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18. Hamilton-Jacobi Theory

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Abstract
Part eighteen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc18]

Hamilton's Principal Function [mln96]

We seek a canonical transformation $H(q,p,t) \;\rightarrow\; K(Q,P) \equiv 0$.
Here q stands for $q_1,\ldots,q_n$ etc.

Canonical equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \rightarrow \quad \dot{Q}_i = \frac{\partial K}{\partial P_i} = 0, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0.$$

Hamilton's principal function: $S(q,P,t)$.

- S is the F_2-type generating function of this canonical transformation.
- S depends on $n+1$ variables $q_1,\ldots,q_n,t$ and n parameters $P_1,\ldots,P_n$.
- $p_j = \frac{\partial S}{\partial q_j}, \quad Q_j = \frac{\partial S}{\partial P_j}, \quad K - H = \frac{\partial S}{\partial t}$.

Hamilton-Jacobi equation: $H\left(q_1,\ldots,q_n;\frac{\partial S}{\partial q_n},\ldots,\frac{\partial S}{\partial q_n};t\right) + \frac{\partial S}{\partial t} = 0.$

- First-order partial differential equation for $S(q,P,t)$.
- Integration constants $P_1,\ldots,P_n$ plus additive constant.
- For given solution: $p_j(q,P,t) = \frac{\partial S}{\partial q_j}, \quad Q_j(p,P,t) = \frac{\partial S}{\partial P_j} = \text{const.}$
- $\Rightarrow\; q_j(Q,P,t), \quad p_j(Q,P,t)$ (transformation relations).
- The constants Q_j, P_j are functions of the initial values $q_j^{(0)}, p_j^{(0)}$.

Physical significance of Hamilton's principal function:

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j}\dot{q}_j + \frac{\partial S}{\partial t} = \sum_j p_j\dot{q}_j - H = L.$$

Hamilton's Characteristic Function [mln97]

Two distinct ways of solving the Hamilton-Jacobi equation become available when the Hamiltonian does not explicitly depend on time.

If $H(q,p) = E = \text{const.}$ then $\dfrac{\partial S}{\partial t} = -E = \text{const.}$

Set $S(q,P,t) = W(q,P) - Et$.

Hamilton's characteristic function: $W(q_1,\ldots,q_n;P_1,\ldots,P_n)$.

Method #1:

- Solve the Hamilton-Jacobi equation $H\left(q, \dfrac{\partial S}{\partial q}\right) + \dfrac{\partial S}{\partial t} = 0.$
- Proceed as in [mln96] but use $S(q,P,t) = W(q,P) - Et$.
- One of the integration constants is reserved: $P_1 = E$.

Method #2:

- Solve the Hamilton-Jacobi equation $H\left(q, \dfrac{\partial W}{\partial q}\right) - E = 0.$
- $W(q,P)$ is a F_2-type generating function of a canonical transformation to action-angle coordinates with $P_1 = K(P) = E$.
- Canonical Equations: $\dot{Q}_j = \dfrac{\partial K}{\partial P_j} = \delta_{j1}, \quad \dot{P}_j = -\dfrac{\partial K}{\partial Q_j} = 0.$
- Solution: $P_j = \text{const.}, \quad Q_j = t\delta_{j1} + Q_j^{(0)}$.
- Transformation to original canonical coordinates:

$$Q_j = \frac{\partial}{\partial P_j} W(q,P), \quad p_j = \frac{\partial}{\partial q_j} W(q,P).$$

$$\Rightarrow \; q_j = q_j\left(Q^{(0)}, P, t\right), \quad p_j = p_j\left(Q^{(0)}, P, t\right).$$

[mex97] Hamilton-Jacobi equation for the harmonic oscillator

Determine the time evolution of the canonical coordinates $q(t), p(t)$ for the harmonic oscillator, $H(q,p) = p^2/2m + \frac{1}{2}m\omega_0^2 q^2$, by solving the Hamilton-Jacobi equation along two different avenues.
(a) Use the ansatz $S(q,E,t) = W(q,E) - Et$ for Hamilton's principal function. Solve the Hamilton-Jacobi equation for $S(q,E,t)$. Use $Q = \partial S/\partial E$ to derive $q(t)$ and $\partial S/\partial q$ to derive $p(t)$.
(b) Solve the Hamilton-Jacobi equation for Hamilton's principal function $W(q,E)$. Use $Q = \partial W/\partial E$ to derive $q(E,Q)$ and $\partial W/\partial q$ to derive $p(E,Q)$. Substitute these results into $H(q,p)$ to obtain the transformed Hamiltonian $K(E) = E$. Solve the canonical equations for the transformed canonical coordinates Q, E and substitute them into $q(E,Q)$ and $p(E,Q)$ to obtain $q(t), p(t)$.

Solution:

[mex98] Hamilton's principal function for central force problem

Consider the one-body central-force problem specified by the Hamiltonian

$$H(r, p, \ell) = \frac{1}{2m}\left(p^2 + \frac{\ell^2}{r^2}\right) + V(r),$$

where $p \equiv p_r$ and $\ell \equiv p_\vartheta$ are the canonical momenta conjugate to r and ϑ, respectively. Solve the Hamilton-Jacobi equation for Hamilton's principal function. (a) Use the ansatz $S(r, \vartheta, \ell, E, t) = W_1(r, \ell, E) + \ell\vartheta - Et$ for the principal function and determine $W_1(r, \ell, E)$. (b) Infer from $\partial S/\partial E = R = \text{const}$ the time evolution of the radial motion $r(t, E, \ell, r_0)$. (c) Infer from $\partial S/\partial \ell = \Theta = \text{const}$ the orbital relation $\vartheta(r, E, \ell, r_0, \vartheta_0)$, which, in combination with $r(t, E, \ell, r_0)$, determines the time evolution of the angular motion.

Solution:

[mex99] Hamilton's characteristic function for central force problem

Consider the one-body central-force problem specified by the Hamiltonian

$$H(r,p,\ell) = \frac{1}{2m}\left(p^2 + \frac{\ell^2}{r^2}\right) + V(r),$$

where $p \equiv p_r$ and $\ell \equiv p_\vartheta$ are the canonical momenta conjugate to r and ϑ, respectively. Solve the Hamilton-Jacobi equation for Hamilton's characteristic function. (a) Use the ansatz $W(r,\vartheta,\ell,E) = W_1(r,\ell,E) + \ell\vartheta$ for the characteristic function and determine $W_1(r,\ell,E)$. (b) The characteristic function $W(r,\vartheta,\ell,E)$ is the generating function of a canonical transformation $(r,\vartheta) \to (R,\Theta)$, which transforms the Hamiltonian as follows: $H(r,p,\ell) = K(E,\ell) = E$. Solve the canonical equations for R,Θ. (c) Infer from $\partial W/\partial E = R = \text{const}$ the time evolution of the radial motion $r(t,E,\ell,r_0)$. (c) Infer from $\partial W/\partial \ell = \Theta = \text{const}$ the orbital relation $\vartheta(r,E,\ell,r_0,\vartheta_0)$, which, in combination with $r(t,E,\ell,r_0)$, determines the time evolution of the angular motion.

Solution:

[mex201] Particle in time-dependent field

Consider the dynamical system described by the time-dependent Hamiltonian

$$H(q, p, t) = \frac{p^2}{2m} - mAtq,$$

where A is a constant. (a) Find Hamilton's principal function $S(q, P, t)$ as the solution of the Hamilton-Jacobi equation. (b) Derive the solutions $q(t), p(t)$ from $S(q, P, t)$ for initial conditions $q(0) = 0$, $p(0) = mv_0$.

Solution:

[mex202] Hamilton-Jacobi theory for projectile motion

Consider a particle of mass m moving in a uniform vertical gravitational field:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + mgy.$$

(a) Find Hamilton's principal function $S(x, y, P_1, P_2, t)$ as the solution of the Hamilton-Jacobi equation. (b) Derive the solutions $x(t), y(t)$ from $S(x, y, P_1, P_2, t)$ for initial conditions $x(0) = y(0) = 0$, $\dot{x}(0) = \dot{x}_0, \dot{y}(0) = \dot{y}_0$.

Solution: