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Global Dynamics and Bifurcations of Two Quadratic Fractional Second Order Difference Equations

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Abstract

We investigate the local stability and the global asymptotic stability of the following two difference equation

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2 + B x_n x_{n-1}}, \quad x_0 + x_{-1} > 0, \quad A + B > 0$$

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}, \quad x_0 > 0, \quad A > 0$$

where all parameters and initial conditions are positive.

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1 Introduction and Preliminaries

We investigate global behavior of the equations:

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2 + B x_n x_{n-1}}, \tag{1}$$

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}, \tag{2}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. Equations (1), (2)) are the special cases of equations

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2 + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, ... \tag{3}$$

and

$$x_{n+1} = \frac{A x_n^2 + B x_n x_{n-1} + C x_{n-1}^2 + D x_n + E x_{n-1} + F}{a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{4}$$

Some special cases of equation (4) have been considered in the series of papers [2, 3, 9, 10, 17]. Some special second order quadratic fractional difference equations have appeared in analysis of competitive and anti-competitive systems of linear fractional difference equations in the plane, see [**?**, 6, **?**, 14]. Local stability analysis of the equilibrium solutions of equation (3) was performed in [8].

Describing the global dynamics of equation (4) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear fractional difference equation

$$x_{n+1} = \frac{D x_n + E x_{n-1} + F}{d x_n + e x_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{5}$$

In this paper we take a different approach based on the theory of monotone maps developed in [12, 13] and use it to describe precisely the basins of attraction of all attractors of this equation.

Our results will be based on the following theorem for a general second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, 2, ..., \tag{6}$$

see [**?**].

Theorem 1 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is non-increasing in the first variable and non-decreasing in the second variable. Then, for ever solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad x_{-1}, x_0 \in I, \ n = 0, 1, 2, ... \tag{7}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution do exactly one of the following:
(i) Eventually they are both monotonically increasing.
(ii) Eventually they are both monotonically decreasing.
(iii) One of them is monotonically increasing and the other is monotonically decreasing.

The consequence of Theorem 1 is that every bounded solution of (7) converges to either equilibrium or period-two solution or to the point on the boundary, and most important question becomes determining the basins of attraction of these solutions as well as the unbounded solutions. The answer to this question follows from an application of theory of monotone maps in the plane which will be presented in Preliminaries.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $x \preceq y$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \tag{8}$$

provided that $\mathcal{R}$ is open and convex.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at x and numbered in a counter-clockwise direction, for example, $Q_1(x) = \{y \in R^2 : \ x_1 \leq y_1, \ x_2 \leq y_2\}$. Basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The next five results, from [13, 12], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [16, 17].

Theorem 2 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 4 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 3 *For the curve $\mathcal{C}$ of Theorem 2 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 2 reduces just to $|\lambda| < 1$. This follows from a change of variables [17] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 4 *Assume the hypotheses of Theorem 2, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{9}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ *is invariant, and* $\operatorname{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ *is invariant, and* $\operatorname{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ *is an interior point of* $\mathcal{R}$ *and* T *is* C^2 *and strongly competitive in a neighborhood of* $\overline{\mathrm{x}}$*, then* T *has no periodic points in the boundary of* $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ *except for* $\overline{\mathrm{x}}$*, and the following statements are true.*

(iii) For every $x \in \mathcal{W}_-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_2(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(x) \in int\, Q_4(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 5 *In addition to the hypotheses of part (B) of Theorem 4, suppose that* $\mu > 1$ *and that the eigenspace* E^μ *associated with* μ *is not a coordinate axis. If the curve* $\mathcal{C}$ *of Theorem 2 has endpoints in* $\partial\mathcal{R}$*, then* $\mathcal{C}$ *is the stable set* $\mathcal{W}^s(\overline{\mathrm{x}})$ *of* $\overline{\mathrm{x}}$*, and the unstable set* $\mathcal{W}^u(\overline{\mathrm{x}})$ *of* $\overline{x}$ *is a curve in* $\mathcal{R}$ *that is tangential to* E^μ *at* $\overline{\mathrm{x}}$ *and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of* $\mathcal{W}^u(\overline{\mathrm{x}})$ *in* $\mathcal{R}$ *are fixed points of* T.

Remark 6 *We say that* $f(u, v)$ *is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative* $D_1 f$ *negative and first partial derivative* $D_2 f$ *positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of equation (7) follows from the fact that if* f *is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to equation (7) is a strictly competitive map on* $I \times I$*, see [13].*

Set $x_{n-1} = u_n$ *and* $x_n = v_n$ *in equation (7) to obtain the equivalent system*

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots.$$

Let $T(u, v) = (v, f(v, u))$*. The second iterate* T^2 *is given by*

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$*, see [13].*

Remark 7 *The characteristic equation of equation (7) at an equilibrium point* $(\bar{x}, \bar{x})$*:*

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{10}$$

has two real roots λ, μ *which satisfy* $\lambda < 0 < \mu$*, and* $|\lambda| < \mu$*, whenever* f *is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 2-5 depends on the nonexistence of minimal period-two solution.*

There are several global attractivity results for equation (7). Some of these results give the sufficient conditions for all solutions to approach a unique equilibrium and they were used efficiently in [11].

The next result is from [4].

Theorem 8 *(See [4]) Consider equation (7) where* $f : I \times I \to I$ *is a continuous function and* f *is decreasing in the first argument and increasing in the second argument. Assume that* $\overline{x}$ *is a unique equilibrium point which is locally*

asymptotically stable and assume that (φ,ψ) *and* (ψ,φ) *are minimal period-two solutions which are saddle points such that*

$$(\varphi,\psi)\preceq_{se}(\overline{x},\overline{x})\preceq_{se}(\psi,\varphi).$$

Then, the basin of attraction $\mathcal{B}((\overline{x},\overline{x}))$ *of* $(\overline{x},\overline{x})$ *is the region between the global stable sets* $\mathcal{W}^s((\varphi,\psi))$ *and* $\mathcal{W}^s((\psi,\varphi))$. *More precisely*

$$\mathcal{B}((\overline{x},\overline{x}))=\{(x,y):\exists y_u,y_l:y_u<y<y_l,(x,y_l)\in\mathcal{W}^s((\varphi,\psi)),(x,y_u)\in\mathcal{W}^s((\psi,\varphi))\}.$$

The basins of attraction $\mathcal{B}((\varphi,\psi))=\mathcal{W}^s((\varphi,\psi))$ *and* $\mathcal{B}((\psi,\varphi))=\mathcal{W}^s((\psi,\varphi))$ *are exactly the global stable sets of* (φ,ψ) *and* (ψ,φ).

If $(x_{-1},x_0)\in\mathcal{W}_+((\psi,\varphi))$ *or* $(x_{-1},x_0)\in\mathcal{W}_-((\varphi,\psi))$, *then* $T^n((x_{-1},x_0))$ *converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region* $I\times I$.

2 Equation $x_{n+1}=\frac{\beta x_n^2+\gamma x_{n-1}}{Ax_n^2+Bx_nx_{n-1}}$

2.1 Local stability analysis

By substitution $x_n=\frac{\beta}{A}y_n$, this equation is reduced to the equation

$$y_{n+1}=\frac{y_ny_{n-1}+\frac{\gamma A}{\beta^2}y_{n-1}}{y_n^2+\frac{B}{A}y_ny_{n-1}},\ n=0,1,...$$

Thus we consider the following equation

$$x_{n+1}=\frac{x_nx_{n-1}+\gamma x_{n-1}}{x_n^2+Bx_nx_{n-1}},\ n=0,1,... \tag{11}$$

Equation (11) has the unique positive equilibrium $\overline{x}$ given by

$$\overline{x}=\frac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}.$$

The partial derivatives associated to the Eq(11) at equilibrium $\overline{x}$ are

$$f'_x=\frac{-x^2y-2\gamma xy-B\gamma y^2}{(x^2+Bxy)^2}\Big|_{\overline{x}}=\frac{-2(1+2(1+B)(2+B)\gamma\sqrt{1+4(1+B)\gamma})}{(1+B)(1+\sqrt{1+4(1+B)\gamma})^2},\quad f'_y=\frac{x+\gamma}{(x+By)^2}\Big|_{\overline{x}}=\frac{1}{1+B}.$$

Characteristic equation associated to the equation (11) at equilibrium is

$$\lambda^2+\frac{2(1+2(1+B)(2+B)\gamma\sqrt{1+4(1+B)\gamma})}{(1+B)(1+\sqrt{1+4(1+B)\gamma})^2}\lambda-\frac{1}{1+B}=0.$$

By applying the linearized stability Theorem we obtain the following result.

Theorem 9 *The unique positive equilibrium point* $\overline{x}=\frac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}$ *of equation (11) is:*

i) locally asymptotically stable when $B>4\gamma+1$;

ii) a saddle point when $B<4\gamma+1$;

ii) a nonhyperbolic point (with eigenvalues $\lambda_1=-1$ *and* $\lambda_2=\frac{1}{2+4\gamma}$*) when* $B=4\gamma+1$.

Lemma 10 *If*

$$B > 1 + 4\gamma$$

then equation (11) possesses a unique minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ *where*

$$\phi = \frac{1}{2} - \tfrac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \text{ and } \psi = \frac{1}{2} + \tfrac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}.$$

The minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ *is a saddle point.*

Proof. Periodic solution is positive solution of the following system

$$\begin{cases} (B-1)y - \gamma = 0 \\ -xy + y = 0. \end{cases} \tag{12}$$

where $\phi + \psi = x$ and $\phi\psi = y$. We have that solutions of system (12) is

$$x = 1 \text{ and } y = \frac{\gamma}{B-1}.$$

Since

$$x^2 - 4y = \frac{B-1-4\gamma}{B-1} > 0$$

if and only if $B > 1 + 4\gamma$, we have a unique minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ where

$$\phi = \frac{1}{2} - \tfrac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \text{ and } \psi = \frac{1}{2} + \tfrac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}.$$

Set

$$u_n = x_{n-1} \text{ and } v_n = x_n, \text{ for } n = 0, 1, ...$$

and write equation (11) in the equivalent form

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{u_n v_n + \gamma u_n}{v_n^2 + B u_n v_n}, \ n = 0, 1, ... \ . \end{aligned}$$

Let T be the function on $(0,\infty) \times (0,\infty)$ defined by

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{uv+\gamma u}{v^2+Buv} \end{pmatrix}.$$

By a starightforward calculation we find that

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

where

$$g(u,v) = \frac{uv + \gamma u}{v^2 + Buv}, \quad h(u,v) = \frac{v^2(Bu+v)(v^2\gamma + u(v+\gamma+Bv\gamma))}{u(v+\gamma)(Bv^3 + u(v+B^2v^2+\gamma))}.$$

We have

$$J_{T^2}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} g'_u(\phi,\psi) & g'_v(\phi,\psi) \\ h'_u(\phi,\psi) & h'_v(\phi,\psi) \end{pmatrix}$$

where

$$\begin{aligned} g'_u &= \tfrac{v+\gamma}{(Bu+v)^2}, \\ g'_v &= -\tfrac{u(v^2+B\gamma u+2\gamma v)}{v^2(Bu+v)^2}, \end{aligned}$$

$$h'_u = -\frac{v^3(B\gamma v^5+2u\gamma v^2(v+\gamma+B^2v^2)+u^2(v^2+v(2+Bv(2+B^2v))\gamma+(1+2Bv)\gamma^2))}{u^2(v+\gamma)(Bv^3+u(v+B^2v^2+\gamma))^2},$$

$$\begin{aligned} h'_v &= \frac{v(B^4u^3v^3\gamma^2+B^3u^2v^4\gamma(v+4\gamma)+B(v+2\gamma)(v^6\gamma+4u^2v^2\gamma(v+\gamma)+u^3(v+\gamma)^2)}{u^2(v+\gamma)(Bv^3+u(v+B^2v^2+\gamma))^2} \\ &\quad + \frac{B^2uv\gamma(u^2(v+\gamma)(v+3\gamma)+v^4(2v+5\gamma))+uv(v+\gamma)(u(v+\gamma)(2v+3\gamma)+v^2\gamma(3v+5\gamma)))}{u^2(v+\gamma)(Bv^3+u(v+B^2v^2+\gamma))^2}. \end{aligned}$$

Set

$$\mathcal{S} = g'_u(\phi,\psi) + h'_v(\phi,\psi), \quad \mathcal{D} = g'_u(\phi,\psi)h'_v(\phi,\psi) - g'_v(\phi,\psi)h'_u(\phi,\psi).$$

After some lengthy calculation one can see that

$$\mathcal{S} = \frac{1+6\gamma+B(-3-6\gamma+B(2+\gamma))}{(B-1)(B+(B-1)\gamma)} \quad \text{and} \quad \mathcal{D} = \frac{\gamma}{(B-1)(B+(B-1)\gamma)}.$$

We have that

$$|\mathcal{S}| > |1+\mathcal{D}| \quad \text{if and only if} \quad B > 1+4\gamma.$$

By applying the linearized stability Theorem we obtain that a unique prime period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ of equation (11) is a saddle point if and only if $B > 1+4\gamma$. ■

2.2 Global results and basins of attraction

In this section we present global dynamics results for equation (11).

Theorem 11 *If $B > 4\gamma+1$ then equation (11) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is locally asymptotically stable and there exist prime period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ where*

$$\phi = \frac{1}{2} - \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \text{ and } \psi = \frac{1}{2} + \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}$$

which is a saddle point.

Furthermore, global stable manifold of the periodic solution $\{P,Q\}$ is given by $\mathcal{W}^s(\{P,Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are continuous increasing curves, that divide the first quadrant into two connected components, namely

$\mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P) : \exists y \in \mathcal{W}^s(P)$ with $y \preceq_{se} x\}$, $\mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P) : \exists y \in \mathcal{W}^s(P)$ with $x \preceq_{se} y\}$, and

$\mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q) : \exists y \in \mathcal{W}^s(Q)$ with $y \preceq_{se} x\}$, $\mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q) : \exists y \in \mathcal{W}^s(Q)$ with $x \preceq_{se} y\}$ respectively such that the following statements are true.

i) If $(u_0,v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q.

ii) If $(u_0,v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P.

iii) If $(u_0,v_0) \in \mathcal{W}_1^-$ (the region above $\mathcal{W}^s(P)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to $(0,\infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to $(\infty,0)$.

iv) If $(u_0,v_0) \in \mathcal{W}_2^+$ (the region below $\mathcal{W}^s(Q)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to $(\infty,0)$ and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to $(0,\infty)$.

v) If $(u_0,v_0) \in \mathcal{W}_1^+ \cap \mathcal{W}_2^-$ (the region between $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$) then the sequence $\{(u_n,v_n)\}$ is attracted to $E(\overline{x},\overline{x})$.

Proof. From Theorem 9 equation (11) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is locally asymptotically stable. Theorem 10 implies that the periodic solution $\{P,Q\}$ is a saddle point. The map $T^2(u,v)=T(T(u,v))$ is competitive on $\mathcal{R}=\mathbb{R}^2\setminus\{(0,0)\}$ and strongly competitive on $int(\mathcal{R})$. It follows from the Perron-Frobenius Theorem and a change of variables that at each point the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, as is well known [12, 13] if the map is strongly competitive then no eigenvector is aligned with a coordinate axis.

i) By Theorem 4 we have that if $(u_0,v_0)\in\mathcal{W}^s(P)$ then $(u_{2n},v_{2n})=T^{2n}(u_0,v_0)\to P$ as $n\to\infty$, which implies that $(u_{2n+1},v_{2n+1})=T(T^{2n}(u_0,v_0))\to T(P)=Q$ as $n\to\infty$, from which it follows the statement i).

ii) The proof of the statement ii) is similar to the proof of the statement i).

iii) A straightforward calculation shows that $(\phi,\psi)\preceq_{se}(\overline{x},\overline{x})\preceq_{se}(\psi,\phi)$. Since equation (11) has no the other equilibrium point or the other minimal-period two solution from Theorem 8 we have if $(x_{-1},x_0)\in\mathcal{W}_1^-$, then

$$(u_{2n},v_{2n})=T^{2n}((u_0,v_0))\to(0,\infty)\text{ and }(u_{2n+1},v_{2n+1})=T^{2n+1}((u_0,v_0))\to(\infty,0).$$

and thence if $(x_{-1},x_0)\in\mathcal{W}_1^-$, then

$$\lim_{n\to\infty}x_{2n}=\infty\text{ and }\lim_{n\to\infty}x_{2n+1}=0.$$

iv) If $(x_{-1},x_0)\in\mathcal{W}_2^+$, then

$$(u_{2n},v_{2n})=T^{2n}((u_0,v_0))\to(\infty,0)\text{ and }(u_{2n+1},v_{2n+1})=T^{2n+1}((u_0,v_0))\to(0,\infty).$$

and hence if $(x_{-1},x_0)\in\mathcal{W}_2^+$, then

$$\lim_{n\to\infty}x_{2n}=0\text{ and }\lim_{n\to\infty}x_{2n+1}=\infty.$$

v) If $(x_{-1},x_0)\in\mathcal{W}_1^+\cap\mathcal{W}_2^-$, then

$$\lim_{n\to\infty}x_n=\tfrac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}.$$

■

Theorem 12 *If $B<4\gamma+1$ then equation (11) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is a saddle point.*

Global stable manifold $\mathcal{W}^s(E)$ which is continuous increasing curve divides the first quadrant such that the following holds:

i) Every initial point (u_0,v_0) in $\mathcal{W}^s(E)$ is attracted to E.

ii) If $(u_0,v_0)\in\mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to $(\infty,0)$ and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to $(0,\infty)$.

iii) If $(u_0,v_0)\in\mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to $(0,\infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to $(\infty,0)$.

Proof. From Theorem 9 equation (11) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is a saddle point. The map T has no fixed points or periodic points of minimal period-two in $\Delta=\mathcal{R}\cap int(Q_1(\overline{x})\cup Q_3(\overline{x}))$. It is easy to see that $detJ_T(E)<0$ and $T(x)=\overline{x}$ only for $x=\overline{x}$. Since the map T is anti-competitive and T^2 is strongly competitive we have that all conditions of Theorem 10 in [7] are satisfied from which the proof follows. ■

Theorem 13 *If $B = 4\gamma + 1$ then equation (11) has a unique equilibrium point $E(\overline{x}, \overline{x}) = (\frac{1}{2}, \frac{1}{2})$ which is a nonhyperbolic point.*

There exists a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and it divides the first quadrant such that the following holds:

i) Every initial point (u_0, v_0) in $\mathcal{C}_E$ is attracted to E.

ii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{C}_E$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(0, \infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(\infty, 0)$.

iii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{C}_E$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(\infty, 0)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(0, \infty)$.

Proof. From Theorem 9 equation (11) has a unique equilibrium point $E(\overline{x}, \overline{x}) = (\frac{1}{2}, \frac{1}{2})$, which is nonhyperbolic. All conditions of Theorem 4 are satisfied, which yields the existence of a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and for every $x \in \mathcal{W}^-(E)$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in intQ_2(\bar{x})$ for $n \geq n_0$ and for every $x \in \mathcal{W}^+(E)$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in intQ_4(\bar{x})$ for $n \geq n_0$.

Set

$$U(t) = \frac{1 - (4\gamma + 1)t + \sqrt{(1 - (4\gamma + 1)t)^2 + 4\gamma}}{2}.$$

It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t < \overline{x}$ and $E \preceq_{se} (t, U(t))$ if $t > \overline{x}$. One can show that

$$T^2(t, U(t)) = \left(t, \frac{2\gamma(t + \gamma)}{t(-t + t^2 + 2\gamma + 4t^2\gamma + 8\gamma^2 + t\sqrt{4\gamma + (-1 + t + 4t\gamma)^2})}\right).$$

Now we have that

$$T^2(t, U(t)) \preceq_{se} (t, U(t)) \text{ if } t < \overline{x}$$

and

$$(t, U(t)) \preceq_{se} T^2(t, U(t)) \text{ if } t > \overline{x}.$$

By monotonicity if $t < \overline{x}$ we obtain that $T^{2n}(t, U(t)) \to (0, \infty)$ as $n \to \infty$ and if $t > \overline{x}$ then we have that $T^{2n}(t, U(t)) \to (\infty, 0)$ as $n \to \infty$.

If $(u', v') \in intQ_2(\bar{x})$ then there exists t_1 such that $(u', v') \preceq_{se} (t_1, U(t_1)) \preceq_{se} E$. By monotonicity of the map T^2 we obtain that $T^{2n}(u', v') \preceq_{se} T^{2n}(t_1, U(t_1)) \preceq_{se} E$ which implies that $T^{2n}(u', v') \to (0, \infty)$ and $T^{2n+1}(u', v') \to T(0, \infty) = (\infty, 0)$ as $n \to \infty$ which proves the statement ii).

If $(u'', v'') \in intQ_4(\bar{x})$ then there exists t_2 such that $E \preceq_{se} (t_2, U(t_2)) \preceq_{se} (u'', v'')$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(t_2, U(t_2)) \preceq_{se} T^{2n}(u'', v'')$ which implies that $T^{2n}(u'', v'') \to (\infty, 0)$ and $T^{2n+1}(u'', v'') \to T(\infty, 0) = (0, \infty)$ as $n \to \infty$ which proves the statement iii), which completes the proof of the Theorem. ■

3 Equation $x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}$

3.1 Local stability analysis

This equation is reduced to the equation

$$x_{n+1} = \frac{x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{x_n^2}, \ n = 0, 1, ... \tag{13}$$

Equation (13) has the unique positive equilibrium $\bar{x}$ given by

$$\bar{x} = \frac{1 + \beta + \sqrt{(1 + \beta)^2 + 4\gamma}}{2}.$$

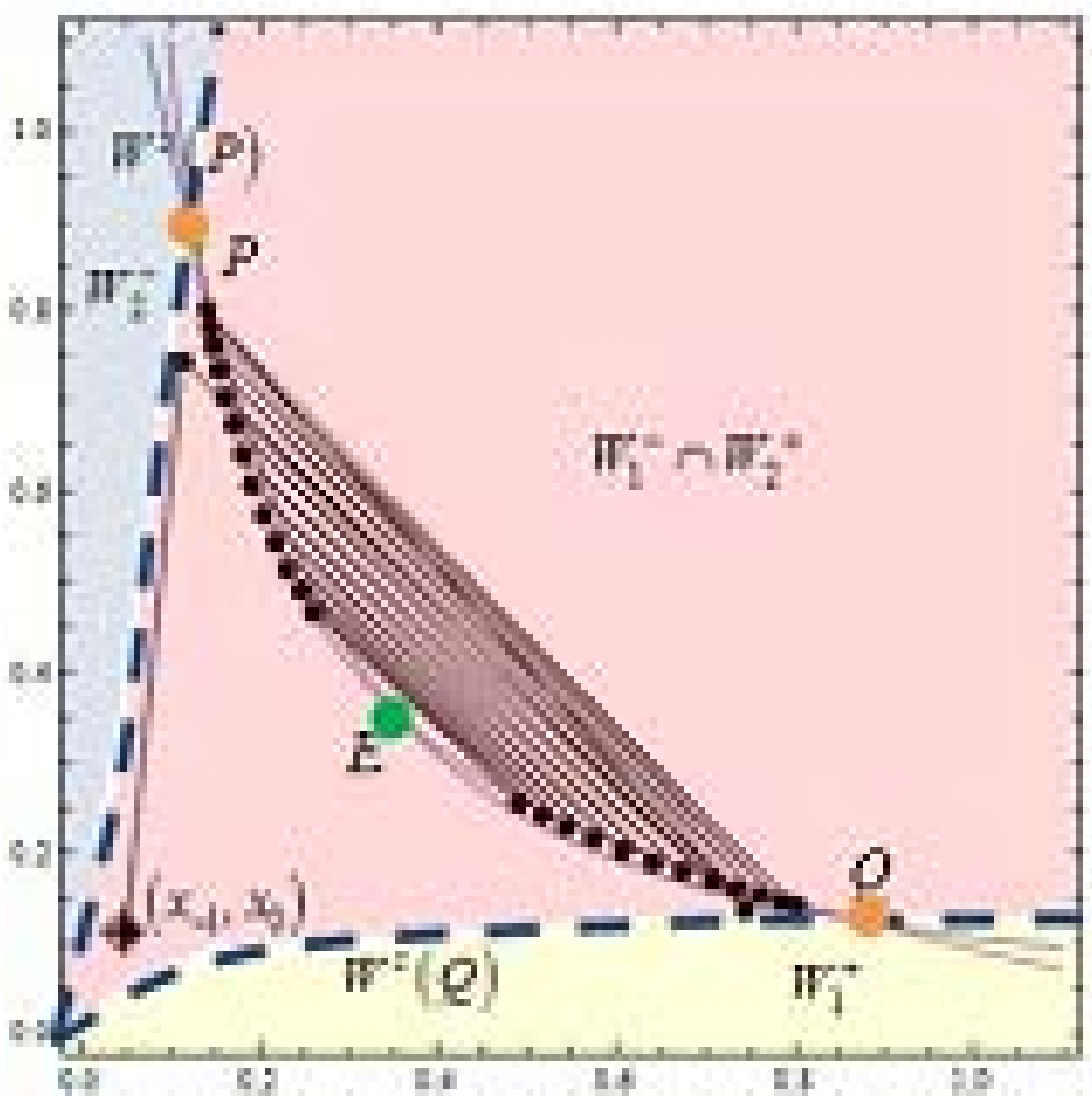

Figure 1: The graph of the function $U_+^2(x,y)=0$ (red curve) and $U_-^2(x,y)=0$ (blue curve) for $a=16.0$, $b=2.0$, $c=0.5$ and $d=0.1$ with the basins of attraction generated by *Dynamica* 3.

The partial derivatives associated to equation (13) at equilibrium $\overline{x}$ are

$$f'_x=\frac{-xy\beta-2\gamma y}{x^3}\Big|_{\overline{x}}=-\frac{2(4\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2},\quad f'_y=\frac{\beta x+\gamma}{x^2}\Big|_{\overline{x}}=\frac{2(2\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2}.$$

Characteristic equation associated to the equation (13) at equilibrium is

$$\lambda^2 + \frac{2(4\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2}\lambda - \frac{2(2\gamma+\beta(1+\beta+\sqrt{(1+\beta)^2+4\gamma}))}{(1+\beta+\sqrt{(1+\beta)^2+4\gamma})^2} = 0.$$

By applying the linearized stability Theorem we obtain the following result.

Theorem 14 *The unique positive equilibrium point* $\bar{x} = \frac{1+\beta+\sqrt{(1+\beta)^2+4\gamma}}{2}$ *of equation (13) is*

i) locally asymptotically stable when $4\gamma + 2\beta + \beta^2 < 3$*;*

ii) a saddle point when $4\gamma + 2\beta + \beta^2 > 3$*;*

iii) a nonhyperbolic point (with eigenvalues $\lambda_1 = -1$ *and* $\lambda_2 = \frac{\beta+1}{\beta+3}$*) when* $4\gamma + 2\beta + \beta^2 = 3$.

Lemma 15 *Equation (13) has the minimal period-two solution* $\{P(\phi,\psi), Q(\psi,\phi)\}$ *where*

$$\phi = \frac{-\gamma+\beta\gamma-\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)} \text{ and } \psi = \frac{-\gamma+\beta\gamma+\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)}$$

if and only if

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

The minimal period-two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$ *s locally asymptotically stable.*

Proof. Two-periodic solution is a positive solution of the following systems

$$\begin{cases} x - y - \gamma = 0 \\ x^2 - xy + (\beta-1)y = 0. \end{cases} \tag{14}$$

where $\phi + \psi = x$ and $\phi\psi = y$. We have that only one solution of system (14) is

$$x = \frac{(\beta-1)\gamma}{\beta+\gamma-1}, \quad y = \frac{-\gamma^2}{\beta+\gamma-1}.$$

Since

$$x^2 - 4y = \frac{\gamma^2(-3+2\beta+\beta^2+4\gamma)}{(\beta+\gamma-1)^2} > 0$$

if and only if

$$\frac{3-2\beta-\beta^2}{4} < \gamma$$

and $x, y > 0$ if and only if $\beta < 1$ and $\gamma < 1-\beta$, we have that ϕ and ψ are solution of the equation

$$t^2 - \frac{(\beta-1)\gamma}{\beta+\gamma-1}t + \frac{-\gamma^2}{\beta+\gamma-1} = 0$$

if and only if

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

The second iterate of the map T is

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

where

$$g(u,v)=\frac{v^2+\beta uv+\gamma u}{v^2},\quad h(u,v)=\frac{v^4\left(1+v(\beta+\gamma)+\frac{u(2+v\beta)(v\beta+\gamma)}{v^2}+\frac{u^2(v\beta+\gamma)^2}{v^4}\right)}{\left(v^2+\beta uv+\gamma u\right)^2}.$$

We have

$$J_{T^2}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}g'_u(\phi,\psi) & g'_v(\phi,\psi)\\ h'_u(\phi,\psi) & h'_v(\phi,\psi)\end{pmatrix}$$

where

$$\begin{aligned} g'_u &= \tfrac{v\beta+\gamma}{v^2},\\ g'_v &= -\tfrac{u(v\beta+2\gamma)}{v^3},\end{aligned}$$

$$h'_u=-\tfrac{v^3(v\beta+\gamma)(uv\beta^2+u\beta\gamma+v^2(\beta+2\gamma))}{(v^2+\beta uv+\gamma u)^3},$$

$$h'_v=\tfrac{v^2(5u^2v\beta^2\gamma+3u^2\beta\gamma^2+v^4(\beta+\gamma)+3uv^3\beta(\beta+\gamma)+uv^2(2u\beta^3+\gamma(u\beta+5\gamma)))}{(v^2+\beta uv+\gamma u)^3}.$$

Set

$$\mathcal{S}=g'_u(\phi,\psi)+h'_v(\phi,\psi)\quad\text{and}\quad\mathcal{D}=g'_u(\phi,\psi)h'_v(\phi,\psi)-g'_v(\phi,\psi)h'_u(\phi,\psi).$$

After some lengthy calculation one can see that

$$\mathcal{S}=\tfrac{4+\beta(-6+\beta+\beta^2)-9\gamma+\beta(7+\beta)\gamma+6\gamma^2}{\gamma^2},\quad\mathcal{D}=\tfrac{(-1+\gamma)(-1+\beta+\gamma)}{\gamma^2}.$$

Applying the linearized stability Theorem we obtain that a unique prime period-two solution $\{P(\phi,\psi),Q(\psi,\phi)\}$ of Eq(13) is locally asymptotically stable when

$$\beta<1\text{ and }\frac{3-2\beta-\beta^2}{4}<\gamma<1-\beta.$$

■

3.2 Global results and basins of attraction

In this section we present global dynamics results for equation (13).

Theorem 16 *If $4\gamma+2\beta+\beta^2<3$ then equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is globally asymptotically stable.*

Proof. From Theorem 14 equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is locally asymptotically stable. Every solution of equation (13) is bounded from above and from below by positive constants. If $\beta+\gamma<1$ then $4\gamma+2\beta+\beta^2<3$ and we have

$$x_{n+1}=\frac{x_n^2+\beta x_nx_{n-1}+\gamma x_{n-1}}{x_n^2}\geq 1$$

and

$$\begin{aligned} x_{n+1} &= 1+\frac{\beta x_{n-1}}{x_n}+\frac{\gamma x_{n-1}}{x_n^2}\leq 1+\beta x_{n-1}+\gamma x_{n-1}=1+(\beta+\gamma)x_{n-1}.\\ x_{2n} &\leq 1+(\beta+\gamma)[1+(\beta+\gamma)x_{2n-4}]\leq ...\\ &\leq 1+(\beta+\gamma)+(\beta+\gamma)^2+...+(\beta+\gamma)^nx_0,\\ &< \frac{1}{1-\alpha-\beta}+(\beta+\gamma)^nx_0,\\ x_{2n-1} &\leq 1+(\beta+\gamma)[1+(\beta+\gamma)x_{2n-5}]\leq ...\\ &\leq 1+(\beta+\gamma)+(\beta+\gamma)^2+...+(\beta+\gamma)^nx_{-1}\\ &< \frac{1}{1-\alpha-\beta}+(\beta+\gamma)^nx_{-1}.\end{aligned}$$

Equation (13) has no other equilibrium points or period two points and using Theorem 1 we have that equilibrium point $E(\overline{x},\overline{x})$ is globally asymptotically stable. ■

Theorem 17 *If $4\gamma+2\beta+\beta^2>3$ and $\beta+\gamma<1$ then equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$ which is a saddle point and minimal period-two solution $\{P(\phi,\psi),Q(\psi,\phi)\}$ which is locally asymptotically stable, where*

$$\phi=\frac{-\gamma+\beta\gamma-\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)},\ \psi=\frac{-\gamma+\beta\gamma+\gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)}.$$

The global stable manifold $\mathcal{W}^s(E)$ which is a continuous increasing curve, divides the first quadrant such that the following holds:

i) Every initial point (u_0,v_0) in $\mathcal{W}^s(E)$ is attracted to E.

ii) If $(u_0,v_0)\in\mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P.

iii) If $(u_0,v_0)\in\mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q.

Proof. From Theorem 14 equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is a saddle point. The map T has no fixed points or periodic points of minimal period-two in $\Delta=\mathcal{R}\cap int(Q_1(\overline{x})\cup Q_3(\overline{x}))$. A straightforward calculation shows that $detJ_T(E)<0$ and $T(x)=\overline{x}$ only for $x=\overline{x}$. Since the map T is anti-competitive and T^2 is strongly competitive we have that all conditions of Theorem 10 in [7] are satisfied from which the proof follows. ■

Theorem 18 *If $4\gamma+2\beta+\beta^2>3$ and $\beta+\gamma\geq 1$ then equation (13) has a unique equilibrium point $E(\overline{x},\overline{x}\ )$ which is a saddle point.*

The global stable manifold $\mathcal{W}^s(E)$, which is a continuous increasing curve divides the first quadrant such that the following holds:

i) Every initial point (u_0,v_0) in $\mathcal{W}^s(E)$ is attracted to E.

ii) If $(u_0,v_0)\in\mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to $(\infty,1)$ and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to $(1,\infty)$.

iii) If $(u_0,v_0)\in\mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to $(1,\infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to $(\infty,1)$.

Proof. The proof is similar to the proof of the previous theorem using the fact that every solution of equation (13) is bounded from below by 1. ■

Theorem 19 *If $4\gamma+2\beta+\beta^2=3$ then equation (13) has a unique equilibrium point $E(\overline{x},\overline{x}\ )$ which is a nonhyperbolic point and a global attractor. There exists a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and it divides the first quadrant such that the following holds:*

Proof. From Theorem 14 equation (13) has a unique equilibrium point $E(\overline{x},\overline{x})$, which is a non-hyperbolic. All conditions of Theorem 4 are satisfies, which yields the existence a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and for every $x\in\mathcal{W}^-(E)$ (the region above $\mathcal{C}_E$) there exists $n_0\in\mathbb{N}$ such that $T^n(x)\in intQ_2(\overline{x})$ for $n\geq n_0$ and for every $x\in\mathcal{W}^+(E)$ (the region below $\mathcal{C}_E$) there exists $n_0\in\mathbb{N}$ such that $T^n(x)\in intQ_4(\overline{x})$ for $n\geq n_0$.

Set

$$U(t)=\frac{\beta t+\sqrt{\beta^2t^2+(3-2\beta-\beta^2)(t^2-t)}}{2(t-1)}.$$

It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t < \overline{x}$ and $E \preceq_{se} (t, U(t))$ if $t > \overline{x}$. One can show that

$$T^2(t, U(t)) = \left(t, \frac{(t\beta + s)^4 (8t^2 + \frac{t\beta + (3 + (-2 + 4t - \beta)\beta)}{t-1} + \frac{(3 + (-2 + 4t - \beta)\beta)s}{t-1}}{8t^4(-3 + t(3 + (-2 + 4t - \beta)\beta) + \beta(2 + \beta + 2s))^2} \right),$$

where

$$s = \sqrt{t(t(3 - 2\beta) + (\beta - 1)(\beta + 3)}$$

Now we have that

$$T^2(t, U(t)) \preceq_{se} (t, U(t)) \text{ if } t > \overline{x}.$$

and

$$(t, U(t)) \preceq_{se} T^2(t, U(t)) \text{ if } t < \overline{x}$$

since

$$\frac{(t\beta+s)^4(8t^2+\frac{t\beta+(3+(-2+4t-\beta)\beta)}{t-1}+\frac{(3+(-2+4t-\beta)\beta)s}{t-1}}{8t^4(-3+t(3+(-2+4t-\beta)\beta)+\beta(2+\beta+2s))^2} - \frac{\beta t+\sqrt{\beta^2t^2+(3-2\beta-\beta^2)(t^2-t)}}{2(t-1)} > 0,$$

if and only if $t > \overline{x}$. By monotonicity if $t < \overline{x}$ then we obtain that $T^{2n}(t, U(t)) \to E$ as $n \to \infty$ and if $t > \overline{x}$ then we have that $T^{2n}(t, U(t)) \to E$ as $n \to \infty$.

If $(u', v') \in intQ_2(\overline{x})$ then there exists t_1 such that $(t_1, U(t_1)) \preceq_{se} (u', v') \preceq_{se} E$. By monotonicity of the map T^2 we obtain that $T^{2n}(t_1, U(t_1)) \preceq_{se} T^{2n}(u', v') \preceq_{se} E$ which implies that $T^{2n}(u', v') \to E$ and $T^{2n+1}(u', v') \to T(E) = E$, as $n \to \infty$ which proves the statement ii).

If $(u'', v'') \in intQ_4(\overline{x})$ then there exists t_2 such that $E \preceq_{se} (u'', v'') \preceq_{se} (t_2, U(t_2))$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(u'', v'') \preceq_{se} T^{2n}(t_2, U(t_2))$ which implies that $T^{2n}(u'', v'') \to E$ and $T^{2n+1}(u'', v'') \to T(E) = E$ as $n \to \infty$ which proves the statement iii), which completes the proof of the Theorem. ■

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