2015

13. Magnetic Field II

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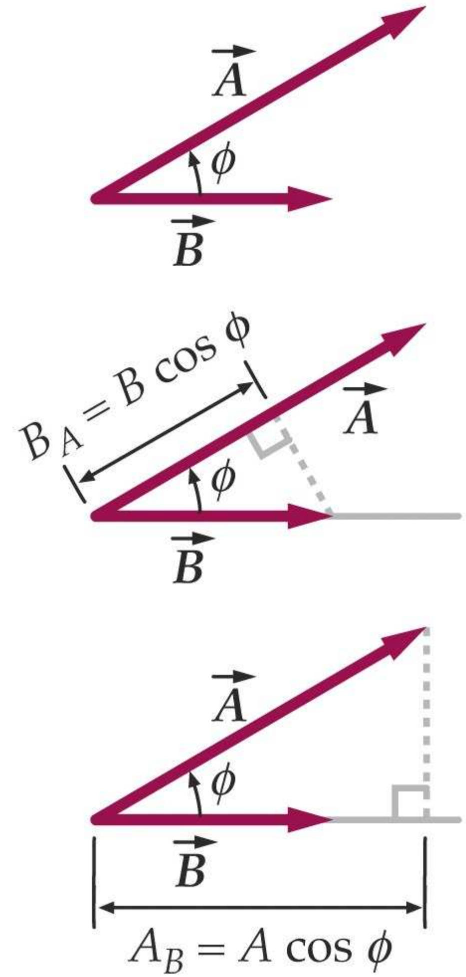
Abstract
Part thirteen of course materials for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.
Some of the slides contain figures from the textbook, Paul A. Tipler and Gene Mosca. *Physics for Scientists and Engineers*, $5^{th}/6^{th}$ editions. The copyright to these figures is owned by W.H. Freeman. We acknowledge permission from W.H. Freeman to use them on this course web page. The textbook figures are not to be used or copied for any purpose outside this class without direct permission from W.H. Freeman.

Dot Product Between Vectors

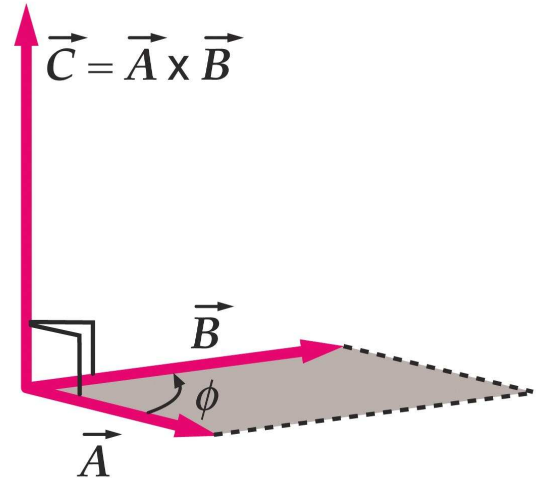
Consider two vectors $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$.

- $\vec{A}\cdot\vec{B} = AB\cos\phi = AB_A = BA_B$.
- $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.
- $\vec{A}\cdot\vec{B} = AB$ if $\vec{A} \parallel \vec{B}$.
- $\vec{A}\cdot\vec{B} = 0$ if $\vec{A} \perp \vec{B}$.
- $$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k}) \\ &\quad + A_yB_x(\hat{j}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z(\hat{j}\cdot\hat{k}) \\ &\quad + A_zB_x(\hat{k}\cdot\hat{i}) + A_zB_y(\hat{k}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k}).\end{aligned}$$
- Use $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$,
  $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.
- $\Rightarrow \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$.



Cross Product Between Vectors

Consider two vectors $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$.



- $\vec{A} \times \vec{B} = AB \sin\phi\, \hat{n}$.
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.
- $\vec{A} \times \vec{A} = 0$.
- $\vec{A} \times \vec{B} = AB\, \hat{n}$ if $\vec{A} \perp \vec{B}$.
- $\vec{A} \times \vec{B} = 0$ if $\vec{A} \parallel \vec{B}$.
- $$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\ &= A_xB_x(\hat{i}\times\hat{i}) + A_xB_y(\hat{i}\times\hat{j}) + A_xB_z(\hat{i}\times\hat{k}) \\ &+ A_yB_x(\hat{j}\times\hat{i}) + A_yB_y(\hat{j}\times\hat{j}) + A_yB_z(\hat{j}\times\hat{k}) \\ &+ A_zB_x(\hat{k}\times\hat{i}) + A_zB_y(\hat{k}\times\hat{j}) + A_zB_z(\hat{k}\times\hat{k}).\end{aligned}$$
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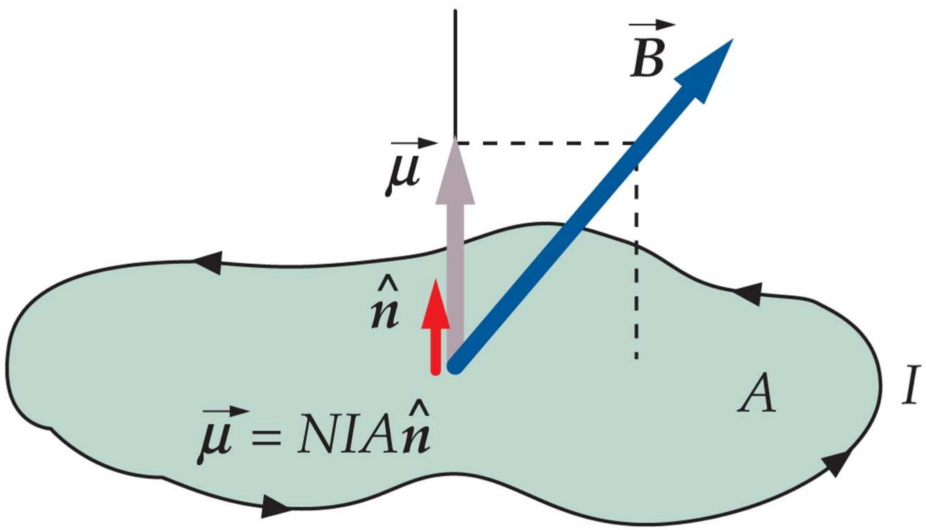
  $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$.
- $\Rightarrow\ \vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$.

Magnetic Dipole Moment of Current Loop

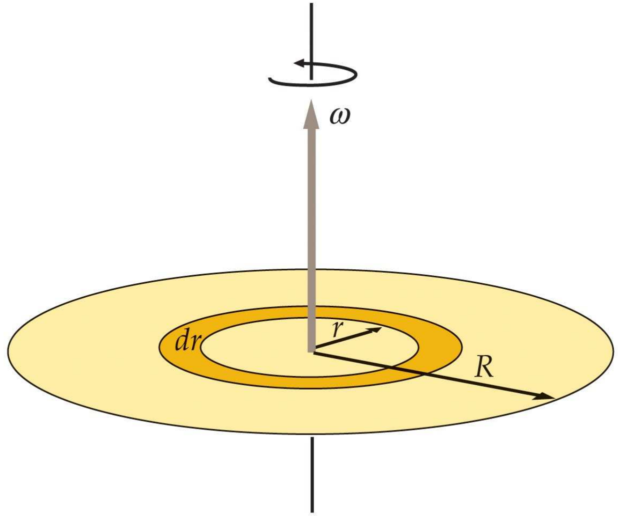
N: number of turns
I: current through wire
A: area of loop
$\hat{n}$: unit vector perpendicular to plane of loop
$\vec{\mu} = NIA\hat{n}$: magnetic dipole moment

$\vec{B}$: magnetic field
$\vec{\tau} = \vec{\mu} \times \vec{B}$: torque acting on current loop



Magnetic Moment of a Rotating Disk

Consider a nonconducting disk of radius R with a uniform surface charge density σ. The disk rotates with angular velocity $\vec{\omega}$.

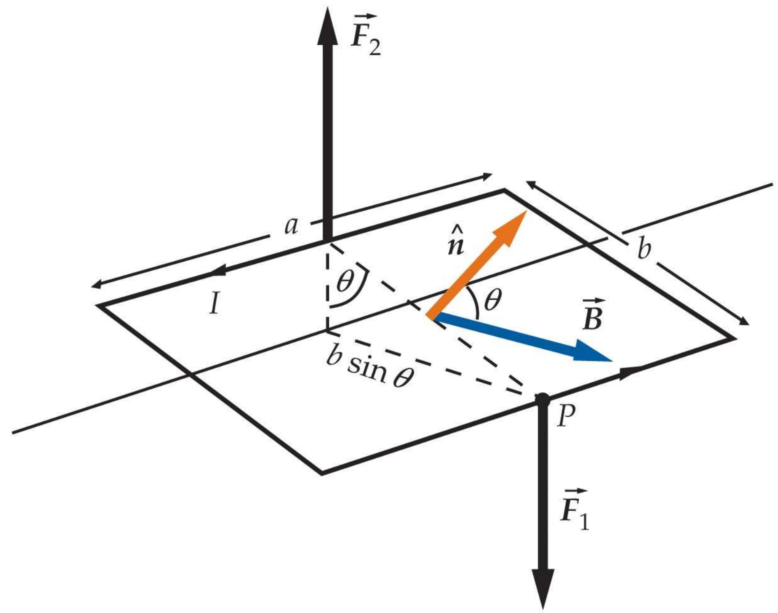


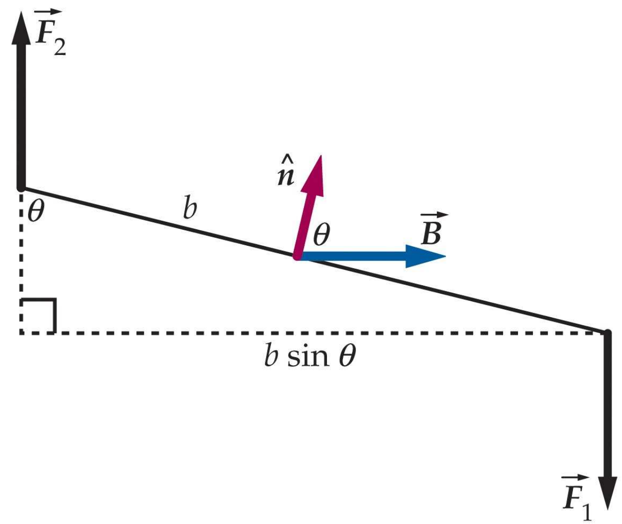
Calculation of the magnetic moment $\vec{\mu}$:

- Total charge on disk: $Q = \sigma(\pi R^2)$.
- Divide the disk into concentric rings of width dr.
- Period of rotation: $T = \dfrac{2\pi}{\omega}$.
- Current within ring: $dI = \dfrac{dQ}{T} = \sigma(2\pi r dr)\dfrac{\omega}{2\pi} = \sigma\omega r dr$.
- Magnetic moment of ring: $d\mu = dI(\pi r^2) = \pi\sigma\omega r^3 dr$.
- Magnetic moment of disk: $\mu = \displaystyle\int_0^R \pi\sigma\omega r^3 dr = \frac{\pi}{4}\sigma R^4 \omega$.
- Vector relation: $\vec{\mu} = \dfrac{\pi}{4}\sigma R^4 \vec{\omega} = \dfrac{1}{4} Q R^2 \vec{\omega}$.

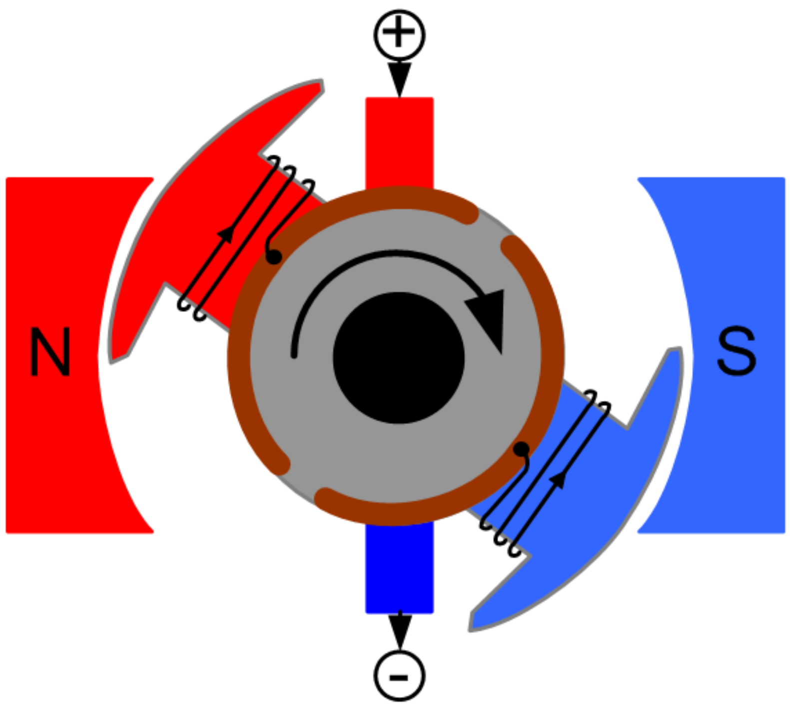
Torque on Current Loop

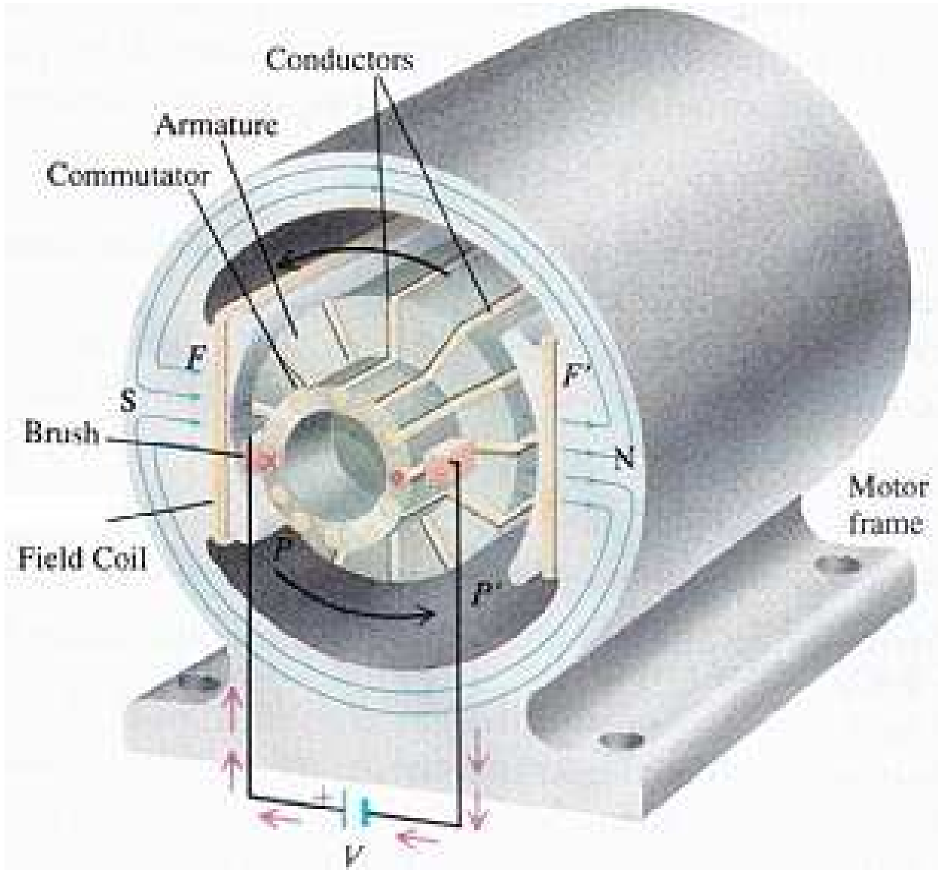
- magnetic field: $\vec{B}$ (horizontal)
- area of loop: $A = ab$
- unit vector $\perp$ to plane of loop: $\hat{n}$
- right-hand rule: $\hat{n}$ points up.
- forces on sides a: $F = IaB$ (vertical)
- forces on sides b: $F = IbB$ (horizontal, not shown)
- torque: $\tau = Fb\sin\theta = IAB\sin\theta$
- magnetic moment: $\vec{\mu} = IA\hat{n}$
- torque (vector): $\vec{\tau} = \vec{\mu} \times \vec{B}$





Direct-Current Motor

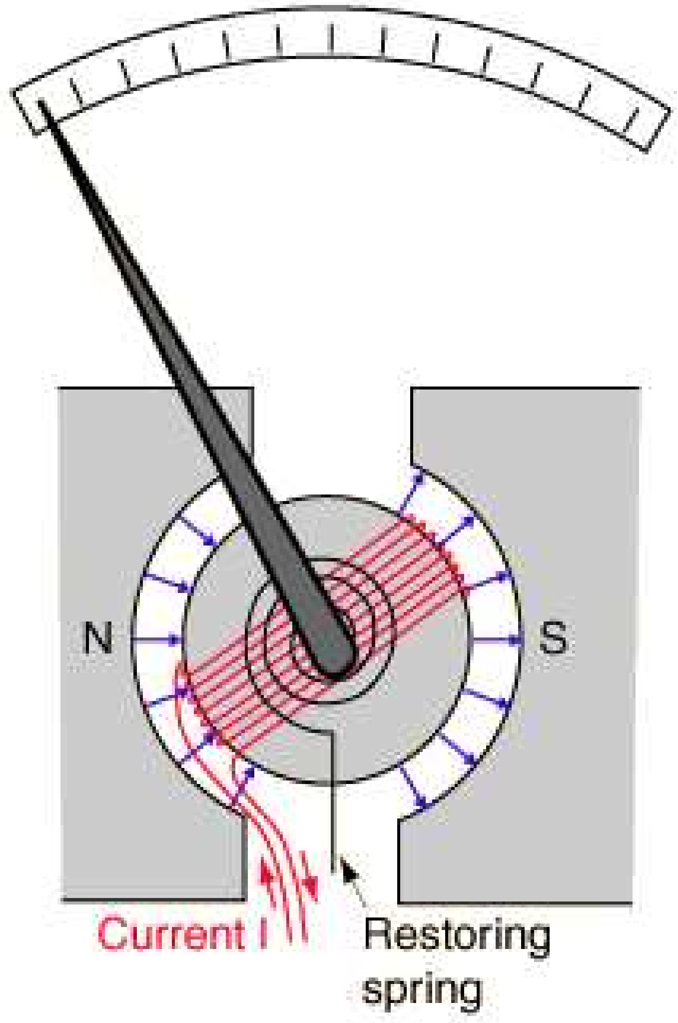




Galvanometer

Measuring direct currents.

- magnetic moment $\vec{\mu}$ (along needle)
- magnetic field $\vec{B}$ (toward right)
- torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ (into plane)

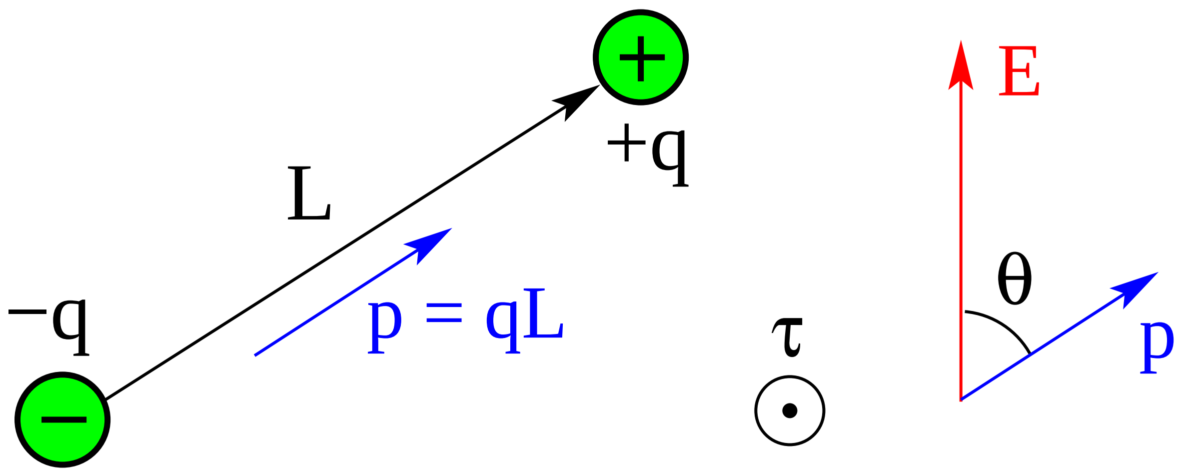


Electric Dipole in Uniform Electric Field

- Electric dipole moment: $\vec{p} = q\vec{L}$
- Torque exerted by electric field: $\vec{\tau} = \vec{p} \times \vec{E}$
- Potential energy: $U = -\vec{p} \cdot \vec{E}$

  $$U(\theta) = -\int_{\pi/2}^{\theta} \tau(\theta) d\theta = pE \int_{\pi/2}^{\theta} \sin\theta d\theta = -pE\cos\theta$$

  Note: $\tau(\theta)$ and $d\theta$ have opposite sign.

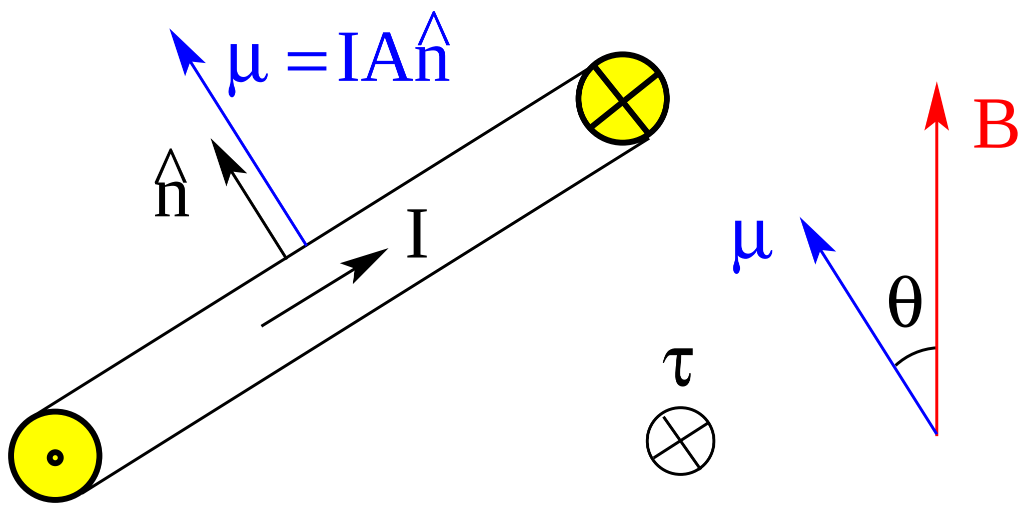


Magnetic Dipole in Uniform Magnetic Field

- Magnetic dipole moment: $\vec{\mu} = IA\hat{n}$
- Torque exerted by magnetic field: $\vec{\tau} = \vec{\mu} \times \vec{B}$
- Potential energy: $U = -\vec{\mu} \cdot \vec{B}$

$$U(\theta) = -\int_{\pi/2}^{\theta} \tau(\theta) d\theta = \mu B \int_{\pi/2}^{\theta} \sin\theta d\theta = -\mu B \cos\theta$$

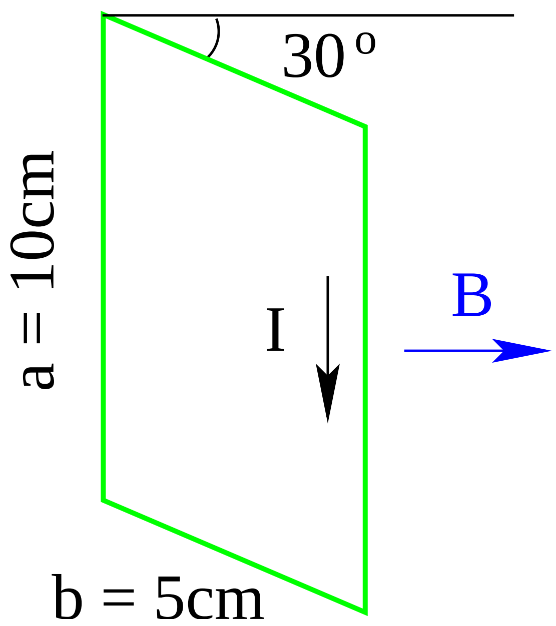
Note: $\tau(\theta)$ and $d\theta$ have opposite sign.



Magnetic Force Application (7)

The rectangular 20-turn loop of wire is 10cm high and 5cm wide.
It carries a current $I = 0.1$A and is hinged along one long side.
It is mounted with its plane at an angle of $30°$ to the direction of a uniform magnetic field of magnitude $B = 0.50$T.

- Calculate the magnetic moment μ of the loop.
- Calculate the torque τ acting on the loop about the hinge line.



Magnetic Force Application (4)

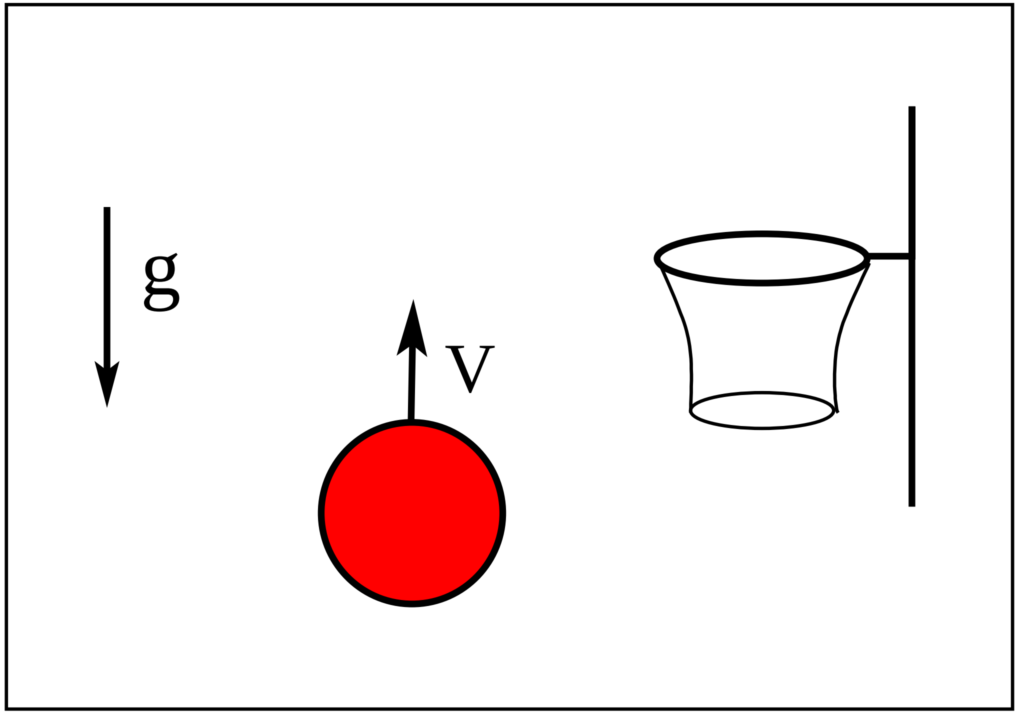
A negatively charged basketball is thrown vertically up against the gravitational field $\vec{g}$.

Which direction of

(a) a uniform electric field $\vec{E}$,

(b) a uniform magnetic field $\vec{B}$

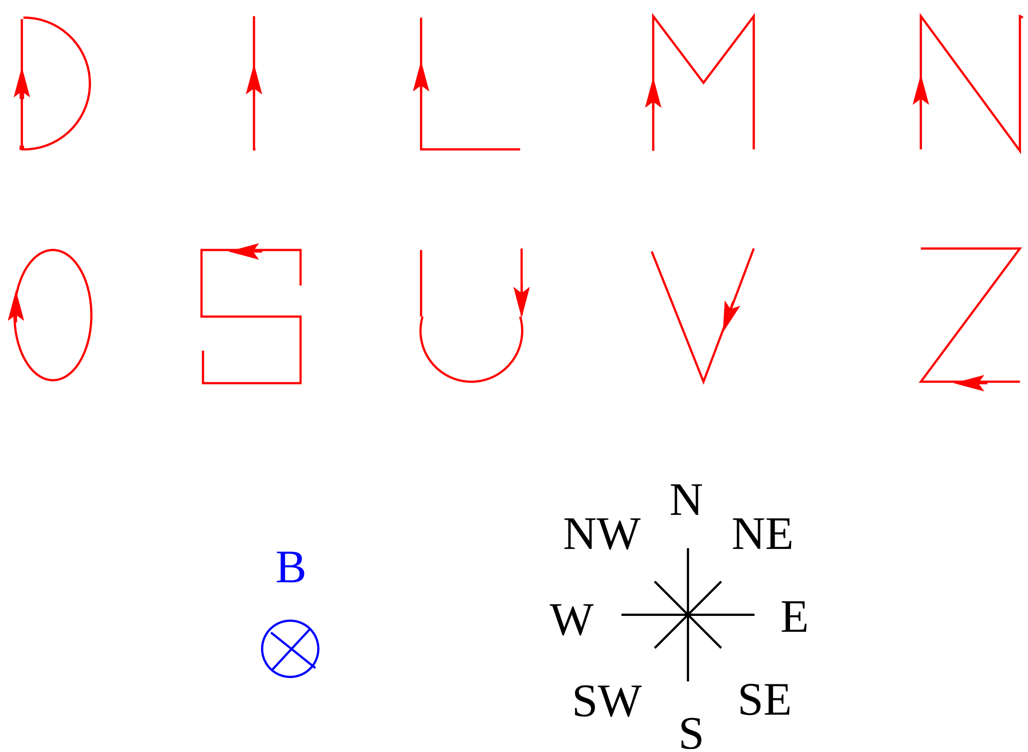
will give the ball a chance
to find its way into the basket?
(up/down/left/right/back/front)



Magnetic Force Application (6)

An electric current flows through each of the letter-shaped wires in a region of uniform magnetic field pointing into the plane.

- Find the direction of the resultant magnetic force on each letter.



Magnetic Force Application (10)

A triangular current loop is free to rotate around the vertical axis PQ.

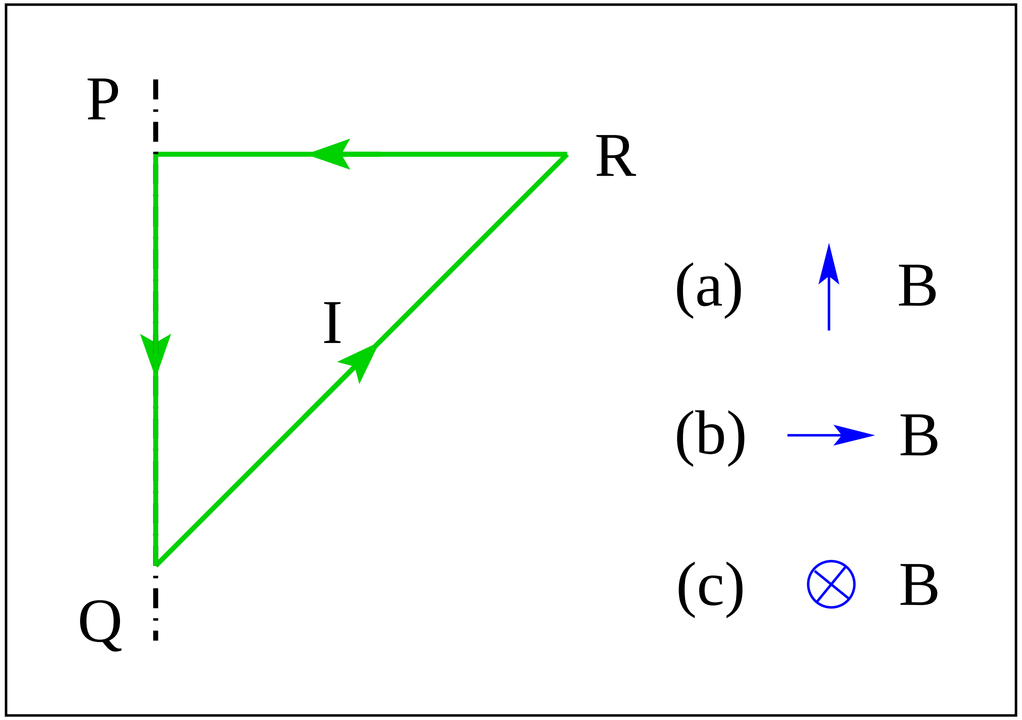
If a uniform magnetic field $\vec{B}$ is switched on, will the corner R of the triangle start to move out of the plane, into the plane, or will it not move at all?

Find the answer for a field $\vec{B}$ pointing

(a) up,

(b) to the right,

(c) into the plane.

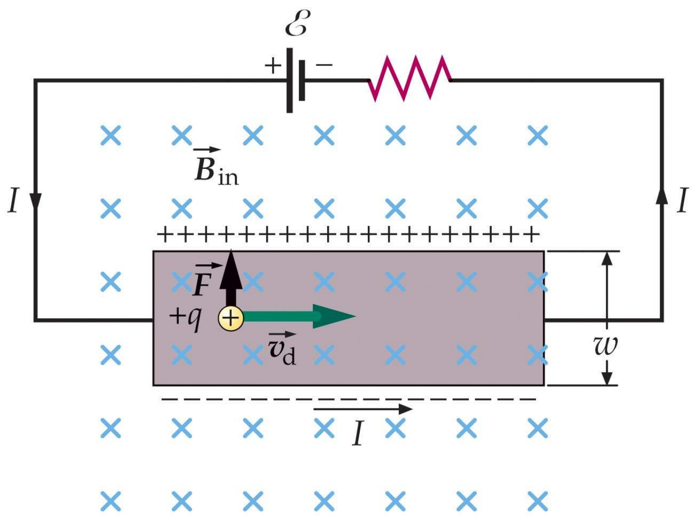


Hall Effect

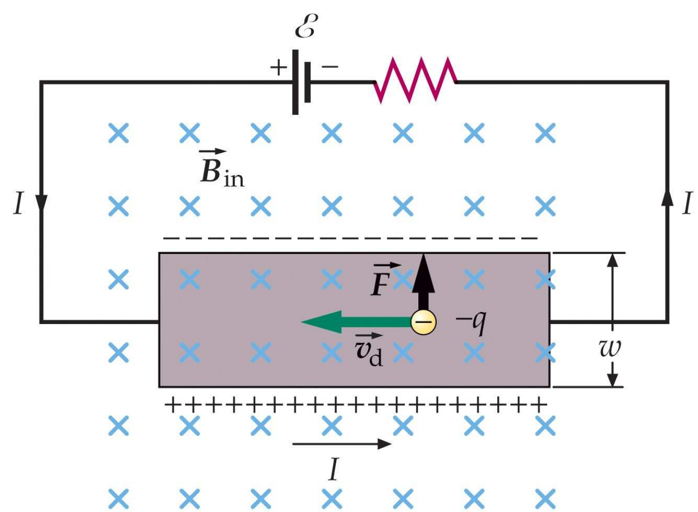
Method for dermining whether charge carriers are positively or negatively charged.

- Magnetic field $\vec{B}$ pulls charge carriers to one side of conducting strip.
- Accumulation of charge carriers on that side and depletion on opposite side produce transverse electric field $\vec{E}$.
- Transverse forces on charge carrier: $F_E = qE$ and $F_B = qv_dB$.
- In steady state forces are balanced: $\vec{F}_E = -\vec{F}_B$.
- Hall voltage in steady state: $V_H = Ew = v_dBw$.

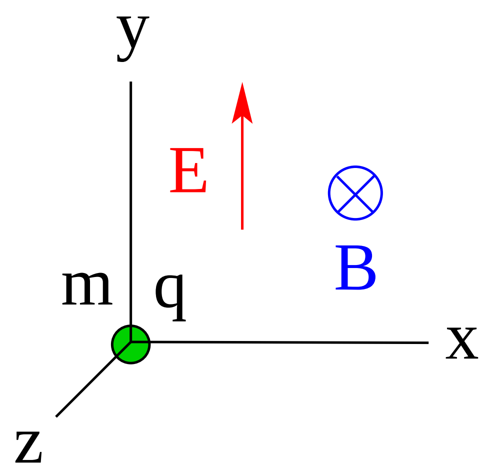
positive charge carriers



negative charge carriers



Charged Particle in Crossed Electric and Magnetic Fields (1)



- Release particle from rest.
- Force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$
- (1) $F_x = m\frac{dv_x}{dt} = -qv_yB \quad \Rightarrow \quad \frac{dv_x}{dt} = -\frac{qB}{m}v_y$
- (2) $F_y = m\frac{dv_y}{dt} = qv_xB + qE \Rightarrow \frac{dv_y}{dt} = \frac{qB}{m}v_x + \frac{qE}{m}$
- Ansatz: $v_x(t) = w_x\cos(\omega_0 t) + u_x,\ v_y(t) = w_y\sin(\omega_0 t) + u_y$
- Substitute ansatz into (1) and (2) to find $w_x, w_y, u_x, u_y, \omega_0$.
- (1) $-\omega_0 w_x \sin(\omega_0 t) = -\frac{qB}{m}w_y\sin(\omega_0 t) - \frac{qB}{m}u_y$
- (2) $\omega_0 w_y \cos(\omega_0 t) = \frac{qB}{m}w_x\cos(\omega_0 t) + \frac{qB}{m}u_x + \frac{qE}{m}$
- $\Rightarrow\ u_y = 0, \quad u_x = -\frac{E}{B}, \quad \omega_0 = \frac{qB}{m}, \quad w_x = w_y \equiv w$
- Initial condition: $v_x(0) = v_y(0) = 0 \Rightarrow\ w = \frac{E}{B}$

Charged Particle in Crossed Electric and Magnetic Fields (2)

- Solution for velocity of particle:

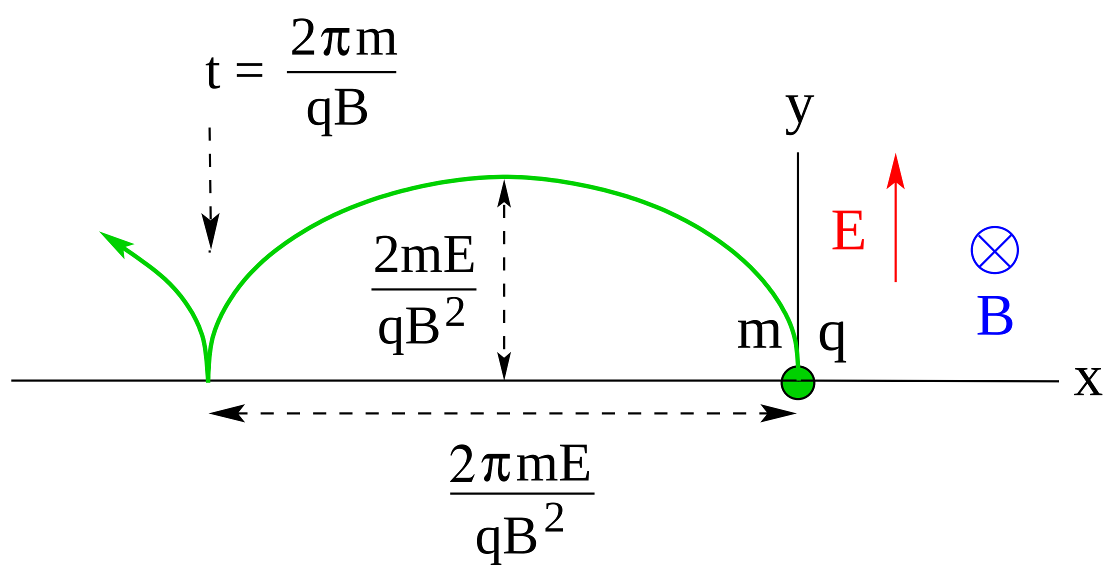
$$v_x(t) = \frac{E}{B}\left[\cos\left(\frac{qBt}{m}\right) - 1\right], \quad v_y(t) = \frac{E}{B}\sin\left(\frac{qBt}{m}\right)$$

- Solution for position of particle:

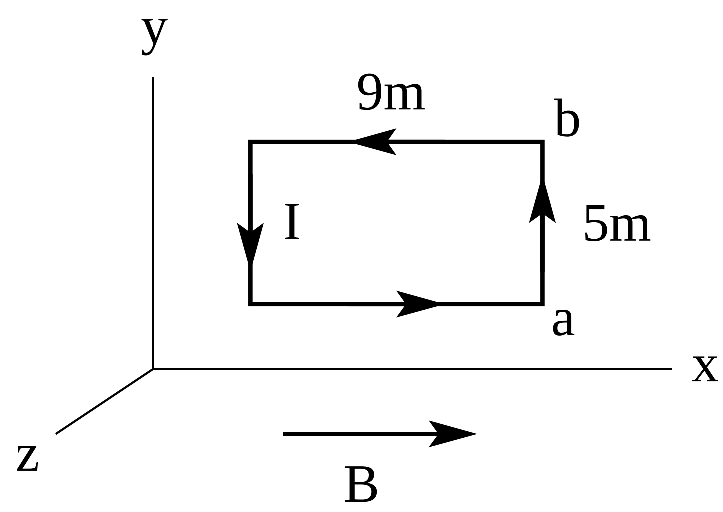
$$x(t) = \frac{E}{B}\int_0^t \left[\cos\left(\frac{qBt}{m}\right) - 1\right] dt = \frac{Em}{qB^2}\sin\left(\frac{qBt}{m}\right) - \frac{Et}{B}$$

$$y(t) = \frac{E}{B}\int_0^t \sin\left(\frac{qBt}{m}\right) dt = \frac{Em}{qB^2}\left[1 - \cos\left(\frac{qBt}{m}\right)\right]$$

- Path of particle in (x, y)-plane: cycloid



Consider a rectangular conducting loop in the xy-plane with a counterclockwise current $I = 7$A in a uniform magnetic field $\vec{B} = 3\text{T}\hat{i}$.
(a) Find the magnetic moment $\vec{\mu}$ (magnitude and direction) of the loop.
(b) Find the force $\vec{F}$ (magnitude and direction) acting on the side ab of the rectangle.
(c) Find the torque $\vec{\tau}$ (magnitude and direction) acting on the loop.



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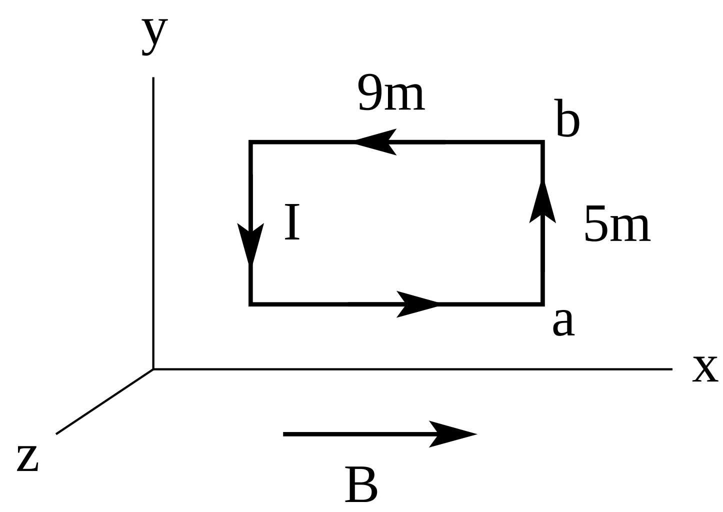

Solution:

(a) $\vec{\mu} = (7\mathrm{A})(45\mathrm{m}^2)\hat{k} = 315\mathrm{Am}^2\hat{k}.$

Consider a rectangular conducting loop in the xy-plane with a counterclockwise current $I = 7$A in a uniform magnetic field $\vec{B} = 3\mathrm{T}\hat{i}$.
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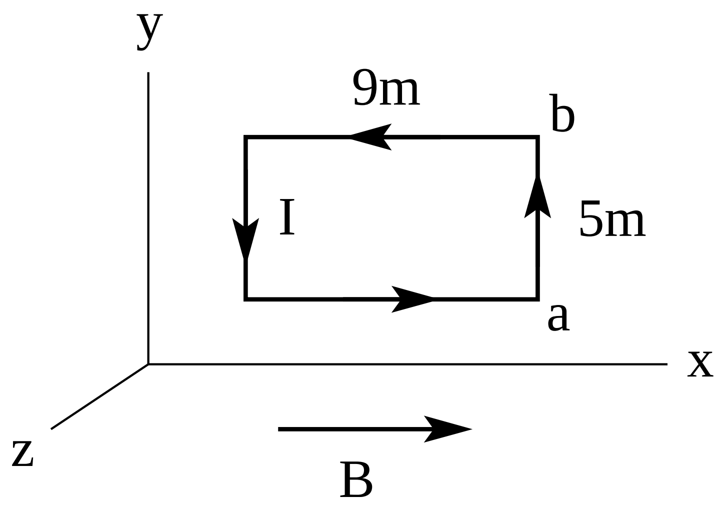

Solution:

(a) $\vec{\mu} = (7\mathrm{A})(45\mathrm{m}^2)\hat{k} = 315\mathrm{Am}^2\hat{k}.$

(b) $\vec{F} = I\vec{L} \times \vec{B} = (7\mathrm{A})(5\mathrm{m}\hat{j}) \times (3\mathrm{T}\hat{i}) = -105\mathrm{N}\hat{k}.$

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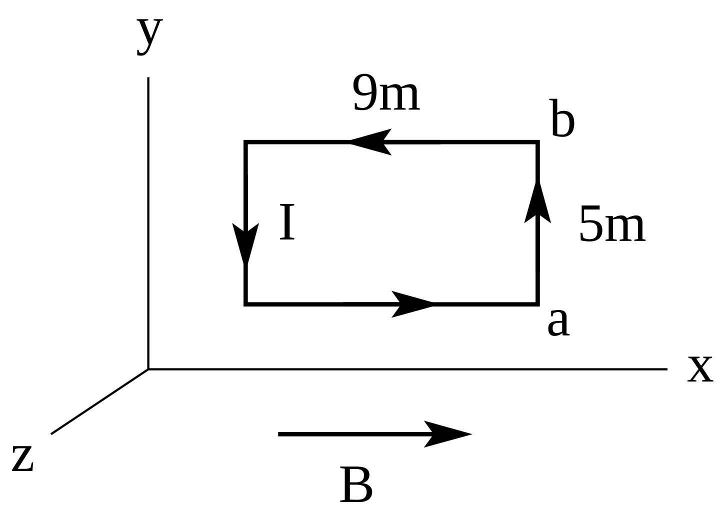

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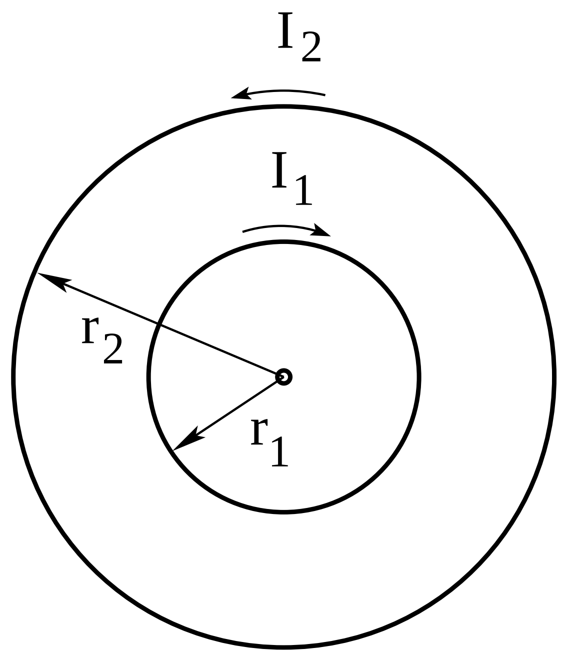
(c) $\vec{\tau} = \vec{\mu} \times \vec{B} = (315\mathrm{Am}^2\hat{k}) \times (3\mathrm{T}\hat{i}) = 945\mathrm{Nm}\hat{j}$

Unit Exam III: Problem #1 (Spring '08)

Consider two circular currents $I_1 = 3$A at radius $r_1 = 2$m and $I_2 = 5$A at radius $r_2 = 4$m in the directions shown.

(a) Find magnitude B and direction $(\odot, \otimes)$ of the resultant magnetic field at the center.

(b) Find magnitude μ and direction $(\odot, \otimes)$ of the magnetic dipole moment generated by the two currents.



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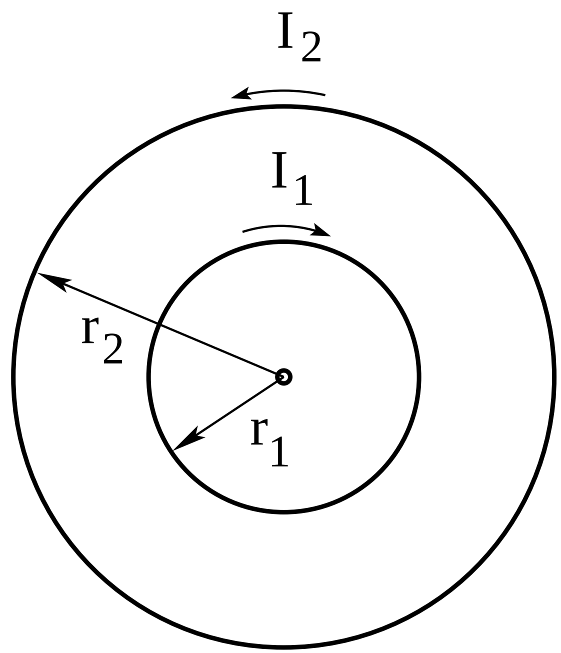

Solution:

(a) $B = \frac{\mu_0(3\text{A})}{2(2\text{m})} - \frac{\mu_0(5\text{A})}{2(4\text{m})} = (9.42 - 7.85) \times 10^{-7}\text{T}$

$\Rightarrow B = 1.57 \times 10^{-7}\text{T} \quad \otimes$

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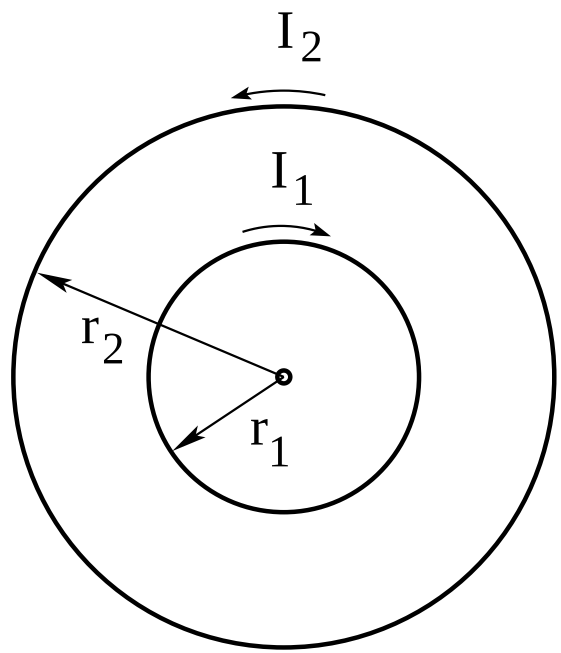

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$\Rightarrow \; B = 1.57 \times 10^{-7}\mathrm{T} \quad \otimes$

(b) $\mu = \pi(4\mathrm{m})^2(5\mathrm{A}) - \pi(2\mathrm{m})^2(3\mathrm{A}) = (251 - 38)\mathrm{Am}^2$

$\Rightarrow \; \mu = 213\mathrm{Am}^2 \quad \odot$

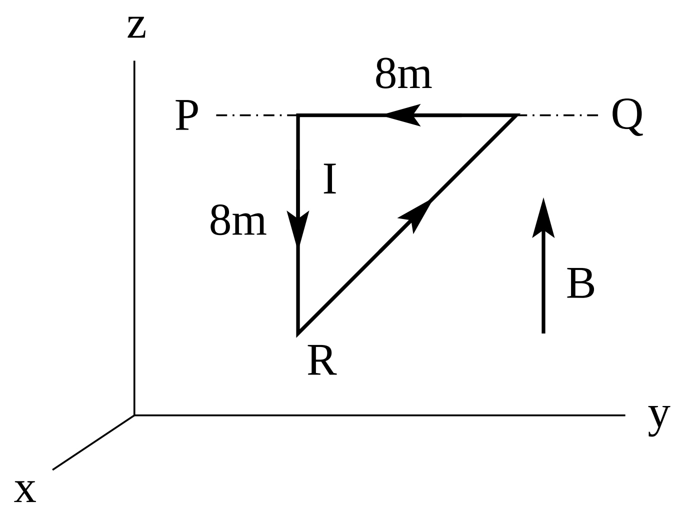
Unit Exam III: Problem #1 (Spring '09)

A triangular conducting loop in the yz-plane with a counterclockwise current $I = 3$A is free to rotate about the axis PQ. A uniform magnetic field $\vec{B} = 0.5\mathrm{T}\hat{k}$ is present. (a) Find the magnetic moment $\vec{\mu}$ (magnitude and direction) of the triangle.

(b) Find the magnetic torque $\vec{\tau}$ (magnitude and direction) acting on the triangle.

(c) Find the magnetic force $\vec{F}_H$ (magnitude and direction) acting on the long side (hypotenuse) of the triangle.

(d) Find the force $\vec{F}_R$ (magnitude and direction) that must be applied to the corner R to keep the triangle from rotating.



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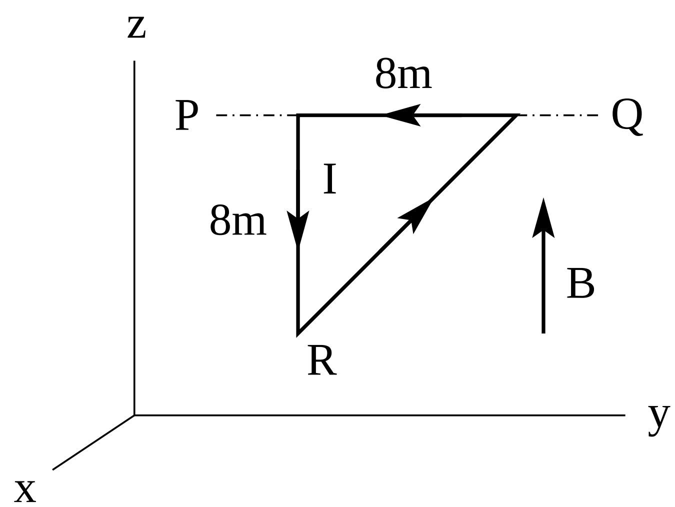

Solution:

(a) $\vec{\mu} = (3\mathrm{A})(32\mathrm{m}^2)\hat{i} = 96\mathrm{Am}^2\hat{i}.$

Unit Exam III: Problem #1 (Spring '09)

A triangular conducting loop in the yz-plane with a counterclockwise current $I = 3$A is free to rotate about the axis PQ. A uniform magnetic field $\vec{B} = 0.5\mathrm{T}\hat{k}$ is present. (a) Find the magnetic moment $\vec{\mu}$ (magnitude and direction) of the triangle.
(b) Find the magnetic torque $\vec{\tau}$ (magnitude and direction) acting on the triangle.
(c) Find the magnetic force $\vec{F}_H$ (magnitude and direction) acting on the long side (hypotenuse) of the triangle.
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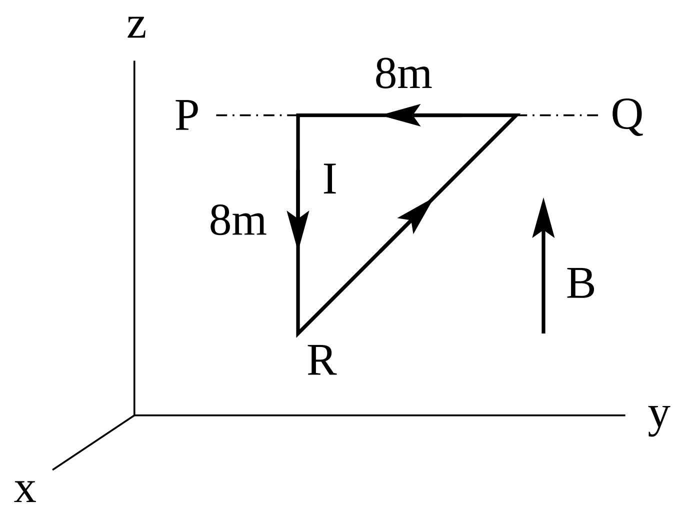

Solution:

(a) $\vec{\mu} = (3\mathrm{A})(32\mathrm{m}^2)\hat{i} = 96\mathrm{Am}^2\hat{i}$.

(b) $\vec{\tau} = \vec{\mu} \times \vec{B} = (96\mathrm{Am}^2\hat{i}) \times (0.5\mathrm{T}\hat{k}) = -48\mathrm{Nm}\hat{j}$.

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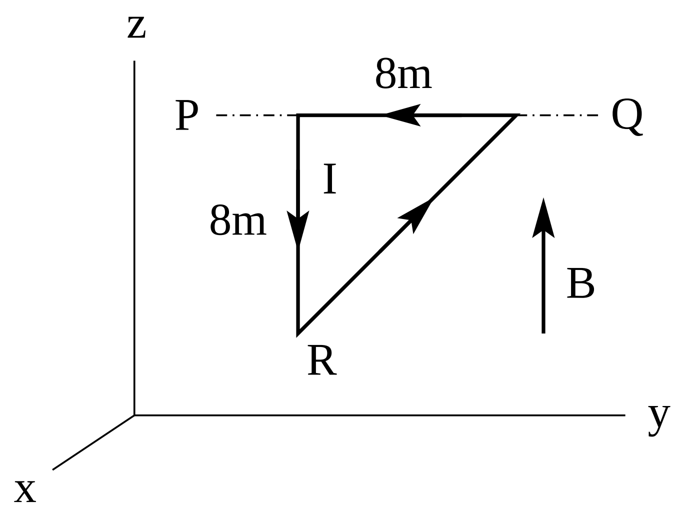

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(c) $F_H = (3\text{A})(8\sqrt{2}\text{m})(0.5\text{T})(\sin 45^\circ) = 12\text{N} \qquad \odot$.

Unit Exam III: Problem #1 (Spring '09)

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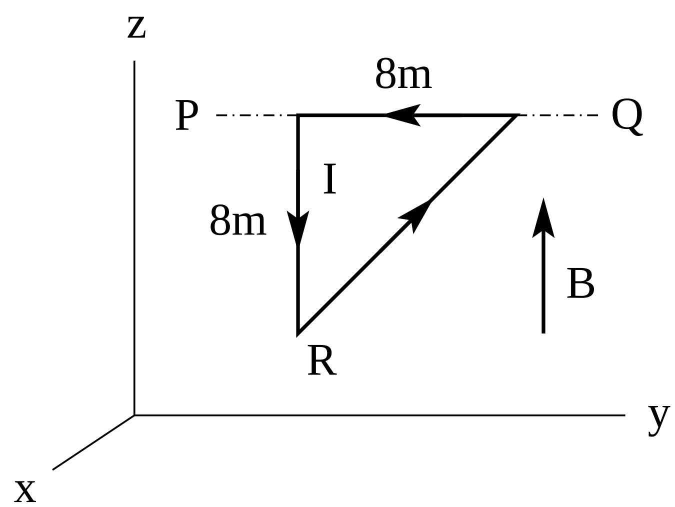

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(c) $F_H = (3\text{A})(8\sqrt{2}\text{m})(0.5\text{T})(\sin 45^\circ) = 12\text{N} \quad \odot$.

(d) $(-8\text{m}\hat{k}) \times \vec{F}_R = -\vec{\tau} = 48\text{Nm}\hat{j} \quad \Rightarrow \quad \vec{F}_R = -6\text{N}\hat{i}$.