2015

03. Electric Field III and Electric Flux

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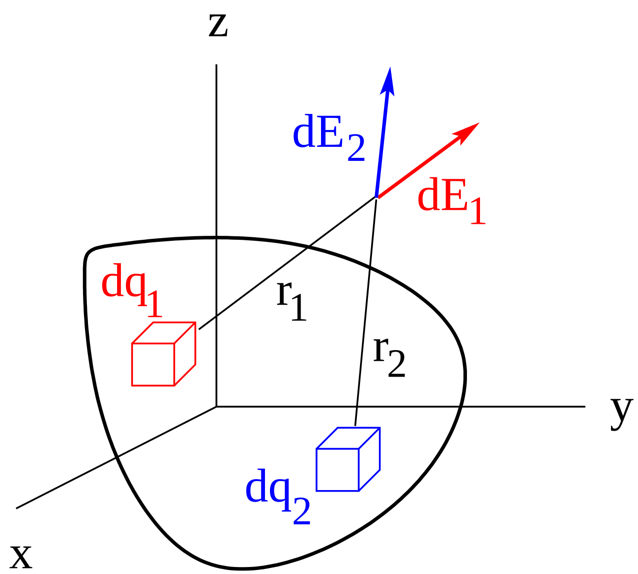
Abstract
Part three of course materials for Elementary Physics II (PHY 204), taught by Gerhard Müller at the University of Rhode Island. Documents will be updated periodically as more entries become presentable.
Some of the slides contain figures from the textbook, Paul A. Tipler and Gene Mosca. *Physics for Scientists and Engineers*, 5th/6th editions. The copyright to these figures is owned by W.H. Freeman. We acknowledge permission from W.H. Freeman to use them on this course web page. The textbook figures are not to be used or copied for any purpose outside this class without direct permission from W.H. Freeman.

Electric Field of Continuous Charge Distribution

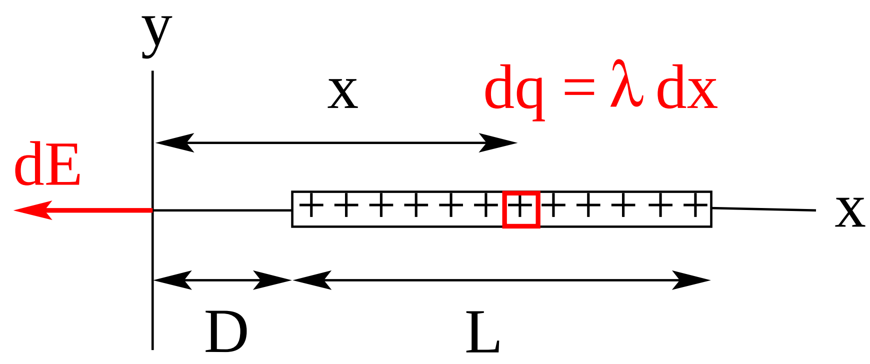
- Divide the charge distribution into infinitesimal blocks.
  - For 3D applications use charge per unit volume: $\rho = \Delta Q/\Delta V$.
  - For 2D applications use charge per unit area: $\sigma = \Delta Q/\Delta A$.
  - For 1D applications use charge per unit length: $\lambda = \Delta Q/\Delta L$.
- Use Coulomb's law to calculate the electric field generated by each block.
- Use the superposition principle to calculate the resultant field from all blocks.
- Use symmetries whenever possible.

$$d\vec{E}_i = k\frac{dq_i}{r_i^2}\,\hat{r}_i$$

$$\vec{E} = \sum_i d\vec{E}_i \rightarrow k\int \frac{dq}{r^2}\,\hat{r}$$



Electric Field of Charged Rod (1)



- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx: $dq = \lambda dx$

- Electric field generated by slice dx: $dE = \frac{kdq}{x^2} = \frac{k\lambda dx}{x^2}$
- Electric field generated by charged rod:

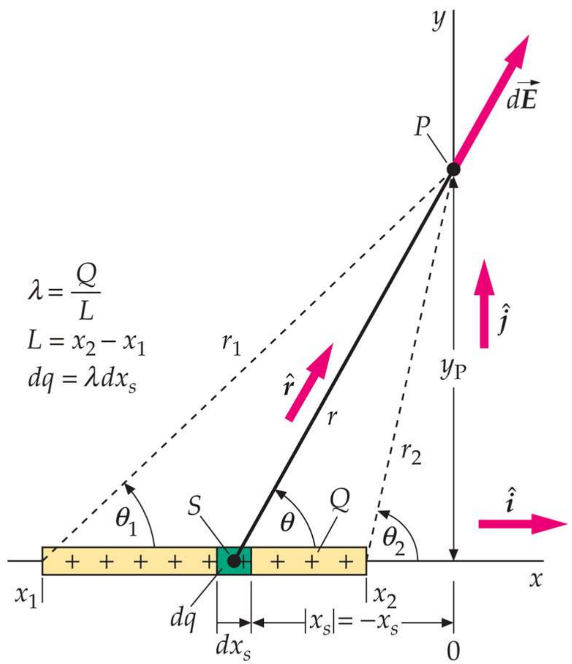
$$E = k\lambda \int_D^{D+L} \frac{dx}{x^2} = k\lambda \left[-\frac{1}{x} \right]_D^{D+L} = k\lambda \left[\frac{1}{D} - \frac{1}{D+L} \right] = \frac{kQ}{D(D+L)}$$

- Limiting case of very short rod $(L \ll D)$: $E \simeq \frac{kQ}{D^2}$
- Limiting case of very long rod $(L \gg D)$: $E \simeq \frac{k\lambda}{D}$

Electric Field of Charged Rod (2)

- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx_s: $dq = \lambda dx_s$
- Trigonometric relations:

$$y_p = r\sin\theta, \quad -x_s = r\cos\theta$$
$$x_s = -y_p\cot\theta, \quad dx_s = \frac{y_p d\theta}{\sin^2\theta}$$

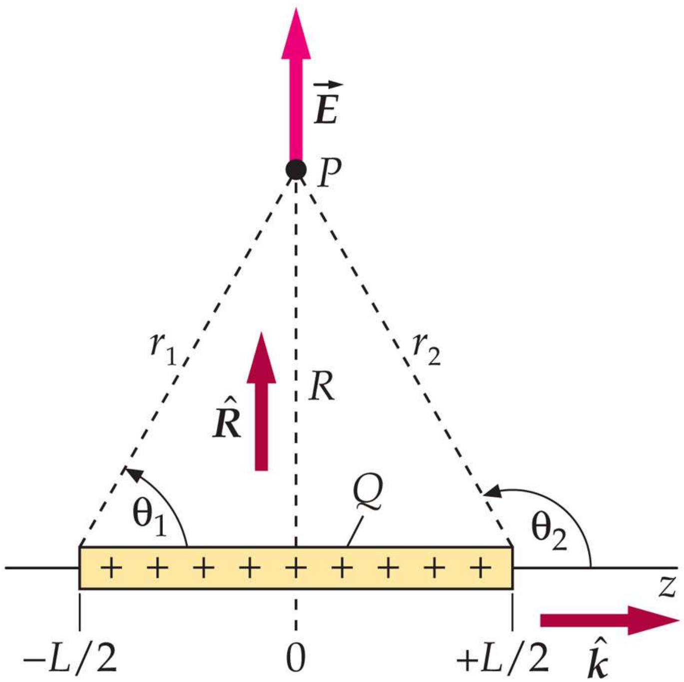


- $dE = \dfrac{k\lambda dx_s}{r^2} = \dfrac{k\lambda dx_s}{y_p^2}\sin^2\theta = \dfrac{k\lambda d\theta}{y_p}$
- $dE_y = dE\sin\theta = \dfrac{k\lambda}{y_p}\sin\theta d\theta \;\Rightarrow\; E_y = \dfrac{k\lambda}{y_p}\displaystyle\int_{\theta_1}^{\theta_2}\sin\theta d\theta = -\dfrac{k\lambda}{y_p}\left(\cos\theta_2 - \cos\theta_1\right)$
- $dE_x = dE\cos\theta = \dfrac{k\lambda}{y_p}\cos\theta d\theta \;\Rightarrow\; E_x = \dfrac{k\lambda}{y_p}\displaystyle\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \dfrac{k\lambda}{y_p}\left(\sin\theta_2 - \sin\theta_1\right)$

Electric Field of Charged Rod (3)

Symmetry dictates that the resulting electric field is directed radially.

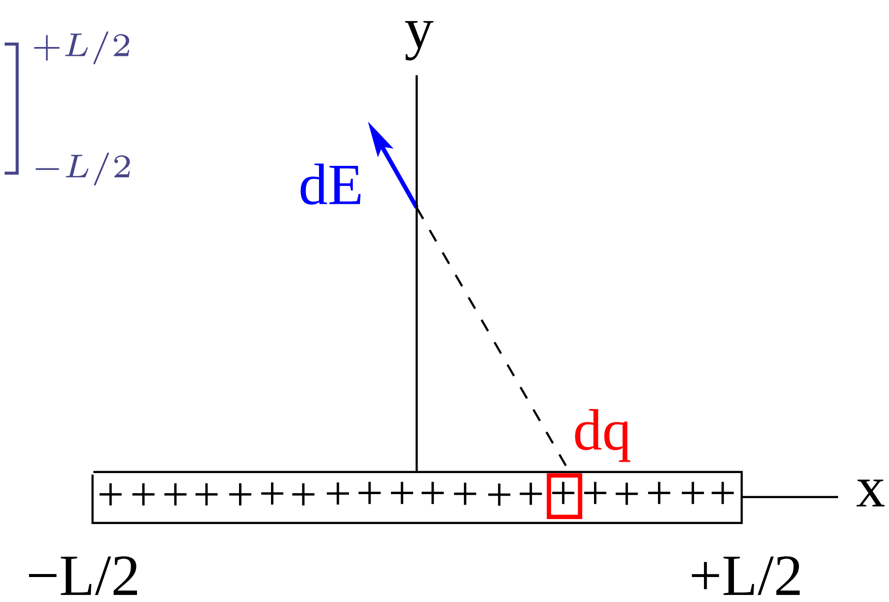
- $\theta_2 = \pi - \theta_1, \quad \Rightarrow \; \sin\theta_2 = \sin\theta_1, \quad \cos\theta_2 = -\cos\theta_1.$
- $\cos\theta_1 = \dfrac{L/2}{\sqrt{L^2/4 + R^2}}.$
- $E_R = -\dfrac{k\lambda}{R}(\cos\theta_2 - \cos\theta_1) = \dfrac{k\lambda}{R}\dfrac{L}{\sqrt{L^2/4 + R^2}}.$
- $E_z = \dfrac{k\lambda}{R}(\sin\theta_2 - \sin\theta_1) = 0.$
- Large distance $(R \gg L)$: $E_R \simeq \dfrac{kQ}{R^2}$.
- Small distances $(R \ll L)$: $E_R \simeq \dfrac{2k\lambda}{R}$
- Rod of infinite length: $\vec{E} = \dfrac{2k\lambda}{R}\hat{R}$.



Electric Field of Charged Rod (4)

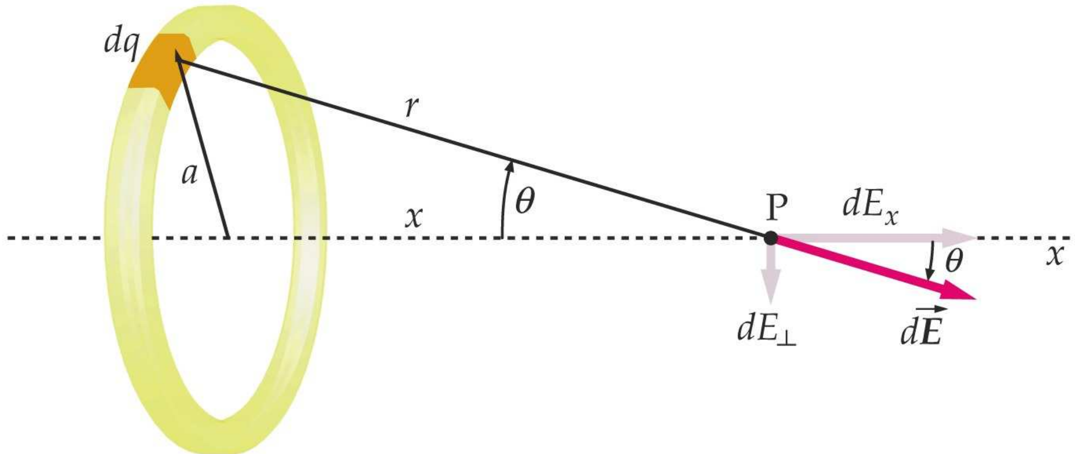
Symmetry dictates that the resulting electric field is directed radially.

- Charge per unit length: $\lambda = Q/L$
- Charge on slice dx: $dq = \lambda dx$
- $dE = \dfrac{kdq}{r^2} = \dfrac{k\lambda dx}{x^2 + y^2}$
- $dE_y = dE \cos\theta = \dfrac{dEy}{\sqrt{x^2 + y^2}} = \dfrac{k\lambda y dx}{(x^2 + y^2)^{3/2}}$
- $E_y = \displaystyle\int_{-L/2}^{+L/2} \frac{k\lambda y dx}{(x^2 + y^2)^{3/2}} = \left[\frac{k\lambda y x}{y^2\sqrt{x^2 + y^2}}\right]_{-L/2}^{+L/2}$
- $E_y = \dfrac{k\lambda L}{y\sqrt{(L/2)^2 + y^2}} = \dfrac{kQ}{y\sqrt{(L/2)^2 + y^2}}$
- Large distance $(y \gg L)$: $E_y \simeq \dfrac{kQ}{y^2}$
- Small distances $(y \ll L)$: $E_y \simeq \dfrac{2k\lambda}{y}$

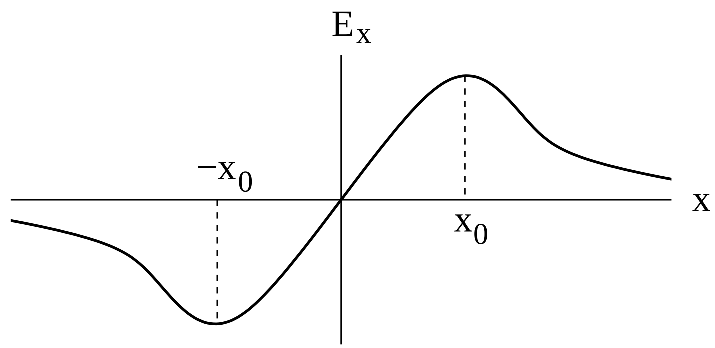


Electric Field of Charged Ring

- Total charge on ring: Q
- Charge per unit length: $\lambda = Q/2\pi a$
- Charge on arc: dq



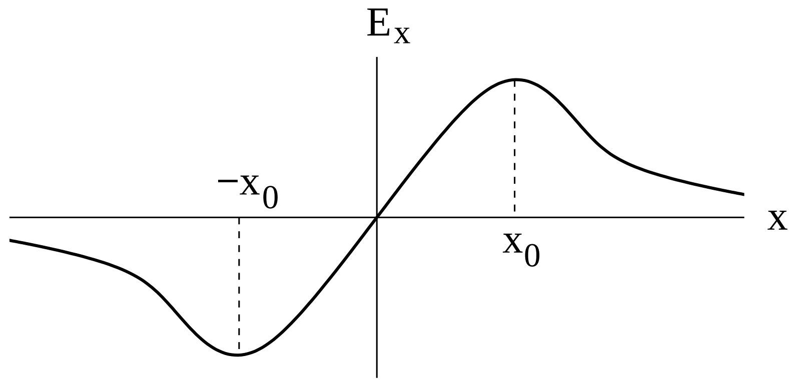
- $dE = \frac{kdq}{r^2} = \frac{kdq}{x^2+a^2}$
- $dE_x = dE\cos\theta = dE\,\frac{x}{\sqrt{x^2+a^2}} = \frac{kxdq}{(x^2+a^2)^{3/2}}$
- $E_x = \frac{kx}{(x^2+a^2)^{3/2}}\int dq \;\Rightarrow\; E_x = \frac{kQx}{(x^2+a^2)^{3/2}}$
- $|x| \ll a:\; E_x \simeq \frac{kQx}{a^3}, \qquad x \gg a:\; E_x \simeq \frac{kQ}{x^2}$
- $(dE_x/dx)_{x=x_0} = 0 \quad\Rightarrow\; x_0 = \pm a/\sqrt{2}$



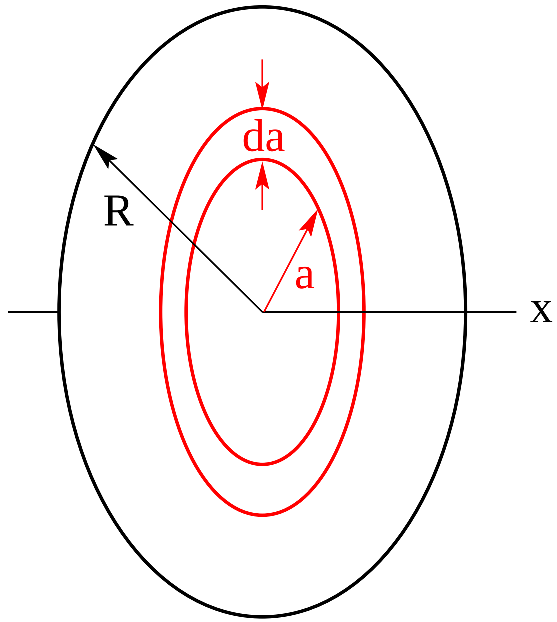
Charged Bead Moving Along Axis of Charged Ring

Consider a negatively charged bead (mass m, charge $-q$) constrained to move without friction along the axis of a positively charged ring.

- Place bead on x-axis near center of ring: $|x| \ll a: \; E_x \simeq \dfrac{kQx}{a^3}$
- Restoring force: $F = -qE_x = -k_s x \text{ with } k_s = \dfrac{kQq}{a^3}$
- Harmonic oscillation: $x(t) = A\cos(\omega t + \phi)$
- Angular frequency: $\omega = \sqrt{\dfrac{k_s}{m}} = \sqrt{\dfrac{kQq}{ma^3}}$



Electric Field of Charged Disk



- Charge per unit area: $\sigma = \dfrac{Q}{\pi R^2}$
- Area of ring: $dA = 2\pi a da$
- Charge on ring: $dq = 2\pi\sigma a da$

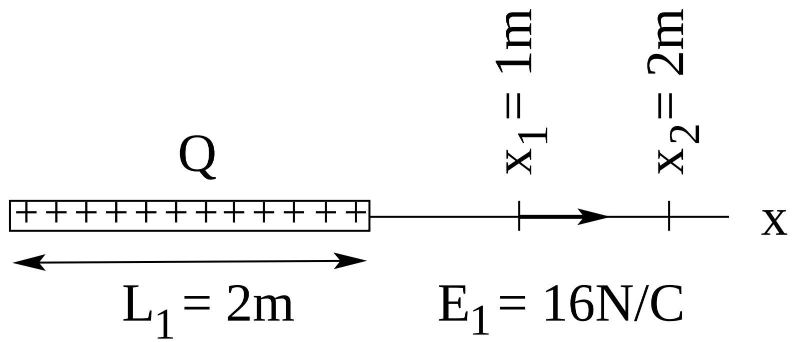
- $dE_x = \dfrac{kxdq}{(x^2+a^2)^{3/2}} = \dfrac{2\pi\sigma kxada}{(x^2+a^2)^{3/2}}$
- $E_x = 2\pi\sigma kx \displaystyle\int_0^R \frac{ada}{(x^2+a^2)^{3/2}} = 2\pi\sigma kx \left[\frac{-1}{\sqrt{x^2+a^2}}\right]_0^R$
- $E_x = 2\pi\sigma k \left[1 - \dfrac{x}{\sqrt{x^2+R^2}}\right]$ for $x > 0$
- $x \ll R: \; E_x \simeq 2\pi\sigma k$
- Infinite sheet of charge produces uniform electric field perpendicular to plane.

Electric Field of Charged Rubber Band

The electric field at position x along the line of a charged rubber band is

$$E = \frac{kQ}{x(x+L)}$$

The value of E at $x_1 = 1$m is $E_1 = 16$N/C.

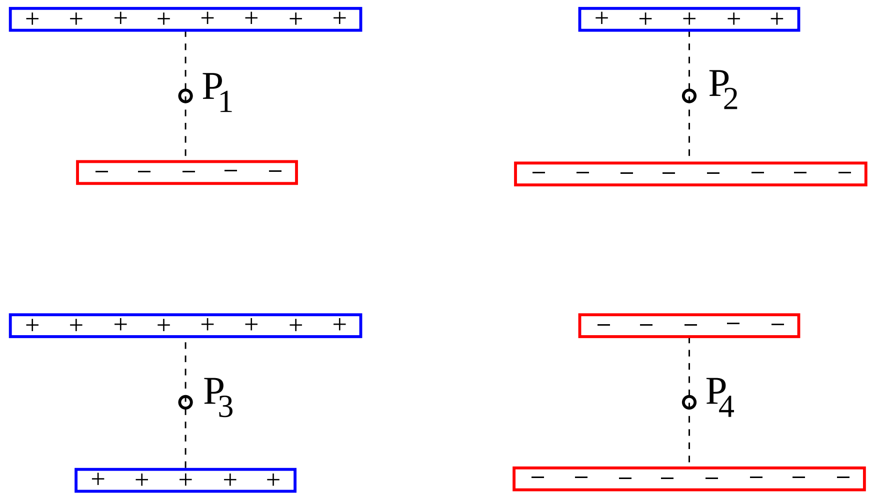


(a) What is the electric field E_2 at a distance $x_2 = 2$m from the edge of the band?

(b) To what length L_2 must the band be stretched (toward the left) such that it generates the field $E_2 = 8$N/C at $x_1 = 1$m?

Electric Field Between Charged Rods

Consider four configurations of two charged rods with equal amounts of charge per unit length $|\lambda|$ on them.



(a) Determine the direction of the electric field at points P_1, P_2, P_3, P_4.

(b) Rank the electric field at the four points according to strength.

Electric Field of Charged Semicircle

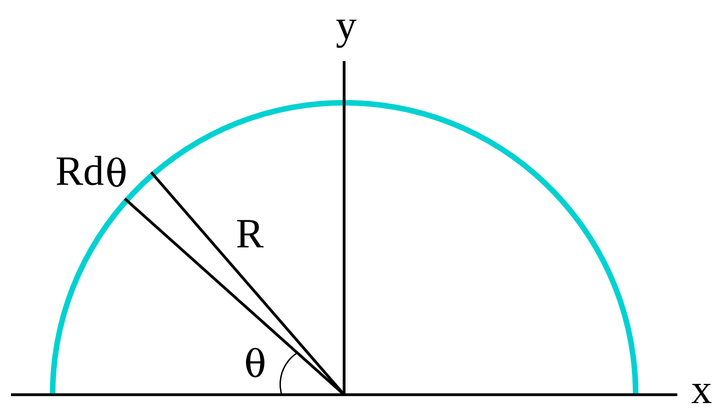
Consider a uniformly charged thin rod bent into a semicircle of radius R.

Find the electric field generated at the origin of the coordinate system.

- Charge per unit length: $\lambda = Q/\pi R$
- Charge on slice: $dq = \lambda R d\theta$ (assumed positive)
- Electric field generated by slice: $dE = k\dfrac{|dq|}{R^2} = \dfrac{k|\lambda|}{R}d\theta$
  directed radially (inward for $\lambda > 0$)
- Components of $d\vec{E}$: $dE_x = dE\cos\theta, \quad dE_y = -dE\sin\theta$
- Electric field from all slices added up:

$$E_x = \frac{k\lambda}{R}\int_0^\pi \cos\theta\, d\theta = \frac{k\lambda}{R}\left[\sin\theta\right]_0^\pi = 0$$

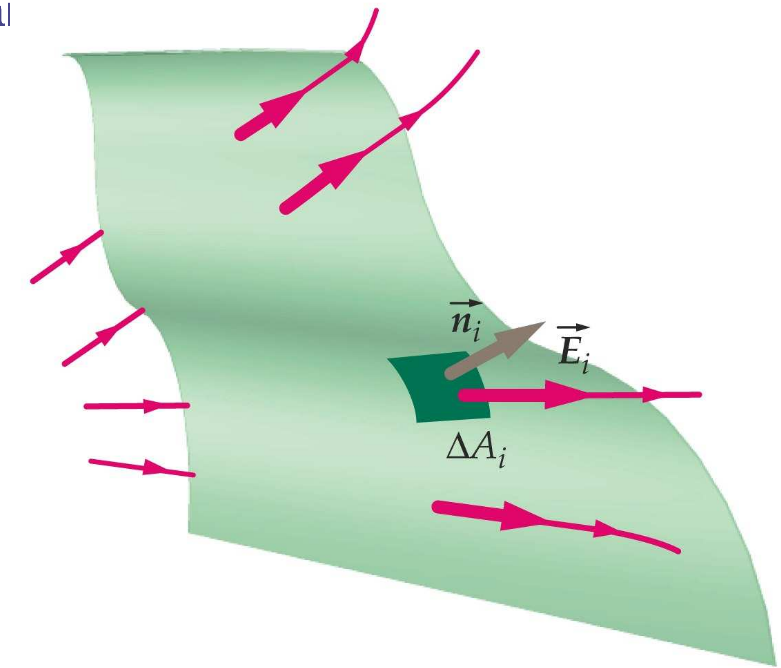
$$E_y = -\frac{k\lambda}{R}\int_0^\pi \sin\theta\, d\theta = \frac{k\lambda}{R}\left[\cos\theta\right]_0^\pi = -\frac{2k\lambda}{R}$$

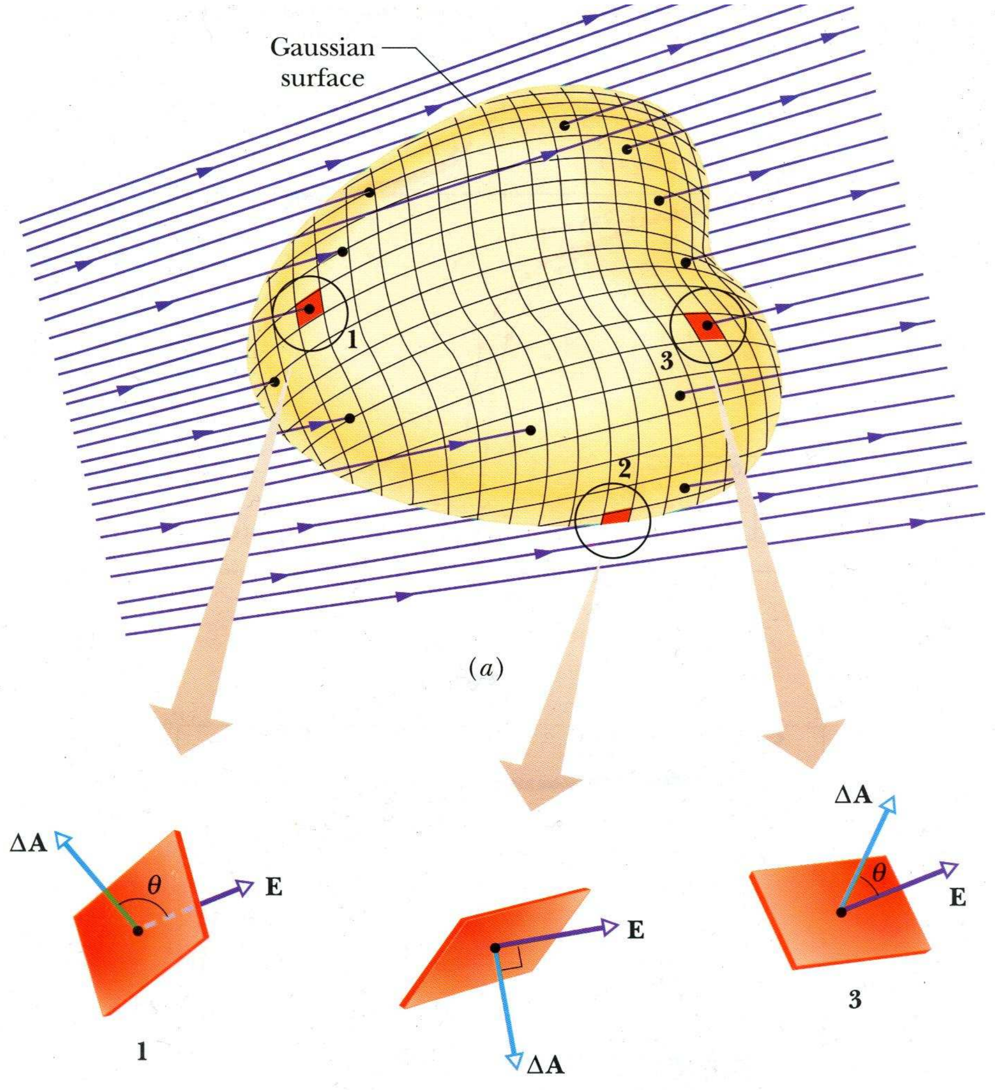


Electric Flux: Definition

Consider a surface S of arbitrary shape in the presence of an electric field $\vec{E}$.
Prescription for the calculation of the electric flux through S:

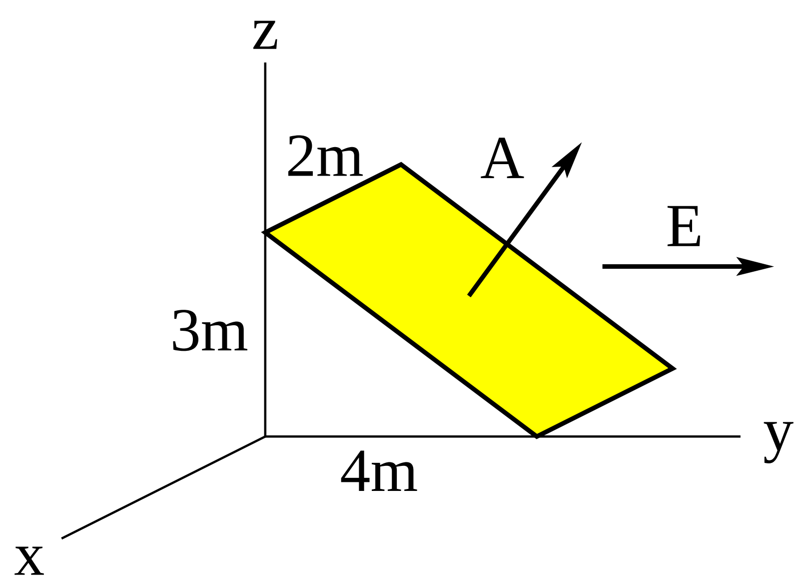
- Divide S into small tiles of area ΔA_i.
- Introduce vector $\Delta\vec{A}_i = \hat{n}_i \Delta A_i$ perpendicular to tile.
  - If S is open choose consistently one of two possible directions for $\Delta\vec{A}_i$.
  - If S is closed choose $\Delta\vec{A}_i$ to be directed outwa
- Electric field at position of tile i: $\vec{E}_i$.
- Electric flux through tile i:
  $\Delta\Phi_i^{(E)} = \vec{E}_i \cdot \Delta\vec{A}_i = E_i \Delta A_i \cos\theta_i$.
- Electric flux through S: $\Phi_E = \sum_i \vec{E}_i \cdot \Delta\vec{A}_i$.
- Limit of infinitesimal tiles: $\Phi_E = \int \vec{E} \cdot d\vec{A}$.
- Electric flux is a scalar.
- The SI unit of electric flux is Nm^2/C.




Gaussian surface
1
2
3
(a)
ΔA
θ
E
1
E
ΔA
ΔA
θ
E
3

Electric Flux: Application (1)

Consider a rectangular sheet oriented perpendicular to the yz plane as shown and positioned in a uniform electric field $\vec{E} = (2\hat{j})$N/C.



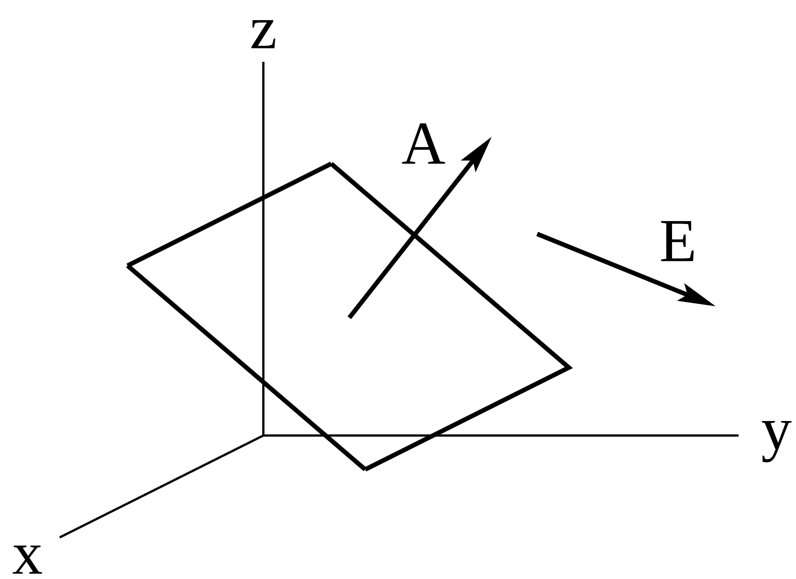
(a) Find the area A of the sheet.

(b) Find the angle between $\vec{A}$ and $\vec{E}$.

(c) Find the electric flux through the sheet.

Electric Flux: Application (2)

Consider a plane sheet of paper whose orientation in space is described by the area vector $\vec{A} = (3\hat{j} + 4\hat{k})\mathrm{m}^2$ positioned in a region of uniform electric field $\vec{E} = (1\hat{i} + 5\hat{j} - 2\hat{k})$N/C.



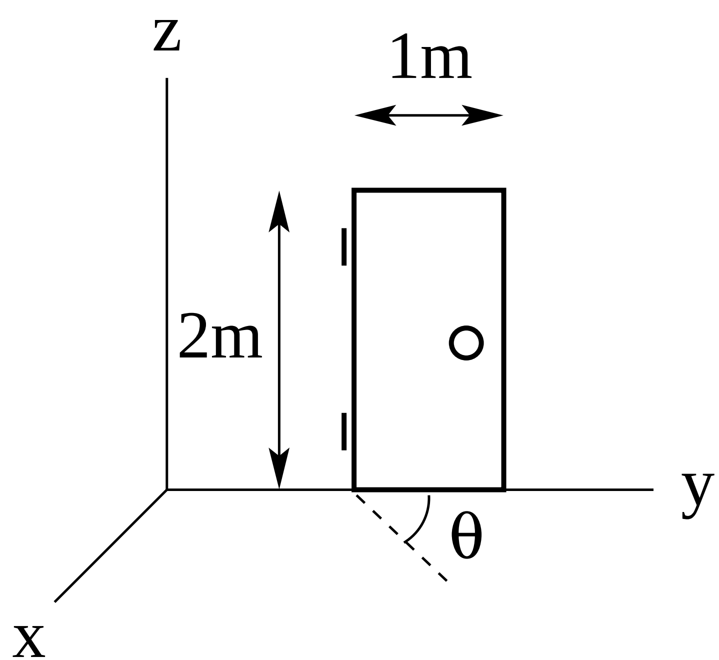
(a) Find the area of the sheet.

(b) Find the magnitude of the electric field.

(c) Find the electric flux through the sheet.

(d) Find the angle between $\vec{A}$ and $\vec{E}$.

The room shown below is positioned in an electric field $\vec{E} = (3\hat{i} + 2\hat{j} + 5\hat{k})$N/C.



(a) What is the electric flux Φ_E through the closed door?

(b) What is the electric flux Φ_E through the door opened at $\theta = 90°$?

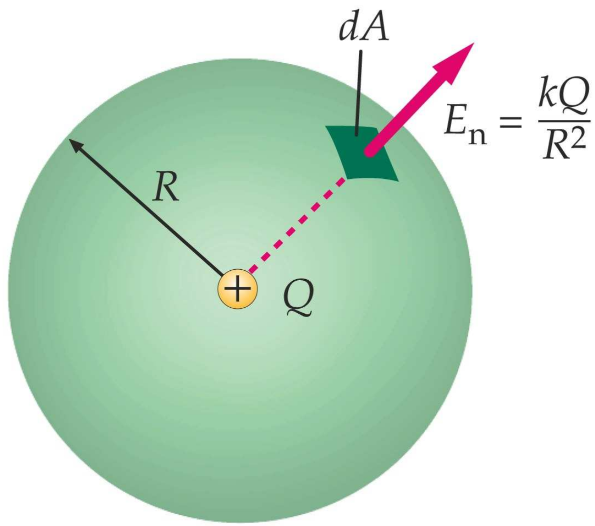
(c) At what angle θ_1 is the electric flux through the door zero?

(d) At what angle θ_2 is the electric flux through the door a maximum?

Electric Flux: Application (4)

Consider a positive point charge Q at the center of a spherical surface of radius R.
Calculate the electric flux through the surface.

- $\vec{E}$ is directed radially outward. Hence $\vec{E}$ is parallel to $d\vec{A}$ everywhere on the surface.
- $\vec{E}$ has the same magnitude, $E = kQ/R^2$, everywhere on the surface.
- The area of the spherical surface is $A = 4\pi R^2$.
- Hence the electric flux is $\Phi_E = EA = 4\pi kQ$.
- Note that Φ_E is independent of R.

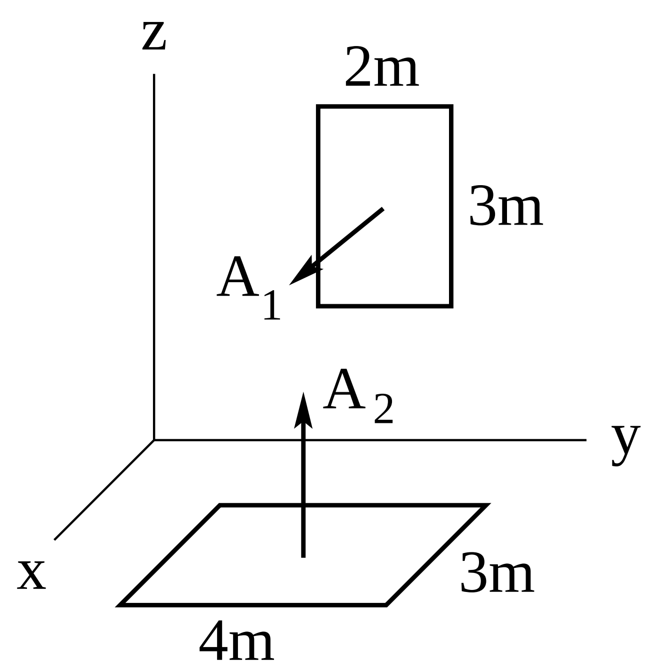


Intermediate Exam I: Problem #3 (Spring '05)

Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive x-direction) and $\vec{A}_2$ (pointing in positive z-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k})$N/C.

(a) Find the electric flux $\Phi_E^{(1)}$ through area A_1.

(b) Find the electric flux $\Phi_E^{(2)}$ through area A_2.



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(a) Find the electric flux $\Phi_E^{(1)}$ through area A_1.

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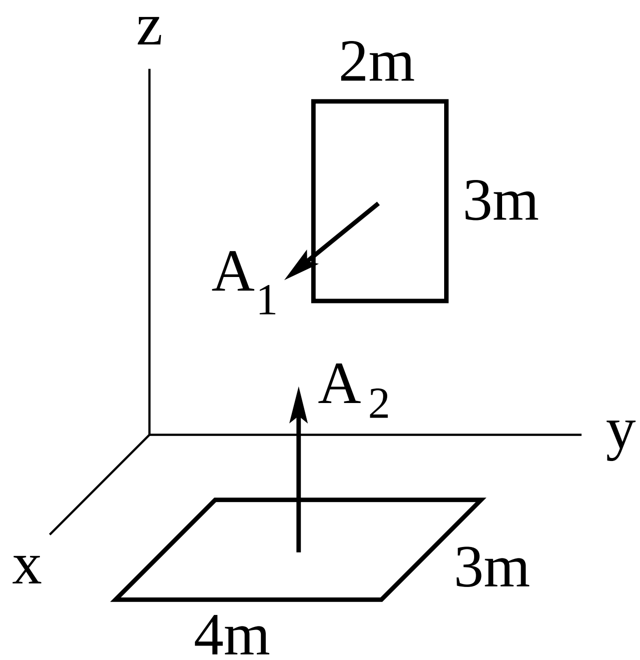


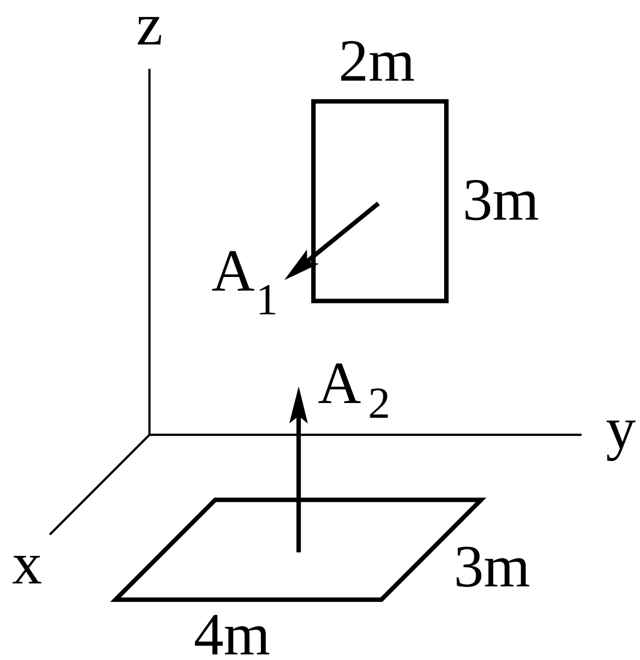
Solution:

(a) $\vec{A}_1 = 6\hat{i}\,\mathrm{m}^2$,
$\Phi_E^{(1)} = \vec{E} \cdot \vec{A}_1 = (2\mathrm{N/C})(6\mathrm{m}^2) = 12\mathrm{Nm}^2/\mathrm{C}$.

Consider two plane surfaces with area vectors $\vec{A}_1$ (pointing in positive x-direction) and $\vec{A}_2$ (pointing in positive z-direction). The region is filled with a uniform electric field $\vec{E} = (2\hat{i} + 7\hat{j} - 3\hat{k})$N/C.

(a) Find the electric flux $\Phi_E^{(1)}$ through area A_1.

(b) Find the electric flux $\Phi_E^{(2)}$ through area A_2.



Solution:

(a) $\vec{A}_1 = 6\hat{i}\,\mathrm{m}^2$,
$\Phi_E^{(1)} = \vec{E}\cdot\vec{A}_1 = (2\mathrm{N/C})(6\mathrm{m}^2) = 12\mathrm{Nm}^2/\mathrm{C}.$

(b) $\vec{A}_2 = 12\hat{k}\,\mathrm{m}^2$,
$\Phi_E^{(2)} = \vec{E}\cdot\vec{A}_2 = (-3\mathrm{N/C})(12\mathrm{m}^2) = -36\mathrm{Nm}^2/\mathrm{C}.$

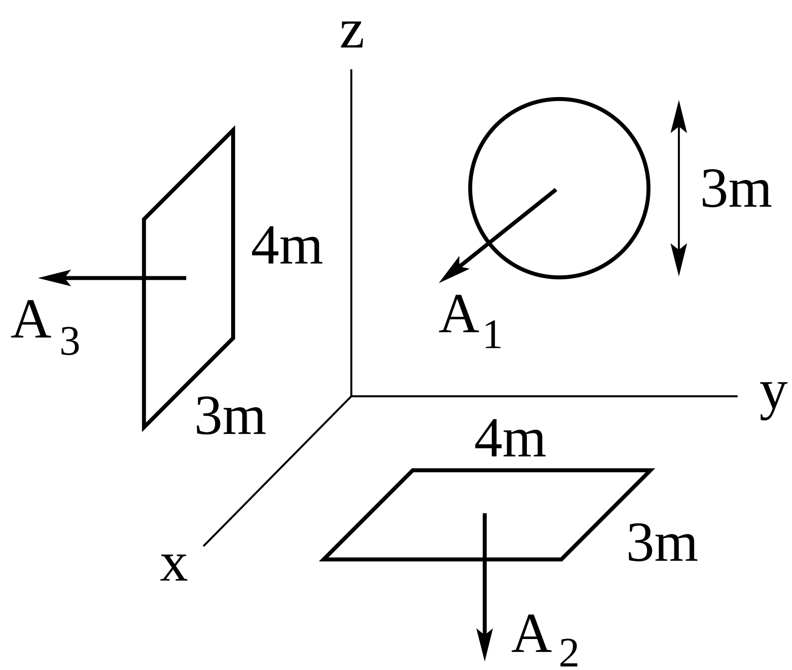
Unit Exam I: Problem #2 (Spring '13)

Consider three plane surfaces (one circle and two rectangles) with area vectors $\vec{A}_1$ (pointing in positive x-direction), $\vec{A}_2$ (pointing in negative z-direction), and $\vec{A}_3$ (pointing in negative y-direction) as shown. The region is filled with a uniform electric field $\vec{E} = (-3\hat{i} + 9\hat{j} - 4\hat{k})$N/C or $\vec{E} = (2\hat{i} - 6\hat{j} + 5\hat{k})$N/C.

(a) Find the electric flux $\Phi_E^{(1)}$ through surface 1.

(b) Find the electric flux $\Phi_E^{(2)}$ through surface 2.

(c) Find the electric flux $\Phi_E^{(3)}$ through surface 3.



Solution:

(a) $\vec{A}_1 = \pi(1.5\text{m})^2\hat{i} = 7.07\text{m}^2\hat{i}, \quad \Phi_E^{(1)} = \vec{E}\cdot\vec{A}_1 = (-3\text{N/C})(7.07\text{m}^2) = -21.2\text{Nm}^2/\text{C}.$

$\vec{A}_1 = \pi(1.5\text{m})^2\hat{i} = 7.07\text{m}^2\hat{i}, \quad \Phi_E^{(1)} = \vec{E}\cdot\vec{A}_1 = (2\text{N/C})(7.07\text{m}^2) = 14.1\text{Nm}^2/\text{C}.$

(b) $\vec{A}_2 = (3\text{m})(4\text{m})(-\hat{k}) = -12\text{m}^2\hat{k}, \quad \Phi_E^{(2)} = \vec{E}\cdot\vec{A}_2 = (-4\text{N/C})(-12\text{m}^2) = 48\text{Nm}^2/\text{C}.$

$\vec{A}_2 = (3\text{m})(4\text{m})(-\hat{k}) = -12\text{m}^2\hat{k}, \quad \Phi_E^{(2)} = \vec{E}\cdot\vec{A}_2 = (5\text{N/C})(-12\text{m}^2) = -60\text{Nm}^2/\text{C}.$

(b) $\vec{A}_3 = (3\text{m})(4\text{m})(-\hat{j}) = -12\text{m}^2\hat{j}, \quad \Phi_E^{(3)} = \vec{E}\cdot\vec{A}_3 = (9\text{N/C})(-12\text{m}^2) = -108\text{Nm}^2/\text{C}.$

$\vec{A}_3 = (3\text{m})(4\text{m})(-\hat{j}) = -12\text{m}^2\hat{j}, \quad \Phi_E^{(3)} = \vec{E}\cdot\vec{A}_3 = (-6\text{N/C})(-12\text{m}^2) = 72\text{Nm}^2/\text{C}.$