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06. Lagrangian Mechanics II

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Abstract
Part six of course materials for Classical Dynamics (Physics 520), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.

Contents of this Document [mtc6]

6. Lagrangian Mechanics II

Constants of the Motion [mln10]

Consider a dynamical system with n degrees of freedom and described by generalized coordinates $q_1, \ldots, q_n$.

Definition: Any function $f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ with $df/dt = 0$ is a *constant of the motion*.

The terms *constant of the motion, conserved quantity, invariant, integral of the motion* are used interchangeably in the literature.

Fact: Any system with n degrees of freedom has $2n$ constants of the motion.

Obvious choices are the $2n$ initial conditions of the Lagrange equations (2^{nd} order ODEs): $c_k(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t), \; k = 1, \ldots, 2n$ derived from $q_i(c_1, \ldots, c_{2n}, t), \dot{q}_i(c_1, \ldots, c_{2n}, t), \; i = 1, \ldots, n$.

Comments:

- Not all constants of the motion are equally important. Some have a stronger impact on the time evolution of the system than others.
- Constants of the motion which are not known prior to the analytic solution of the system are, in general, not useful.
- Integration constants in particular (initial conditions, boundary values) are only meaningful constants of the motion if an analytic solution is available.
- Meaningful constants of the motion may very well be identified in systems for which no analytic solution exists.
- Certain constants of the motion can be used to reduce the number of degrees of freedom by factoring out single degrees of freedom (one per invariant). These special constants of the motion are best described in the context of Hamiltonian mechanics.
- The existence of n such constants of the motion preclude the dynamical system from behaving chaotically. Such system are called integrable.
- Some constants of the motion can be derived from known symmetries of the dynamical system, others point to obscure or hidden symmetries.

Conservation Laws and Symmetry [mln11]

Consider an isolated system described by generalized coordinates in an inertial reference frame: $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. The following conservation laws can be derived from general properties of space and time.

- Homogeneity of time leads to conservation of energy.

$$\frac{\partial L}{\partial t} = 0 \;\Rightarrow\; L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n).$$

$$\frac{dL}{dt} = \sum_j \left[\frac{\partial L}{\partial q_j}\, \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\, \ddot{q}_j \right] = \sum_j \left[\dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right] = \sum_j \frac{d}{dt}\left[\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right].$$

$$\frac{d}{dt}\left[L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right] = 0 \;\Rightarrow\; \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = E(q_j, \dot{q}_j) = \text{const.}$$

$$L(q_j, \dot{q}_j) = T(q_j, \dot{q}_j) - V(q_j), \quad \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = 2T \quad \text{[mex155]}$$

$$E(q_j, \dot{q}_j) = T(q_j, \dot{q}_j) + V(q_j).$$

A generalized coordinate q_l which does not appear in the Lagrangian $L(q_j, \dot{q}_j)$ is called *cyclic*. The generalized momentum p_l conjugate to a cyclic coordinate is conserved:

$$\frac{\partial L}{\partial q_l} = 0 \;\Rightarrow\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} = 0 \;\Rightarrow\; \frac{\partial L}{\partial \dot{q}_l} \doteq p_l = \text{const.}$$

- Homogeneity of space leads to conservation of linear momentum.

  The Lagrangian is invariant under global translations.
  Therefore, the center-of-mass coordinates are cyclic.
  Therefore, the total linear momentum vector is conserved.

- Isotropy of space leads to conservation of angular momentum.

  The Lagrangian is invariant under global rotations.
  Therefore, the angle of a global rotation about any axis is cyclic.
  Therefore, the total angular momentum vector is conserved.

[mex155] Kinetic energy in Lagrangian mechanics

Show that the kinetic energy of a dynamical system of N particles subject to k scleronomic constraints $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_n),\; i = 1, \ldots, N$ is a homogeneous quadratic function of the generalized coordinates:

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\mathbf{r}}_i|^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\dot{q}_k\dot{q}_k.$$

Identify the coefficients a_{jk} and show that $\sum_l \dot{q}_l(\partial T/\partial \dot{q}_l) = 2T$.

Solution:

[mex156] Spherical pendulum: reduction to quadrature

A particle of mass m in a uniform gravitational field g is constrained to move on the surface of a sphere of radius ℓ.
(a) Find the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$, where the range of the polar angle is $0 \leq \theta \leq \pi$ and the range of the azimuthal angle is $0 \leq \phi \leq 2\pi$.
(b) Derive the two Lagrange equations.
(c) Identify two conservation laws.
(d) Reduce the general solution for $\theta(t), \phi(t)$ to quadrature.

Solution:

Routhian Function [mln39]

Goal: systematic elimination of cyclic coordinates in the Lagrangian formulation of mechanics.

Consider a system with n generalized coordinates of which the first k are cyclic.

Lagrangian: $L(q_{k+1},\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t) \;\Rightarrow\; q_1,\ldots,q_k$ are cyclic.

Routhian: $R(q_{k+1},\ldots,q_n,\dot{q}_{k+1},\ldots,\dot{q}_n,\beta_1,\ldots,\beta_k,t) = L - \sum_{i=1}^{k}\beta_i\dot{q}_i.$

where the relations $\beta_i = \dfrac{\partial L}{\partial \dot{q}_i} = \text{const},\ i=1,\ldots,k$ are to be inverted into
$\dot{q}_i = \dot{q}_i(q_{k+1},\ldots,q_n,\dot{q}_{k+1},\ldots,\dot{q}_n,\beta_1,\ldots,\beta_k,t), \quad i=1,\ldots,k.$

Compare coefficients of the variations

$$\delta R = \sum_{i=k+1}^{n}\frac{\partial R}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n}\frac{\partial R}{\partial \dot{q}_i}\delta\dot{q}_i + \sum_{i=1}^{k}\frac{\partial R}{\partial \beta_i}\delta\beta_i + \frac{\partial R}{\partial t}\delta t,$$

$$\delta\left(L - \sum_{i=1}^{k}\beta_i\dot{q}_i\right) = \sum_{i=k+1}^{n}\frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i - \sum_{i=1}^{k}\dot{q}_i\delta\beta_i + \frac{\partial L}{\partial t}\delta t.$$

Resulting relations between partial derivatives:

$$\frac{\partial R}{\partial q_i} = \frac{\partial L}{\partial q_i}, \quad \frac{\partial R}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}, \quad i=k+1,\ldots,n,$$

$$\frac{\partial R}{\partial t} = \frac{\partial L}{\partial t}; \quad \dot{q}_i = -\frac{\partial R}{\partial \beta_i}, \quad i=1,\ldots,k.$$

Lagrange equations for the noncyclic coordinates:

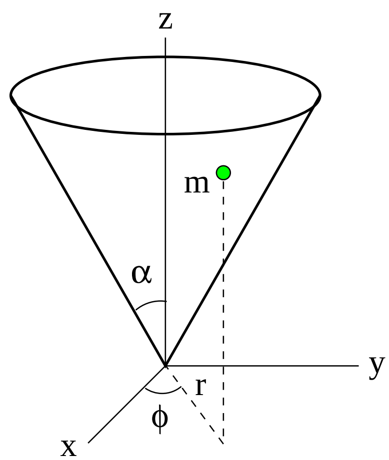
$$\frac{\partial R}{\partial q_i} - \frac{d}{dt}\frac{\partial R}{\partial \dot{q}_i} = 0, \quad i=k+1,\ldots,n.$$

Time evolution of cyclic coordinates:

$$q_i(t) = -\int dt\,\frac{\partial R}{\partial \beta_i}, \quad i=1,\ldots,k.$$

[mex157] Routhian function for heavy particle sliding inside cone

Consider a conical surface with vertical axis (z) and apex with angle 2α at the bottom in a uniform gravitational field g. A particle of mass m is free to slide on the inside of the cone.
(a) Express the Lagrangian in the generalized coordinates r, ϕ.
(b) Identify the cyclic coordinate and identify the Routhian function which eliminates the cyclic coordinate.
(c) Derive the equation of motion for the noncyclic coordinate and an integral expression for the cyclic coordinate.



Solution:

Noether's Theorem I [mln12]

Symmetries indicated by cyclic variables in the Lagrangian lead to conservation laws. Some symmetries may be obscure or hidden. Noether's theorem derives conservation laws from a general class of continuous symmetries.

Consider a Lagrangian system $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Theorem (restricted case):

If a transformation $Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon),\ i = 1, \ldots, n$ with $Q_i = q_i$ at $\epsilon = 0$ can be found such that

$$\left.\frac{\partial L'}{\partial \epsilon}\right|_{\epsilon=0} = 0$$

is satisfied, where

$$L'(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon) \doteq L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t),$$

then the following quantity is conserved:

$$I(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \left.\frac{\partial q_i}{\partial \epsilon}\right|_{\epsilon=0}.$$

Proof:

Use the inverse transformation $q_i(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon),\ i = 1, \ldots, n$ and keep the variables $Q_i, \dot{Q}_i$ fixed.

$$\frac{\partial L'}{\partial \epsilon} = \sum_i \left[\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon}\right] = \sum_i \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial q_i}{\partial \epsilon}\right)\right] = 0.$$

$$\Rightarrow \frac{\partial L'}{\partial \epsilon} = \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \epsilon}\right] = 0.$$

Applications:

- translation in space [mex35]
- rotation in space [mex36]

[mex121] Routhian function of 2D harmonic oscillator

Consider the 2D harmonic oscillator with kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and potential energy $V = \frac{1}{2}k(x^2 + y^2)$. (a) Express the Lagrangian of this system in polar coordinates r, θ. (b) Identify the cyclic coordinate and construct the Routhian function which eliminates the cyclic coordinate. (c) Derive the equation of motion for the noncyclic coordinate and an integral expression for the cyclic coordinate.

Solution:

[mex35] Noether's theorem I: translation in space

Consider the Lagrangian $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(y, z)$ of a particle with mass m moving in 3D space under the influence of a scalar potential.
(a) Identify an infinitesimal symmetry transformation.
(b) Apply Noether's theorem to determine the associated constant of the motion.

Solution:

[mex36] Noether's theorem II: rotation in space

Consider the Lagrangian $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x^2 + y^2, z)$ of a particle with mass m moving in 3D space under the influence of a scalar potential.
Identify an infinitesimal symmetry transformation. Then apply Noether's theorem to determine the associated constant of the motion. Perform the calculation using (a) Cartesian coordinates x, y, z, (b) cylindrical coordinates r, ϕ, z.

Solution:

Noether's Theorem II [mln13]

A more general class of symmetry transformations leaves the Lagrange equations invariant but not the Lagrangian itself.

Consider again a Lagrangian system $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Theorem (more general case):

If a transformation $Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon),\ i = 1, \ldots, n$ with

$Q_i = q_i$ at $\epsilon = 0$ can be found such that

$$\left.\frac{\partial L'}{\partial \epsilon}\right|_{\epsilon=0} = \frac{d}{dt}\left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0}$$

is satisfied, where

$$L'(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon) \doteq L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

and $F(Q_1, \ldots, Q_n, t, \epsilon)$ is an arbitrary differentiable function, then the following quantity is conserved:

$$I(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \left.\frac{\partial q_i}{\partial \epsilon}\right|_{\epsilon=0} - \left.\frac{\partial F}{\partial \epsilon}\right|_{\epsilon=0}.$$

Proof:

Use inverse transformation $q_i(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon),\ i = 1, \ldots, n$ and keep the variables $Q_i, \dot{Q}_i$ fixed. Then use gauge invariance (see [mex21]),

$$L'(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t, \epsilon) = L(Q_1, \ldots, Q_n, \dot{Q}_1, \ldots, \dot{Q}_n, t) + \frac{d}{dt} F(Q_1, \ldots, Q_n, t, \epsilon)$$

and the steps of the proof in [mln12].

$$\left[\frac{\partial L'}{\partial \epsilon} - \frac{d}{dt}\frac{\partial F}{\partial \epsilon}\right]_{\epsilon=0} = \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} - \frac{\partial F}{\partial \epsilon}\right]_{\epsilon=0}.$$

Applications:

- pure Galilei transformation [mex37]

[mex37] Noether's theorem III: pure Galilei transformation

Consider the Lagrangian $L(z, \dot{z}) = \frac{1}{2}m\dot{z}^2 - mgz$ of a particle with mass m moving vertically in 3D space under the influence of a uniform gravitational field. Show that the transformation $X = x,\ Y = y,\ Z = z + \epsilon t$ is a symmetry transformation by establishing the relation

$$L'(Z, \dot{Z}, t, \epsilon) = L(Z, \dot{Z}) + \frac{d}{dt}F(Z, t, \epsilon).$$

Find the function $F(Z, t, \epsilon)$ and the conserved quantity $I(z, \dot{z}, t)$ associated with this symmetry.

Solution:

Noether's Theorem III [mln42]

The continuous symmetry transformation may also involve the time.

Consider again a Lagrangian system $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Theorem (most general case):

If a transformation $Q_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon)$, $T(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t, \epsilon)$, $i = 1, \ldots, n$ with $Q_i = q_i$ and $T = t$ at $\epsilon = 0$ can be found such that

$$\frac{\partial}{\partial \epsilon}\left[L\left(Q_1, \ldots, Q_n, \frac{dQ_1}{dT}, \ldots, \frac{dQ_n}{dT}, T\right)\frac{dT}{dt}\right]_{\epsilon=0} = \frac{d}{dt}\, G(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$$

is satisfied (for an arbitrary function G), then the following quantity is conserved:

$$I = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\left(\frac{\partial T}{\partial \epsilon}\right)_{\epsilon=0} - G + \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\left[\left(\frac{\partial Q_i}{\partial \epsilon}\right)_{\epsilon=0} - \dot{q}_i\left(\frac{\partial T}{\partial \epsilon}\right)_{\epsilon=0}\right].$$

Proof:

$$\dot{G} = \left\{\left[\sum_i \frac{\partial L}{\partial Q_i}\frac{\partial Q_i}{\partial \epsilon} + \sum_i \frac{\partial L}{\partial(dQ_i/dT)}\frac{\partial(dQ_i/dT)}{\partial \epsilon} + \frac{\partial L}{\partial T}\frac{\partial T}{\partial \epsilon}\right]\dot{T} + L\frac{\partial \dot{T}}{\partial \epsilon}\right\}_{\epsilon=0}.$$

Define: $A_i \equiv \left(\dfrac{\partial Q_i}{\partial \epsilon}\right)_{\epsilon=0}$, $\quad B \equiv \left(\dfrac{\partial T}{\partial \epsilon}\right)_{\epsilon=0}$.

Expand $Q_i = q_i + A_i\epsilon + \ldots$, $\quad T = t + B\epsilon + \ldots$.

$$\Rightarrow\ \dot{Q}_i = \dot{q}_i + \dot{A}_i\epsilon + \ldots, \quad \dot{T} = 1 + \dot{B}\epsilon + \ldots, \quad \frac{dQ_i}{dT} = \frac{\dot{Q}_i}{\dot{T}} = \frac{\dot{q}_i + \dot{A}_i\epsilon + \ldots}{\dot{t} + \dot{B}\epsilon + \ldots}.$$

At $\epsilon = 0$: $\dot{Q}_i = \dot{q}_i$, $\dot{T} = 1$, $\dfrac{dQ_i}{dT} = \dot{q}_i$, $\dfrac{\partial \dot{T}}{\partial \epsilon} = \dot{B}$, $\dfrac{\partial}{\partial \epsilon}\dfrac{dQ_i}{dT} = \dot{A}_i - \dot{q}_i\dot{B}$.

Use $\dfrac{\partial L}{\partial q_i} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i}$, $\quad \dfrac{dL}{dt} = \dfrac{\partial L}{\partial t} + \dfrac{\partial L}{\partial q_i}\dot{q}_i + \dfrac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$.

$$\Rightarrow\ \frac{d}{dt}\left[LB - G + \sum_i \frac{\partial L}{\partial \dot{q}_i}(A_i - \dot{q}_i B)\right] = 0\ \Rightarrow\ I = \text{const.}$$

Dissipative forces in Lagrangian mechanics [mln9]

A dissipative force counteracts motion. Its direction is opposite to the direction of the velocity vector. Hence any dissipative force depends on velocity, be it on its direction only or also on its magnitude.

Dissipative forces are (by definition) non-conservative; they cannot be derived from a potential, not even from a velocity-dependent potential. However, Lagrangian mechanics allows the derivation of purely velocity dependent dissipative forces from a *dissipation function*.

Dissipative forces in Cartesian coordinates: $\mathbf{R}_i \doteq -h_i(v_i)\dfrac{\mathbf{v}_i}{v_i}, \quad i = 1, \ldots, N$

Transformation to generalized coordinates $q_1, \ldots, q_n$:

$$R_j = -\sum_{i=1}^{N} h_i(v_i)\frac{\mathbf{v}_i}{v_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_{i=1}^{N} h_i(v_i)\frac{\mathbf{v}_i}{v_i} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = -\sum_{i=1}^{N} h_i(v_i)\frac{\partial v_i}{\partial \dot{q}_j}.$$

We have used: $\dfrac{\partial \mathbf{r}_i}{\partial q_j} = \dfrac{\partial \mathbf{v}_i}{\partial \dot{q}_j}, \quad \mathbf{v}_i \cdot \dfrac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \dfrac{1}{2}\dfrac{\partial \mathbf{v}_i \cdot \mathbf{v}_i}{\partial \dot{q}_j} = \dfrac{1}{2}\dfrac{\partial v_i^2}{\partial \dot{q}_j} = v_i\,\dfrac{\partial v_i}{\partial \dot{q}_j}.$

Dissipation function: $P \doteq \displaystyle\sum_{i=1}^{N}\int_0^{v_i} dv_i\, h_i(v_i).$

$$\Rightarrow \; R_j = -\sum_{i=1}^{N} h_i(v_i)\frac{\partial v_i}{\partial \dot{q}_j} = -\frac{\partial}{\partial \dot{q}_j}\sum_{i=1}^{N}\int_0^{v_i} dv_i\, h_i(v_i) = -\frac{\partial P}{\partial \dot{q}_j}, \quad j = 1, \ldots, n$$

Lagrange equations: $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} - \dfrac{\partial L}{\partial q_j} + \dfrac{\partial P}{\partial \dot{q}_j} = 0, \quad j = 1, \ldots, n.$

Common dissipative forces:

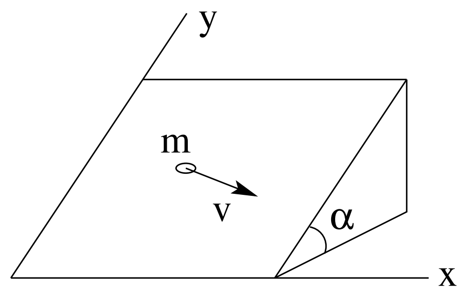
- kinetic friction (Coulomb damping): $\mathbf{R} = -\mu N\,\dfrac{\mathbf{v}}{v}$.
- linear damping (more common at low velocity): $\mathbf{R} = -\beta v\,\dfrac{\mathbf{v}}{v}$.
- quadratic damping (more common at high velocity): $\mathbf{R} = -\gamma v^2\,\dfrac{\mathbf{v}}{v}$.

Examples:

- Motion with friction on inclined plane [mex151]
- Linearly damped spherical pendulum [mex158]

[mex151] Motion with friction on inclined plane

Consider a particle moving on an inclined plane as shown. The motion is impeded by kinetic friction. Find the Lagrangian $L(x, y, \dot{x}, \dot{y})$ and the dissipation function $P(\dot{x}, \dot{y})$ and derive the Lagrange equations for the variables x, y from these functions.



Solution:

[mex158] Linearly damped spherical pendulum

Consider the spherical pendulum with Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$ as analyzed in [mex156]. Now we assume that the motion is subject to a linear damping force $\mathbf{R} = -\beta v(\mathbf{v}/v)$. Find the dissipation function $P(\theta, \phi, \dot{\theta}, \dot{\phi})$ representing this kind of attenuation and derive from it the damping torques $R_\theta = -\partial P/\partial\dot{\theta},\ R_\phi = -\partial P/\partial\dot{\phi}$ acting on the angular coordinates θ, ϕ, respectively.

Solution:

Generalized Forces of Constraint in Lagrangian Mechanics [mln15]

Lagrangian: $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Differential constraints: $\displaystyle\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, \quad j = 1, \ldots, m.$

Relations between virtual displacements: $\displaystyle\sum_{i=1}^{n} a_{ji} \delta q_i = 0, \quad j = 1, \ldots, m.$

The generalized forces of constraint, Q_i, do not perform any work.

D'Alembert's principle $\Rightarrow \displaystyle\sum_{i=1}^{n} Q_i \delta q_i = 0.$

$$\Rightarrow \sum_{i=1}^{n} \left(Q_i - \sum_{j=1}^{m} \lambda_j a_{ji} \right) \delta q_i = 0 \text{ for arbitrary values of } \lambda_j.$$

Choose the Lagrange multipliers λ_j to satisfy $Q_i = \displaystyle\sum_{j=1}^{m} \lambda_j a_{ji}, \; i = 1, \ldots, n.$

The δq_i can now be chosen independently because the constraints are enforced by the generalized forces Q_i.

The solution of the dynamical problem is then determined by the following $n + m$ equations for the n dynamical variables q_i and the m Lagrange multipliers λ_j:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m} \lambda_j a_{ji} = 0, \quad i = 1, \ldots, n.$$

$$\sum_{i=1}^{n} a_{ji} \dot{q}_i + a_{jt} = 0, \quad j = 1, \ldots, m.$$

For holonomic constraints, $f_j(q_1, \ldots, q_n, t) = 0, \; j = 1, \ldots, m$, we have

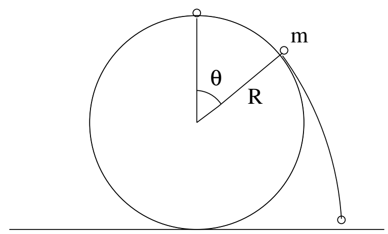
$$a_{ji} = \frac{\partial f_j}{\partial q_i}, \quad a_{jt} = \frac{\partial f_j}{\partial t}, \quad Q_i = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_i}.$$

Whereas holonomic constraints can be handled *kinematically*, i.e. via the elimination of redundant coordinates, nonholonomic constraints must be handled *dynamically*, i.e. via the explicit use of constraint forces.

In some cases, the generalized forces of constraint Q_j can be determined without integrating the equations of motion.

[mex34] Particle sliding down sphere (revisited)

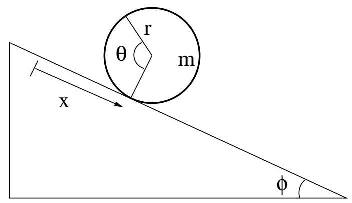
A tiny particle of mass m slides without friction down a spherical surface of radius R. The particle starts at the top with negligible speed. (a) Determine the Lagrangian in polar coordinates, $L(r, \theta, \dot{r}, \dot{\theta})$, and the holonomic constraint $f(r, \theta) = 0$ of the sliding motion for as long as it lasts. (b) Use the results of (a) and the conservation of energy to determine the force of constraint (normal force) during the sliding part of the motion. (c) Determine the angle at which the particle leaves the sphere from the criterion that the force of constraint vanishes.



Solution:

[mex32] Static frictional force of constraint

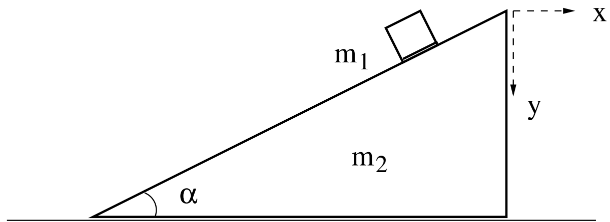
Consider a hoop of mass m and radius r rolling without slipping down an incline. (a) Determine the Lagrangian $L(x, \dot{x})$ of this one-degree-of-freedom system. Derive from it the Lagrange equation and its solution for initial condition $x_0 = 0, \dot{x}_0 = 0$. (b) Determine the alternative Lagrangian $L(x, \theta, \dot{x}, \dot{\theta})$ and the holonomic constraint $f(x, \theta) = 0$ that must accompany it. Derive the associated three equations of motion for the two unknown dynamical variables x, θ and the undetermined Lagrange multiplier λ. Solve these equations for the same initial conditions as in (a) and determine the static frictional force of constraint between the hoop and the incline.



Solution:

[mex33] Normal force of constraint

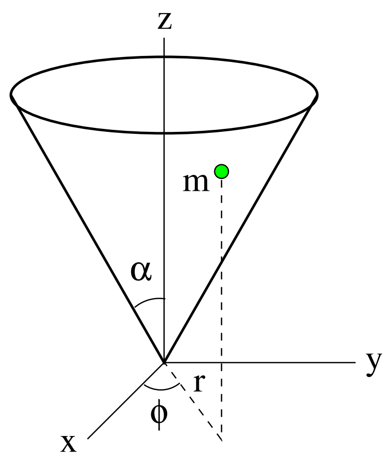
A particle of mass m_1 slides without friction on a wedge of angle α and mass m_2. The wedge in turn is free to slide without friction on a smooth horizontal surface. (a) Determine the Lagrangian $L(x_1, y_1, x_2, \dot{x}_1, \dot{y}_1, \dot{x}_2)$ and the holonomic constraint $f(x_1, y_1, x_2) = 0$ that goes with it. (b) Derive the associated four equations of motion for the three dynamical variables x_1, y_1, x_2 and the Lagrange multiplier λ. (c) Find the solutions and the forces of constraint acting on the particle and on the wedge. (d) Discuss the solutions in the limits $m_2 \to 0$ and $m_2 \to \infty$.



Solution:

[mex159] Particle sliding inside cone: normal force of constraint

Consider a conical surface with vertical axis (z) and apex with angle 2α at the bottom in a uniform gravitational field g. A particle of mass m is free to slide on the inside of the cone.
(a) Write the equation of holonomic constraint $f(z, r, \phi) = 0$ between the cylindrical coordinates and the Lagrangian $L(z, r, \phi)$
(b) Derive the three Lagrange equations. Along with the equation $f = 0$, they determine the generalized coordinates z, r, ϕ and the Lagrange multiplier λ.
(c) By using the conservation law $p_\phi = mr^2\dot{\phi} = \text{const}$, solve the equations of motion (without integrating $\ddot{z}$ and $\ddot{r}$) for the Lagrange multiplier $\lambda(p_\phi, r)$ and infer from it the three components $Q_z = \lambda\partial f/\partial z$, $Q_r = \lambda\partial f/\partial r$, $Q_\phi = \lambda\partial f/\partial\phi$ of the normal force of constraint.



Solution: