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13. Rigid Body Dynamics II

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Abstract
Part thirteen of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Entries listed in the table of contents, but not shown in the document, exist only in handwritten form. Documents will be updated periodically as more entries become presentable.

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13. Rigid Body Dynamics II

[mex70] Stability of rigid body rotations about principal axes

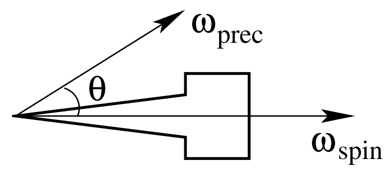
Consider a rigid body with principal moments of inertia $I_1 < I_2 < I_3$ undergoing a torque-free rotation about one of the principal axes. Investigate the stability of this motion against small perturbations as follows: (a) Use the vector $\vec{\omega} = \omega_i \hat{\mathbf{e}}_i + \delta_j \hat{\mathbf{e}}_j + \delta_k \hat{\mathbf{e}}_k$ with $\delta_j, \delta_k \ll \omega_i$ for $\{i, j, k\} = \text{cycl}\{1, 2, 3\}$ in Euler's equations and linearize them in δ_j, δ_k. (b) Solve the linearized equations exactly. (c) Describe the motion of $\vec{\omega}$ separately for $i = 1, 2, 3$ in the range of the approximations made.

Solution:

[mex176] Steady precession of symmetric top

A symmetric top with moments of inertia $I_3, I_\perp$ rotates with constant angular velocity ω_{spin} about its symmetry axis, which, in turn, precesses with constant angular velocity ω_{prec} at an angle θ about a direction fixed in the inertial frame. Use Euler's equations to show that the torque $\mathbf{N}$ causing this precessional motion is

$$\mathbf{N} = \left[I_3 + (I_3 - I_\perp)\frac{\omega_{prec}}{\omega_{spin}}\cos\theta \right] \vec{\omega}_{prec} \times \vec{\omega}_{spin}.$$



Solution:

Heavy symmetric top: general solution [mln47]

Lagrangian: $L = T(\theta, \dot{\phi}, \dot{\theta}, \dot{\psi}) - V(\theta)$. The coordinates ϕ, ψ are cyclic.

$$T = \frac{1}{2}I_\perp(\omega_1^2+\omega_2^2)+\frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_\perp(\sin^2\theta\dot{\phi}^2+\dot{\theta}^2)+\frac{1}{2}I_3(\cos\theta\dot{\phi}+\dot{\psi})^2, \quad V = mg\ell\cos\theta.$$

Conserved generalized momenta:

$$\alpha_\phi \equiv \frac{\partial L}{\partial\dot{\phi}} = (I_\perp\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\cos\theta\dot{\psi} = \text{const.}$$

$$\alpha_\psi \equiv \frac{\partial L}{\partial\dot{\psi}} = I_3(\dot{\psi} + \cos\theta\dot{\phi}) = I_3\omega_3 = \text{const} \;\Rightarrow\; \omega_3 = \text{const.}$$

$$\Rightarrow\; \dot{\phi} = \frac{\alpha_\phi - \alpha_\psi\cos\theta}{I_\perp\sin^2\theta}, \quad \dot{\psi} = \frac{\alpha_\psi}{I_3} - \frac{(\alpha_\phi - \alpha_\psi\cos\theta)\cos\theta}{I_\perp\sin^2\theta}.$$

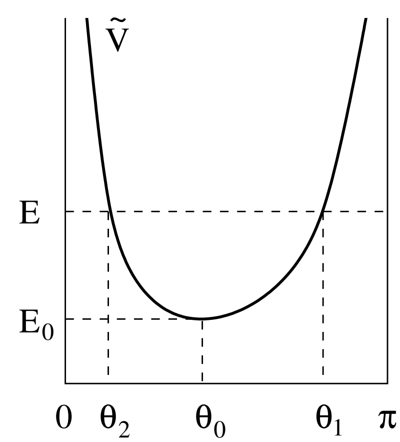
Routhian function: $R(\theta, \dot{\theta}; \alpha_\phi, \alpha_\psi) = \tilde{T}(\dot{\theta}) - \tilde{V}(\theta)$.

$$\tilde{T}(\dot{\theta}) = \frac{1}{2}I_\perp\dot{\theta}^2, \quad \tilde{V}(\theta) = \frac{\alpha_\psi^2}{2I_3} + \frac{(\alpha_\phi - \alpha_\psi\cos\theta)^2}{2I_\perp\sin^2\theta} + mg\ell\cos\theta.$$

Conserved energy: $E = \tilde{T}(\dot{\theta}) + \tilde{V}(\theta) = \text{const.}$

Solution by quadrature: $\dfrac{d\theta}{dt} = \sqrt{\dfrac{2}{I_\perp}\left[E - \tilde{V}(\theta)\right]}$.

- Nutation: $t(\theta) = \displaystyle\int \frac{d\theta}{\sqrt{\frac{2}{I_\perp}\left[E - \tilde{V}(\theta)\right]}}$.
- Precession: $\phi(t) = \displaystyle\int dt\,\dot{\phi}(t)$.
- Rotation: $\psi(t) = \displaystyle\int dt\,\dot{\psi}(t)$.



Specification of general solution:

- integrals of the motion $\alpha_\psi,\ \alpha_\phi,\ E$,
- starting values $\theta_s,\ \phi_s,\ \psi_s$.

Physical solution for given α_ψ, α_ϕ requires $E \geq E_0 = \tilde{V}(\theta_0)$.

For energies $E > E_0$ the angle of inclination θ oscillates between θ_1 and θ_2.

Heavy symmetric top: steady precession [mln81]

Special case: $E = E_0 \;\Rightarrow\; \theta = \theta_0 = \text{const} \;\Rightarrow\; \dot\phi = \text{const},\; \dot\psi = \text{const}.$

Steady angle of inclination θ_0 determined by condition $(d\tilde V/d\theta)_{\theta_0} = 0$.
$\Rightarrow$ Quadratic equation for $\beta_0 \doteq \alpha_\phi - \alpha_\psi \cos\theta_0$:

$$(\cos\theta_0)\beta_0^2 - (\alpha_\psi \sin^2\theta_0)\beta_0 + mg\ell I_\perp \sin^4\theta_0 = 0.$$

Solution: $\beta_0^\pm = \dfrac{\alpha_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \dfrac{4mg\ell I_\perp \cos\theta_0}{\alpha_\psi^2}}\,\right].$

Interpretation: For given θ_0 and α_ψ there exist two values $\alpha_\phi^\pm$ for which steady precession is realized.

Distinguish frequencies of fast precession (+) and slow precession (−):

$$\dot\phi_0^\pm = \frac{\beta_0^\pm}{I_\perp \sin^2\theta_0}.$$

Distinguish hanging top ($\theta_0 > \pi/2$) and standing top ($\theta_0 < \pi/2$):

- $\theta_0 > \pi/2$: Steady precession exists without restrictions on α_ψ.
- $\theta_0 < \pi/2$: Steady precession requires that angular velocity about figure axis exceeds threshold value:

$$\alpha_\psi^2 \geq 4mg\ell I_\perp \cos\theta_0 \quad \Rightarrow\; \omega_3 = \frac{\alpha_\psi}{I_3} \geq \frac{2}{I_3}\sqrt{mg\ell I_\perp \cos\theta_0}.$$

Consider fast top ($\alpha_\psi \gg 2\sqrt{mg\ell I_\perp}$):

$$\beta_0^\pm \simeq \frac{\alpha_\psi \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm 1 \mp \frac{2mg\ell I_\perp \cos\theta_0}{\alpha_\psi^2}\right].$$

- Fast precession: $\beta_0^\pm \simeq \dfrac{\alpha_\psi \sin^2\theta_0}{\cos\theta_0} \quad \Rightarrow\; \dot\phi_0^+ \simeq \dfrac{I_3\omega_3}{I_\perp \cos\theta_0}.$
- Slow precession: $\beta_0^\pm \simeq \dfrac{mg\ell I_\perp \sin^2\theta_0}{\alpha_\psi} \quad \Rightarrow\; \dot\phi_0^- \simeq \dfrac{mg\ell}{I_3\omega_3}.$

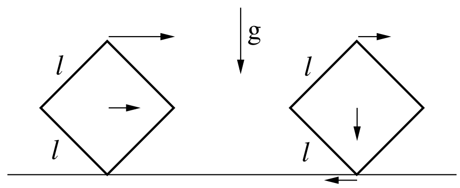
[mex177] Stability of sleeping top

A symmetric top (with principal moments of inertia $I_\perp, I_3$) is standing in an upright position ($\theta = 0$) and rotating with angular velocity ω_3 about the symmetry axis. This motion is only stable under small perturbations if ω_3 exceeds a critical value ω_c. Find ω_c.

Solution:

[mex72] Cube standing on edge

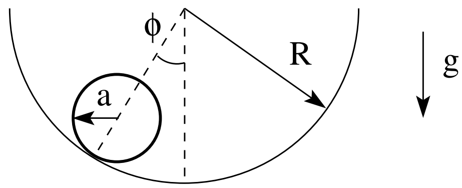
A homogeneous cube of side ℓ is initially in a position of unstable equilibrium with one edge on a horizontal plane. The cube then falls on one side. Calculate the angular velocity ω of the cube when the face strikes the plane, (a) if the lowest edge remains fixed, (b) if the lowest edge can slide on the plane without friction. Express the results as functions of g and ℓ.



Solution:

[mex178] Rolling pendulum

Consider a homogeneous cylinder of mass m and radius a rolling on the inside of a cylindrical surface with radius R. The cylinder axes are horizontal. There is a uniform, vertical gravitational field $\mathbf{g}$. (a) Find the Lagrangian $L(\phi, \dot{\phi})$. (b) Find the period T of small-amplitude oscillations about the stable equilibrium position.



Solution:

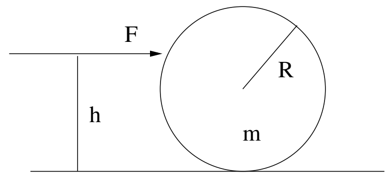
[mex74] Cone on the roll

A cone of mass M, height h, and angle 2α at the apex rolls without slipping on a horizontal plane. As it rolls in a circle about its apex, the cone rotates with angular velocity Ω about the figure axis. Calculate the total kinetic energy of the cone. Express the result as a function of M, h, α, Ω.

Solution:

[mex4] Make the billiard ball roll

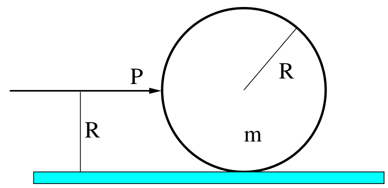
Find the height h at which a billiard ball should be struck (via horizontal impulse) to make it roll without slipping on a surface with negligible friction. The billiard ball is to be regarded as a homogeneous sphere with radius R and mass m.



Solution:

[mex220] From sliding to rolling motion

A billiard ball (rigid homogeneous sphere of mass m and radius R) is initially at rest on a flat table. A cue then imparts a horizontal impulse $\mathbf{P}$ in a very short time at height R. The coefficient of kinetic friction between table and ball is μ. (a) Find the time t_r that elapses before the motion of the billiard ball turns into pure rolling. (b) Find the speed v_r of the rolling billiard ball.



Solution:

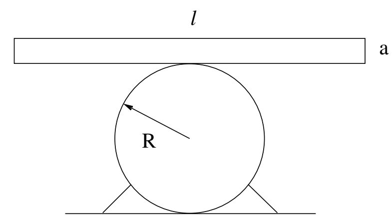
[mex179] Rolling inhomogeneous disk

Consider a disk of mass m and radius R composed of two homogeneous halves connected along a diameter. One half has twice the density of the other half.
(a) Find the distance b between the center of mass and the geometric center of the disk.
(b) Find the moment of inertia I_{cm} for rotations about the center of mass.
(c) Find the Lagrangian $L(\phi, \dot{\phi})$ for the rolling motion of the disk on a flat surface. Use $\phi = 0$ for the stable equilibrium position.
(d) Consider the disk being pulled by a horizontal force at constant speed across the surface. What is the maximum speed v_{max} at which the disk can roll without jumping of the ground?

Solution:

[mex75] Balancing act of board on cylinder

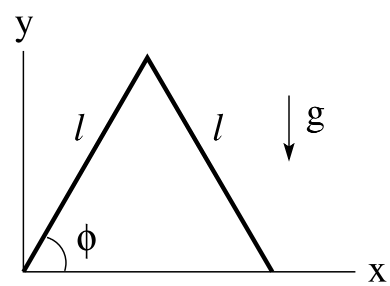
A homogeneous rigid board of thickness a, width w, and length ℓ is placed symmetrically atop a rigid and fixed cylinder of radius R and horizontal axis. (a) Show that the condition for stable equilibrium of the board in its horizontal position is $a < 2R$. (b) Show that the angular frequency of small oscillations of the board about this stable equilibrium as obtained from the linearized equation of motion is $\omega_0^2 = [6g(2R - a)]/[4a^2 + \ell^2]$, where g is the acceleration due to gravity. The assumption is that the board rolls back and forth without slipping.



Solution:

[mex256] Falling flat

Two rods of mass m and length l each are connected at one end by a hinge. The opposite end of one rod is hinged to the origin of the coordinate system. The opposite end of the other rod is free to slide along the horizontal axis. Starting from rest at initial angle $\phi = 60°$ the triangular configuration collapses under the influence of the gravitational field g. Find the speed v of the hinge that connects rods when it hits the horizontal axis.



Solution:

[mex258] Rod off balance

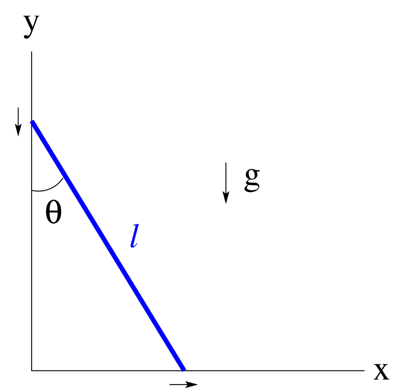
A uniform rod of mass m and length l is positioned upright ($\theta = 0$), initially at rest, on a slippery floor (x-axis) against a slippery wall (y-axis). The unstable equilibrium is upset when, in the absence of friction, the two ends of the rod begin to slide as shown under the influence of a uniform gravitational field g.

(a) Find the kinetic energy T as a function of the angle θ.

(b) Find the the components p_x, p_y of the center-of-mass momentum as functions of the angle θ.

(c) Identify an attribute in the results of parts (a) or (b) that can be used as a criterion to determine if and when the rod loses contact with the wall during its fall.

(d) Find (or show how to determine) the angle θ_c at which the rod does indeed lose wall contact.



Solution:

[mex260] Solid sphere rolling on plane

A solid sphere of mass m and radius a is rolling without slipping on the xy-plane under the influence of an external force $\mathbf{F} = (F_x, F_y, F_z)$ and an external torque $\mathbf{N} = (N_x, N_y, N_z)$, both acting on its center of mass. The rolling motion is described by the instantaneous velocity $\mathbf{V} = (V_x, V_y, V_z)$ of the center of mass and the instantaneous angular velocity $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$ about its center of mass. In [mln106] we have established the equations of motion,

$$m\frac{d\mathbf{V}}{dt} = \mathbf{F} + \mathbf{F}^{\mathrm{c}}, \quad I\frac{d\vec{\omega}}{dt} = \mathbf{N} - a\hat{\mathbf{n}} \times \mathbf{F}^{\mathrm{c}},$$

and the equation of constraint,

$$\dot{\mathbf{V}} = a\,\dot{\vec{\omega}} \times \hat{\mathbf{n}}$$

Eliminate the contact force of constraint, $\mathbf{F}^{\mathrm{c}}$, from these relations to arrive at the equations of motion for $\mathbf{V}$ and $\vec{\omega}$ reduced to quadrature as stated in [mln106].

Solution: