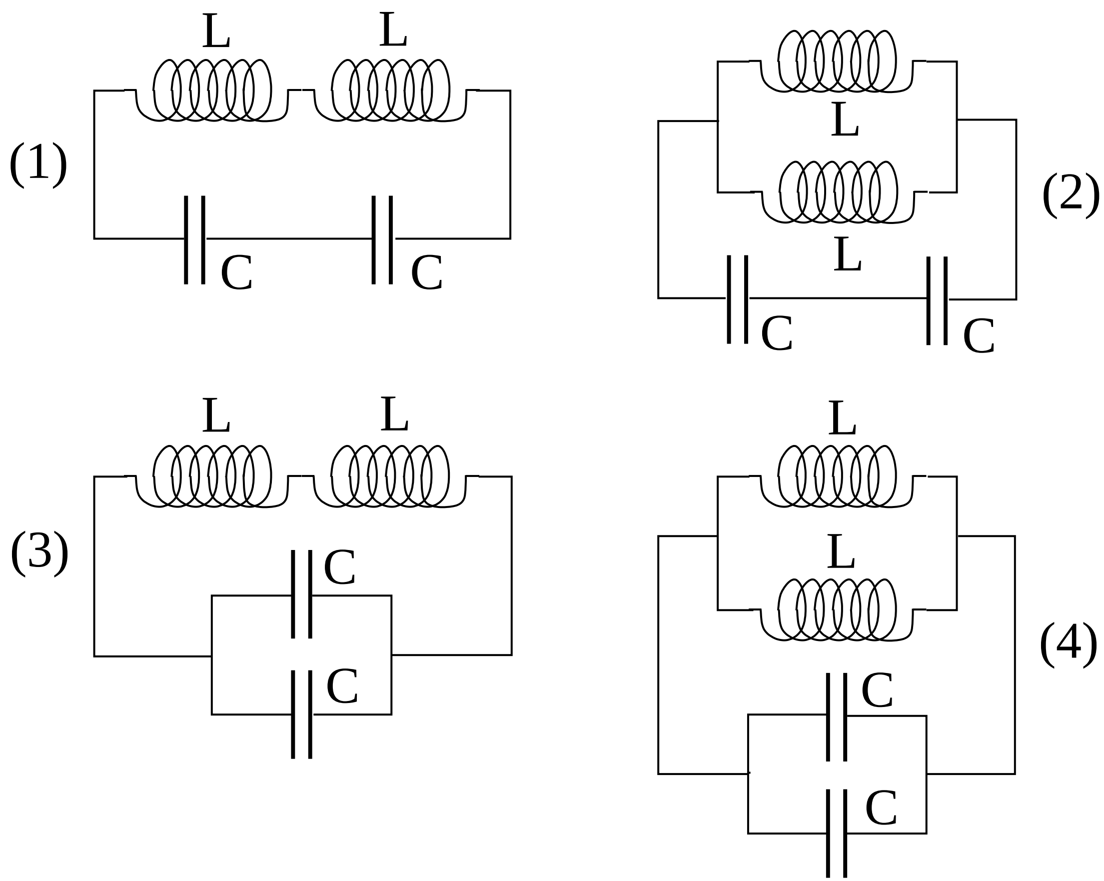
$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[\cos(\omega' t) + \frac{R}{2L\omega'}\sin(\omega' t)\right] \quad \text{with} \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

(b) **overdamped motion**: $R^2 > \frac{4L}{C}$

$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[\cosh(\Omega' t) + \frac{R}{2L\Omega'}\sinh(\Omega' t)\right] \quad \text{with} \quad \Omega' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

LC Circuit: Application (1)

Name the LC circuit with the highest and the lowest angular frequency of oscillation.



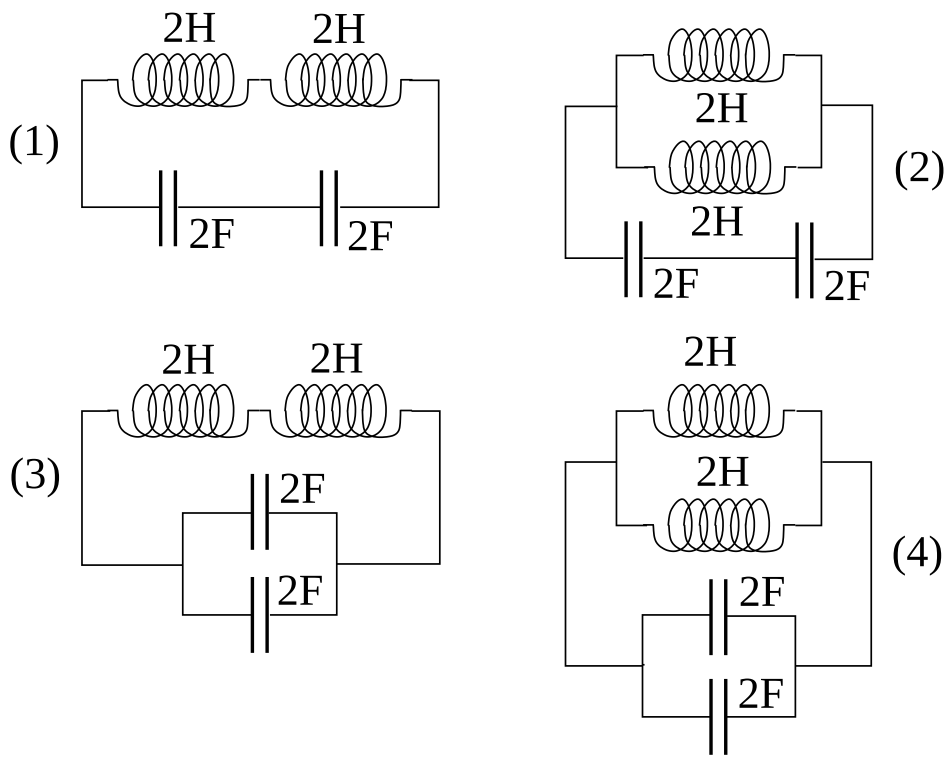
LC Circuit: Application (2)

At time $t = 0$ a charge $Q = 2$C is on each capacitor and all currents are zero.

(a) What is the energy stored in the circuit?

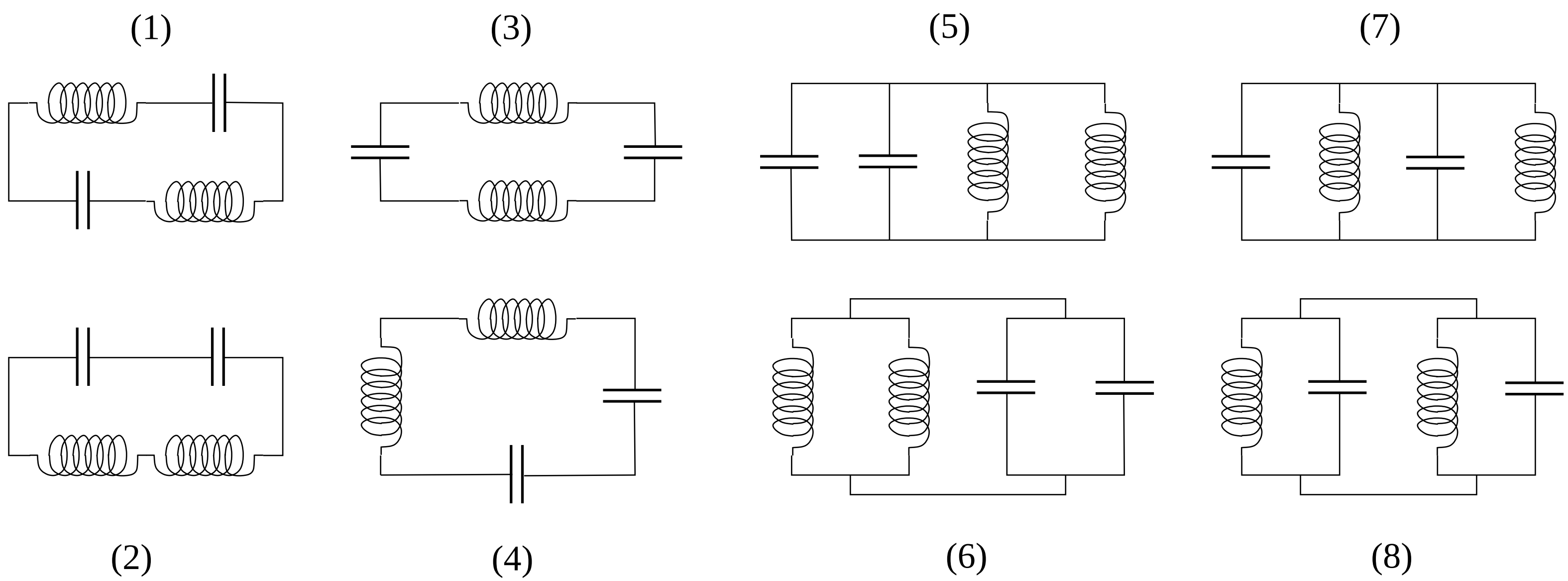
(b) At what time t_1 are the capacitors discharged for the first time?

(c) What is the current through each inductor at time t_1?



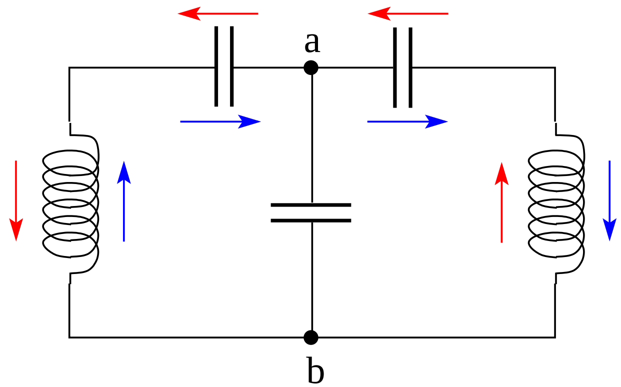
LC Circuit: Application (3)

In these LC circuits all capacitors have equal capacitance C and all inductors have equal inductance L. Sort the circuits into groups that are equivalent.



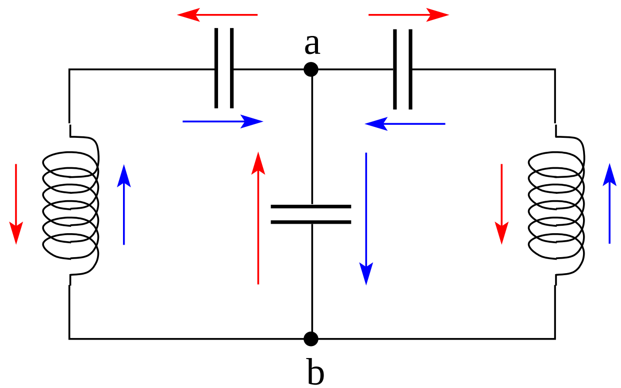
Oscillator with Two Modes

Electromagnetic:



mode #1: $L\frac{dI}{dt} + \frac{Q}{C} + \frac{Q}{C} + L\frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt}$

$\Rightarrow \frac{dI}{dt} = -\frac{Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}}$



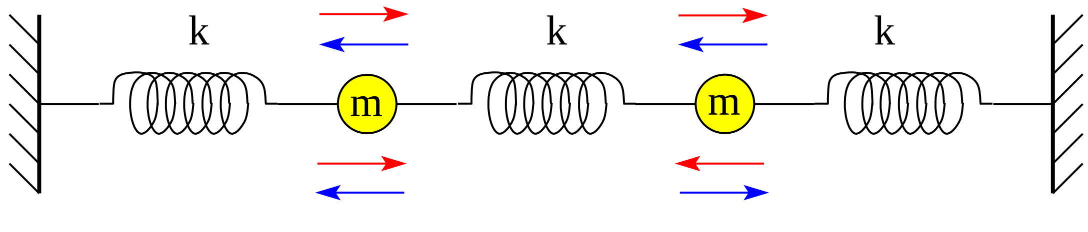
mode #2: $L\frac{dI}{dt} + \frac{Q}{C} + \frac{2Q}{C} = 0, \quad I = \frac{dQ}{dt}$

$\Rightarrow \frac{dI}{dt} = -\frac{3Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \sqrt{\frac{3}{LC}}$

Mechanical:

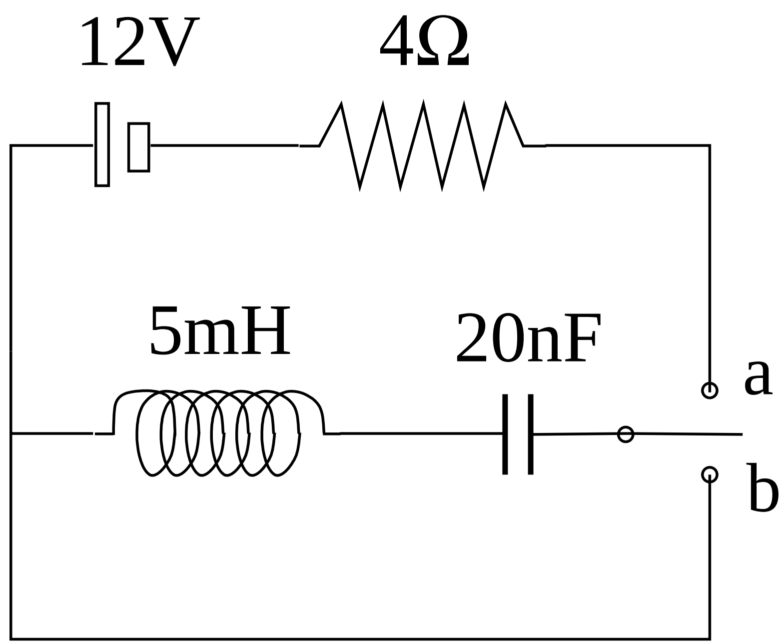
mode #1: $\omega = \sqrt{\frac{k}{m}}$

mode #2: $\omega = \sqrt{\frac{3k}{m}}$



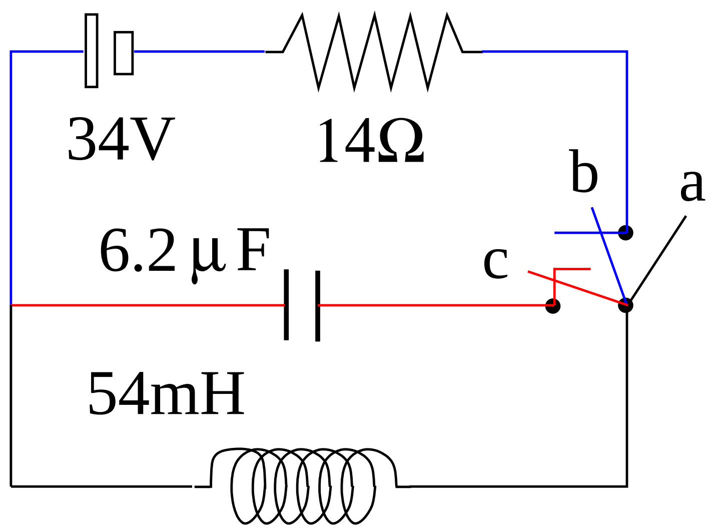
RLC Circuit: Application (1)

In the circuit shown the capacitor is without charge.
When the switch is closed to position a...
(a) find the initial rate dI/dt at which the current increases from zero,
(b) find the charge Q on the capacitor after a long time.
Then, when the switch is thrown from a to b...
(c) find the time t_1 it takes the capacitor to fully discharge,
(d) find the maximum current I_{max} in the process of discharging.



RLC Circuit: Application (2)

In the circuit shown the capacitor is without charge and the switch is in position a.
(i) When the switch is moved to position b we have an RL circuit with the current building up gradually: $I(t) = (\mathcal{E}/R)[1 - e^{-t/\tau}]$.
Find the time constant τ and the current I_{max} after a long time.
(ii) Then we reset the clock and move the switch from b to c with no interruption of the current through the inductor. We now have a an LC circuit: $I(t) = I_{max}\cos(\omega t)$.
Find the angular frequency of oscillation ω and the maximum charge Q_{max} that goes onto the capacitor periodically.



RLC Circuit: Application (3)

In the circuit shown the capacitor is without charge and the switch is in position a.
(i) When the switch is moved to position b we have an RC circuit with the capacitor being charged up gradually: $Q(t) = \mathcal{E}C[1 - e^{-t/\tau}]$.
Find the time constant τ and the charge Q_{max} after a long time.
(ii) Then we reset the clock and move the switch from b to c.
We now have a an LC circuit: $Q(t) = Q_{max}\cos(\omega t)$.
Find the angular frequency of oscillation ω and the maximum current I_{max} that flows through the inductor periodically.